Algebraic approach to entanglement and entropy

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We present a general approach to quantum entanglement and entropy that is based on algebras of observables and states thereon. In contrast to more standard treatments, Hilbert space is an emergent concept, appearing as a representation space of the observable algebra, once a state is chosen. In this approach, which is based on the Gel'fand-Naimark-Segal construction, the study of subsystems becomes particularly clear. We explicitly show how the problems associated with the partial trace for the study of entanglement of identical particles are readily overcome. In particular, a suitable entanglement measure is proposed that can be applied to systems of particles obeying Fermi, Bose, para-, and even braid-group statistics. The generality of the method is also illustrated by study of the time evolution of subsystems emerging from restriction to subalgebras. Problems related to anomalies and quantum epistemology are also discussed.

I. INTRODUCTION

A. On states and algebras

In a physical theory, the theoretical description of observations contains two ingredients. The first is the state of the system being observed; it contains the data specifying the system. The second is the specific observable of the system being measured. The output of observations involves a suitable pairing of the state to the observable, which yields a number.

In both classical and quantum physics, the state ω provides a probability distribution and observables form an (associative) algebra. In classical physics, the state ω_c gives a probability distribution in phase space whereas the algebra $\mathcal{A}_c$ of observables consists of real-valued functions on phase space.[1] The product of two functions α, β on phase space is pointwise multiplication: for a point $x = (q, p)$ in phase space, $(\alpha\beta)(x) = \alpha(x)\beta(x)$. The algebra $\mathcal{A}_c$ is commutative.

The pairing of ω_c to α produces the mean value of α, as follows. The mean value of α for the state ω_c is

$$\omega_c(\alpha) = \int d\mu(x)\rho_c(x)\alpha(x), \qquad (1.1)$$

$$d\mu(p) = \text{Liouville measure on phase space}, \qquad (1.2)$$

where ρ_c is the probability density on phase space associated with ω_c. We note that, being a probability density, ρ_c is normalized as

$$\omega_c(\mathbb{1}) \equiv \int d\mu(x)\rho_c(x) = 1, \qquad (1.3)$$

where $\mathbb{1}$ is a constant function with value 1:

$$\mathbb{1}(x) = 1. \qquad (1.4)$$

In quantum theory we still have a state ω and an algebra $\mathcal{A}$. The state ω on an observable $\alpha \in \mathcal{A}$ is generally representable in terms of a density matrix ρ_ω and an operator $\pi_\omega(\alpha)$ representing α on a Hilbert space. The mean value of the observable α is then

$$\omega(\alpha) = \text{Tr}(\rho_\omega \alpha) \equiv \text{Tr}[\rho_\omega \pi_\omega(\alpha)]. \qquad (1.5)$$

From now on, we represent $\pi_\omega(\alpha)$ by α itself, if there is no ambiguity. The state and its density matrix are normalized just as in the classical case:

$$\omega(\mathbb{1}) = \text{Tr}\, \rho_\omega = 1. \qquad (1.6)$$

The basic mathematical difference between classical and quantum physics lies in this: *classical physics is a probability theory on a commutative algebra $\mathcal{A}_c$, quantum physics is a probability theory on a noncommutative algebra $\mathcal{A}$.*

A state ω need not be presented using a density matrix to extract numbers from theory. It is enough that ω is a linear map from $\mathcal{A}$ to $\mathbb{C}$ with the properties

$$\omega(\mathbb{1}) = 1, \quad \omega(\alpha^*) = \overline{\omega(\alpha)}, \quad \omega(\alpha^*\alpha) \geqslant 0 \quad \text{for all} \quad \alpha \text{ in } \mathcal{A},$$
$$(1.7)$$

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[1]More generally one talks of observables as 'Hermitian' functions on a complexified phase space.

where "∗" represents a Hermitian conjugation ("antilinear involution") that $\mathcal{A}$ should have. A state is best understood in this manner. State vectors and Hilbert spaces play no role at this point. Where, then, do they come from?

B. On the Gel'fand-Naimark-Segal construction

State vectors and Hilbert spaces are best thought of as emergent concepts in quantum physics. The primary concepts are states ω and the algebra $\mathcal{A}$ of observables.

In the 1940s, Gel'fand, Naimark, and Segal described the reconstruction of the Hilbert space $\mathcal{H}_\omega$ from the data $(\mathcal{A},\omega)$. The algebra $\mathcal{A}$ acts by a representation π_ω on $\mathcal{H}_\omega$. This reconstruction, known as the GNS construction, has played a foundational role in the theory of operator algebras. It has also been an important tool for studies in quantum field theory [1].

We suggest in this paper that the GNS construction is the proper framework for the study of entanglement as well. A brief account of our ideas has appeared before [2]. In Sec. II, we describe the GNS reconstruction of $(\mathcal{H}_\omega,\pi_\omega)$ from $(\mathcal{A},\omega)$. We do not aspire to rigor but to concepts and computations. Whatever we say is correct in finite dimensions. But our presentation omits the fine points of topology for infinite-dimensional $\mathcal{A}$.

The GNS construction presents the state ω as a density matrix ρ. We can expand ρ in terms of orthogonal rank-1 density matrices ρ_i:

$$\rho_i \rho_j = \delta_{ij}\rho_i, \quad \mathrm{Tr}\,\rho_i = 1, \qquad (1.8)$$

and accordingly write

$$\rho = \sum_i \lambda_i \rho_i, \quad \omega = \sum_i \lambda_i \omega_i, \quad \lambda_i > 0, \quad \sum_i \lambda_i = 1, \tag{1.9}$$

where ω_i is the state associated with the density matrix ρ_i. The von Neumann entropy for ω is then

$$S(\omega) = -\mathrm{Tr}\,\rho \ln \rho = -\sum_i \lambda_i \ln \lambda_i. \qquad (1.10)$$

We can thus associate an entropy with a pair $(\mathcal{A},\omega)$ of a state and an algebra of observables. This result is important for us.

C. What is entanglement?

For a system of *nonidentical* constituents A_i with Hilbert spaces $\mathcal{H}_i$, "entanglement" can be understood in terms of "partial trace" as follows. It is enough to consider a bipartite system and the Hilbert space

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2. \qquad (1.11)$$

Then a density matrix

$$\rho_{12} = |\psi\rangle\langle\psi| \qquad (1.12)$$

with a normalized vector state $|\psi\rangle \in \mathcal{H}$ is entangled when the density matrix

$$\rho_i = \mathrm{Tr}_j\,\rho_{12} \quad (j,i = 1,2, \quad i \neq j) \qquad (1.13)$$

obtained by partial tracing has nonzero von Neumann entropy $S(\rho_i)$:

$$S(\rho_i) = -\mathrm{Tr}\,\rho_i \ln \rho_i \neq 0.$$

There is no entanglement if $S(\rho_i) = 0$.

In Sec. III, we consider the restriction to subalgebras [2] which has relevance to the case of identical fermions, for which there is no general agreement regarding entanglement properties [3]. Our approach will allow us to propose an entanglement measure for identical particles that encompasses Fermi, Bose, and even braid-group statistics. We thus hope to contribute to the ongoing debate around entanglement of identical particles [4–13].

In this section we reformulate the notion of partial tracing in terms of the restriction $\omega_0 := \omega|_{\mathcal{A}_0}$ of a state ω on an algebra $\mathcal{A}$ to a subalgebra $\mathcal{A}_0$. The example we choose is very simple: a bipartite system of nonidentical particles. This restatement is possible whenever partial tracing can be understood in terms of a restriction. When such an interpretation is not possible, partial tracing loses meaning. But the restriction of ω to subalgebras is always sensible.

The notion of a "local operation" (as used in quantum information theory) is implemented here through the choice of an appropriate subalgebra $\mathcal{A}_0 \subseteq \mathcal{A}$. The importance of focusing on subsystems using subalgebras has also been emphasized in [14–17]. In our approach, the restriction of a state ω on $\mathcal{A}$ to the subalgebra $\mathcal{A}_0$ will give rise to a "reduced" or restricted state ω_0, whose entropy compared to that of ω provides a measure of the entanglement of $\mathcal{A}_0$ with $\mathcal{A}$ in the state ω. For the purposes of this paper, this is what is meant by entanglement.

If ω is pure, that is, $S(\omega) = 0$, the entropy of entanglement is obtained from the expansion

$$\omega_0 = \sum_i \lambda_i \omega_{0,i}, \quad \lambda_i > 0, \quad \sum_i \lambda_i = 1 \qquad (1.14)$$

of ω_0 in terms of pure states and using the formula (1.10). There are orderly ways to calculate the expansion (1.9) and that will be taken up in the following sections.

If ω itself is not pure, we can write it as a convex combination of pure states. If these resultant states can be chosen to be nonentangled, then the state will be called nonentangled. In the present paper we will consider entanglement only for pure states, leaving the mixed-state case for future work.

The operation of partial trace is very important in practical applications concerning identical particles. For example, in atomic physics and quantum chemistry, among other fields, the concept of the one-particle reduced density matrix $\rho^{(1)}$ contains important physical information (e.g., the kinetic energy). The two-particle reduced density matrix $\rho^{(2)}$ is also extremely important. For instance, for a fermionic system with pairwise interaction, the energy of a state can be expressed in terms of only $\rho^{(2)}$; hence its relevance for the N-representability problem, which can be considered as a special case of the quantum marginal problem [18].

However, the use of von Neumann entropy obtained from the partial trace is not completely satisfactory if one wants to interpret it as an entanglement measure. Consider, for illustration, the von Neumann entropy of the reduced density matrix ρ of an N-fermion state computed using the partial trace. The value of this entropy for a Slater rank-1 state is $\ln N$ [9]. But correlations due solely to the antisymmetric character of the state should not contribute to the entanglement of the system. There are strong physical reasons for this. For

instance, it has been shown in [9] that a Slater rank-1 state cannot be used to violate Bell's inequality. Also, in the context of many-body physics, one should expect vanishing entropy for a one-body function (whose marginals give precisely the electronic density) of the Hartee-Fock type [19]. Thus, in the case of fermions, due to the ln N term the criterion for deciding whether a state is entangled or not looks quite different from in the case of nonidentical particles. A similar remark applies to the case of bosons [9,20]. It will become clear, from the examples discussed in this paper, that it is indeed possible to keep a single entanglement criterion, independently of the statistics of the particles being considered.

D. Examples

We illustrate the GNS approach to entropy by several examples in Sec. IV. The first is simple and involves just the matrix algebra $\mathcal{A} = M_2(\mathbb{C})$.

This example is followed up using those from identical fermions and bosons. We clarify what is meant by observing single-particle operators of a boson or a fermion in a two-particle system. For this purpose, we need at the simplest level that the single-particle algebra $\mathcal{A}$ is Hopf with a coproduct Δ compatible with the permutation group which here defines the statistics[2] [21]. This Δ allows us to identify the correct "single-particle" subalgebra $\Delta(\mathcal{A})$ of the full two-particle algebra and to calculate $\omega|_{\Delta(\mathcal{A})}$ and its entropy for any two-body state ω. It does not agree with what one finds from partial tracing.

We can also illustrate all this for parabosons and parafermions. For these cases, Δ is not changed. So we instead consider the braid group B_N to be the N-particle statistics group. In that case Δ is changed to Δ_b. We calculate the restriction of a chosen three-particle state to single-particle observables, obtained from Δ_b in a systematic manner, and calculate its entropy. This illustrates the effectiveness of the GNS approach when combined with concepts from Hopf algebras, in quantum information theory.

If the subalgebra $\mathcal{A}_0$ comes from a k-particle subsystem in an N-particle system, it is necessary that $\mathcal{A}_0$ is at least a coalgebra with a coproduct Δ to extend its meaning beyond N particles.

Second quantization also requires the notion of the coproduct [21].

E. Time evolution

In the GNS approach, given the time evolution $\omega(t)$ of a state $\omega \equiv \omega(0)$ on $\mathcal{A}$, it induces a time evolution on the restriction $\omega_0 = \omega|_{\mathcal{A}_0}$ of ω to $\mathcal{A}_0$. It is

$$\omega_0(t) = \omega(t)|_{\mathcal{A}_0}. \tag{1.15}$$

If $\omega(0)$ is represented by a density matrix $\rho(0)$ with unitary time evolution according to

$$\rho(t) = U(t)^{-1}\rho(0)U(t), \quad U(t) = e^{-itH} \tag{1.16}$$

(where H is the Hamiltonian), there is no assurance that $\rho_0(t) = \rho(t)|_{\mathcal{A}_0}$ also undergoes a unitary evolution. The generic

situation will rather be that of an evolution by positive maps. This follows from Stinespring's theorem [22].

We illustrate the evolution $\rho_0(t) \to \rho_0(t + \tau)$ by completely positive maps, using our algebraic approach, in Sec. V. Here we only remark that this type of evolution arises very naturally from the restriction to a subalgebra. This is the basic mechanism behind quantum decoherence.

F. Anomalies

It has been suggested [23,24] that unwanted anomalies, such as the T violation associated with the quantum chromodynamics (QCD) angle θ, can be eliminated by using mixed states. The appearance of such mixed states for restoring anomalies can also be explained in terms of restrictions of states to subalgebras, as we show in Sec. VI. This argument provides further strong evidence for the power of the proposition for the use of mixed states in that context. The GNS theory is thus useful in many different physical contexts.

G. On state restrictions and quantum epistemology

We suggest in Sec. VII of this paper that the restriction $\omega_0 = \omega|_{\mathcal{A}_0}$ of a state ω on $\mathcal{A}$ can also be understood in terms of "collapse of wave packets" and an interpretative superstructure of quantum theory. Thus suppose that we have an algebra $\mathcal{A}$ of observables acting on a Hilbert space $\mathcal{H}$ and we observe a projector $p \in \mathcal{A}$. Let $\mathcal{A}_0 \subset \mathcal{A}$ be the maximal subalgebra which commutes with p (and hence includes p). Then we can interpret ω_0 as the restriction of ω to $\mathcal{A}_0$. We explain how this is so in that section.

II. THE GNS CONSTRUCTION

For completeness, we begin with a few definitions. They are all effortlessly fulfilled in all our examples where we consider only finite-dimensional matrix algebras.

A $*$-algebra $\mathcal{A}$ is an associative algebra over $\mathbb{C}$ with an antilinear involution ("Hermitian conjugation"):

$$* : \mathcal{A} \to \mathcal{A}, \quad *^2 = \text{id}. \tag{2.1}$$

A C^* norm $\| \cdot \|$ on such an algebra $\mathcal{A}$ is a norm fulfilling the property

$$\|\alpha^*\alpha\| = \|\alpha\|^2, \quad \forall \, \alpha \in \mathcal{A}. \tag{2.2}$$

If it exists, it is unique.

Algebras $M_d(\mathbb{C})$ of $d \times d$ matrices can all be regarded as C^* algebras, with the norm fixed by

$$\|\alpha\|^2 = \text{ largest eigenvalue of } \alpha^*\alpha. \tag{2.3}$$

The algebra $\mathcal{A}$ of observables in quantum theory is a C^* algebra. Indeed, the algebra of all bounded operators $\mathcal{B}(\mathcal{H})$ on a separable Hilbert space admits a norm to turn it into a C^* algebra.

We will assume that $\mathcal{A}$ is unital with unity $\mathbb{1}_\mathcal{A}$. That is needed to discuss completeness relations, for example.

The data we are given are thus $(\mathcal{A}, \omega)$. We can construct $(\mathcal{H}_\omega, \pi_\omega(\mathcal{A}))$ from these data as follows.

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[2]It is perhaps enough if $\mathcal{A}$ is quasi-Hopf or just a coalgebra.

With each element $\alpha \in \mathcal{A}$, we associate a vector $|\alpha\rangle$ in a complex vector space $\hat{\mathcal{A}}$ with the property

$$|\lambda\alpha + \mu\beta\rangle = \lambda|\alpha\rangle + \mu|\beta\rangle, \quad \lambda,\mu \in \mathbb{C}, \quad \alpha,\beta \in \mathcal{A}. \quad (2.4)$$

Next, an inner product is introduced in $\hat{\mathcal{A}}$ using ω:

$$\langle\beta|\alpha\rangle = \omega(\beta^*\alpha). \quad (2.5)$$

It fulfills

$$\langle\mathbb{1}_\mathcal{A}|\alpha\rangle = \omega(\alpha) = \overline{\langle\alpha|\mathbb{1}_\mathcal{A}\rangle} = \overline{\omega(\alpha^*)} \quad (2.6)$$

and the Schwarz inequality as well in view of (1.7) $[\omega(\alpha^*\alpha) \geqslant 0]$:

$$|\langle\beta|\alpha\rangle|^2 \leqslant \langle\beta|\beta\rangle\langle\alpha|\alpha\rangle. \quad (2.7)$$

But it may not be a scalar product from which we can build a Hilbert space, as there may be $0 \neq \alpha \in \mathcal{A}$ giving vectors $|\alpha\rangle$ of zero norm:

$$\langle\alpha|\alpha\rangle = 0. \quad (2.8)$$

Let $\mathcal{N}_\omega$ denote the subspace of $\mathcal{A}$ whose image $\hat{\mathcal{N}}_\omega \subset \hat{\mathcal{A}}$ contains vectors of zero norm:

$$\mathcal{N}_\omega = \{\alpha \in \mathcal{A} \mid \langle\alpha|\alpha\rangle = 0\}. \quad (2.9)$$

Observe that, from the Schwarz inequality,

$$\langle a|\alpha\rangle = 0, \quad \forall \, a \in \mathcal{A}, \; \alpha \in \mathcal{N}_\omega. \quad (2.10)$$

Hence, $\mathcal{N}_\omega$ is a left ideal. That is,

$$a\mathcal{N}_\omega \subseteq \mathcal{N}_\omega \quad \forall \; a \in \mathcal{A}. \quad (2.11)$$

This follows from (2.10): if $a \in \mathcal{A}$ and $\alpha \in \mathcal{N}_\omega$, $\langle a\alpha|a\alpha\rangle = \langle a^*a\alpha|\alpha\rangle = 0$ by (2.10). The subspace $\mathcal{N}_\omega$ is called the Gel'fand ideal.

We next consider the vector space

$$\hat{\mathcal{A}}/\hat{\mathcal{N}}_\omega = \{|[a]\rangle := |a + \mathcal{N}_\omega\rangle, \; a \in \mathcal{A}\}. \quad (2.12)$$

The label $[a]$ of a vector denotes an equivalence class $a + \mathcal{N}_\omega$, a set, in $\mathcal{A}$.

Now:

(a) $\hat{\mathcal{A}}/\hat{\mathcal{N}}_\omega$ has a well-defined scalar product $\langle\cdot|\cdot\rangle$ given by

$$\langle[a]|[b]\rangle = \omega(a^*b). \quad (2.13)$$

In particular the vector $|\mathcal{N}_\omega\rangle$, playing the role of zero, is the only vector that has zero norm. Note in this connection that since $\omega(a^*(b + \alpha)) = \omega(a^*b)$ for all $\alpha \in \mathcal{N}_\omega$, the right-hand side of (2.13) does not depend on the choices of representatives a,b from $[a]$ and $[b]$.

We denote the Hilbert space obtained by the closure of $\hat{\mathcal{A}}/\hat{\mathcal{N}}_\omega$ under (2.13) as $\mathcal{H}_\omega$.

(b) Because of (2.11), $\mathcal{H}_\omega$ carries a representation π_ω of $\mathcal{A}$:

$$\pi_\omega(a)|[b]\rangle := |[ab]\rangle. \quad (2.14)$$

We have now obtained $(\mathcal{H}_\omega, \pi_\omega(\mathcal{A}))$ from $(\mathcal{A}, \omega)$, thereby completing the GNS construction.

A. Properties of the GNS representation

Consider the vector $|[\mathbb{1}_\mathcal{A}]\rangle$. Then, if $\pi_\omega(\mathcal{A})$ denotes the set $\{\pi_\omega(a) : \forall \, a \in \mathcal{A}\}$,

$$\pi_\omega(\mathcal{A})|[\mathbb{1}_\mathcal{A}]\rangle = \{|[a]\rangle \mid a \in \mathcal{A}\}, \quad (2.15)$$

so that from $|[\mathbb{1}_\mathcal{A}]\rangle$ we can generate all vectors of $\mathcal{H}_\omega$ by acting with $\pi_\omega(\mathcal{A})$ and closure. Such a vector $|[\mathbb{1}_\mathcal{A}]\rangle$ is said to be *cyclic*. The representation π_ω is a cyclic representation.

The state ω can now be represented as a density matrix ρ_ω,

$$\rho_\omega = |[\mathbb{1}_\mathcal{A}]\rangle\langle[\mathbb{1}_\mathcal{A}]|. \quad (2.16)$$

That is because

$$\begin{aligned}
\omega(a^*b) = \langle[a]|[b]\rangle &= \langle[\mathbb{1}_\mathcal{A}]|\pi_\omega(a^*)\pi_\omega(b)|[\mathbb{1}_\mathcal{A}]\rangle \\
&= \mathrm{Tr}[\rho_\omega\pi_\omega(a^*)\pi_\omega(b)]. \quad (2.17)
\end{aligned}$$

B. On irreducibility and entropy

The representation π_ω may not be irreducible. In finite dimensions at least, which are our concern here, it can be reduced to a direct sum of irreducible representations $\pi_\omega^{(\alpha)}$:

$$\pi_\omega = \oplus_\alpha \pi_\omega^{(\alpha)}. \quad (2.18)$$

That is because $\mathcal{A}$ is a $*$-algebra. The proof is similar to that for finite groups G, their group algebras also being $*$-algebras.

The proof of (2.18) goes as follows. If $\mathcal{H}_\omega^{(1)} \subset \mathcal{H}_\omega$ is a nontrivial invariant subspace under $\pi_\omega(\mathcal{A})$, then so is its orthogonal complement $\mathcal{H}_\omega^{(1)\perp}$. For if

$$|[\alpha]\rangle \in \mathcal{H}_\omega^{(1)}, \quad |[\beta]\rangle \in \mathcal{H}_\omega^{(1)\perp}, \quad (2.19)$$

so that $\langle[\alpha]|[\beta]\rangle = 0$, then

$$\langle[\alpha]|[a\beta]\rangle = 0, \quad \forall \, a \in \mathcal{A}, \quad (2.20)$$

for the left-hand side is $\langle[a^*\alpha]|[\beta]\rangle$. But $a^* \in \mathcal{A}$ ($\mathcal{A}$ being $*$) and hence $|[a^*\alpha]\rangle \in \mathcal{H}_\omega^{(1)}$. The statement follows.

Now, $\mathcal{H}_\omega^{(1)}$ (and similarly $\mathcal{H}_\omega^{(1)\perp}$) is either irreducible or has a nontrivial invariant subspace. If the latter is the case, we repeat the process, ending up with

$$\mathcal{H}_\omega = \oplus_\alpha \mathcal{H}_\omega^{(\alpha)}, \quad (2.21)$$

where the sum is the orthogonal direct sum and $\mathcal{H}_\omega^{(\alpha)}$ carries $\pi_\omega^{(\alpha)}(\mathcal{A})$.

We now show how to correspondingly decompose ρ_ω into a convex sum of orthogonal rank-1 density matrices:

$$\begin{aligned}
\rho_\omega = \sum_\alpha \lambda_\alpha \rho_\omega^{(\alpha)}, \; \lambda_\alpha > 0, \;\; \sum_\alpha \lambda_\alpha = 1, \\
\rho_\omega^{(\alpha)}\rho_\omega^{(\beta)} = \delta_{\alpha\beta}\rho_\omega^{(\alpha)}. \quad (2.22)
\end{aligned}$$

For this purpose, we write

$$|[\mathbb{1}_\mathcal{A}]\rangle = \sum_\alpha |[\mathbb{1}_\mathcal{A}^{(\alpha)}]\rangle, \quad |[\mathbb{1}_\mathcal{A}^{(\alpha)}]\rangle \in \mathcal{H}_\omega^{(\alpha)}, \quad (2.23)$$

that is, we decompose the left-hand side into its components in $\mathcal{H}_\omega^{(\alpha)}$. Now, since for $\alpha \neq \beta$

$$\langle[\mathbb{1}_\mathcal{A}^{(\alpha)}]|\pi_\omega(a)|[\mathbb{1}_\mathcal{A}^{(\beta)}]\rangle = 0, \quad (2.24)$$

the density matrix ρ_ω can be rewritten as

$$\rho_\omega = \sum_\alpha |[\mathbb{1}_\mathcal{A}^{(\alpha)}]\rangle\langle[\mathbb{1}_\mathcal{A}^{(\alpha)}]|. \quad (2.25)$$

If we set

$$\lambda_\alpha = \langle[\mathbb{1}_\mathcal{A}^{(\alpha)}]|[\mathbb{1}_\mathcal{A}^{(\alpha)}]\rangle, \quad (2.26)$$

we can rewrite ρ_ω in terms of normalized vectors, as follows. We define

$$|\chi^{(\alpha)}\rangle = \frac{1}{\sqrt{\lambda_\alpha}} |[\mathbb{1}_{\mathcal{A}}^{(\alpha)}]\rangle. \tag{2.27}$$

Then,

$$\langle \chi^{(\alpha)} | \chi^{(\beta)} \rangle = \delta_{\alpha\beta}. \tag{2.28}$$

With

$$\rho_\omega^{(\alpha)} := |\chi^{(\alpha)}\rangle\langle\chi^{(\alpha)}|, \tag{2.29}$$

we then obtain the decomposition of ρ_ω in terms of pure states as

$$\rho_\omega = \sum_\alpha \lambda_\alpha \rho_\omega^{(\alpha)}, \tag{2.30}$$

where

$$\lambda_\alpha > 0, \quad \sum_\alpha \lambda_\alpha = 1, \quad \rho_\omega^{(\alpha)}\rho_\omega^{(\beta)} = \delta_{\alpha\beta}\rho_\omega^{(\alpha)}. \tag{2.31}$$

The von Neumann entropy of ρ_ω is

$$S(\rho_\omega) = -\operatorname{Tr}\rho_\omega \ln \rho_\omega = -\sum_\alpha \lambda_\alpha \ln \lambda_\alpha. \tag{2.32}$$

Corresponding to (2.30), we have the decomposition of ω into extremal or pure states $\omega^{(\alpha)}$ [Recall that $\omega(\cdot) = \operatorname{Tr}\rho\cdot$]:

$$\omega = \sum_\alpha \lambda_\alpha \omega^{(\alpha)}. \tag{2.33}$$

It has entropy

$$S(\omega) = -\sum_\alpha \lambda_\alpha \ln \lambda_\alpha. \tag{2.34}$$

There are important issues related to the uniqueness of the decomposition (2.30) and hence of the entropy of ω as observed by Sorkin [25] [cf. Eq. (4.27)]. For a detailed discussion of these issues we refer to [26].

III. ON ENTANGLEMENT AND SUBALGEBRAS

For a bipartite system of nonidentical particles A and B with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, a vector state $|\psi\rangle \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, which in general is of the form

$$|\psi\rangle = \sum_{i,j} C_{ij} |\chi_{A,i}\rangle \otimes |\eta_{B,j}\rangle, \tag{3.1}$$

is said to be entangled if it cannot be reduced to the form

$$|\psi\rangle = |\chi'_A\rangle \otimes |\eta'_B\rangle \tag{3.2}$$

by a change of basis. A measure of entanglement is the von Neumann entropy of the reduced density matrix

$$\rho_A = \operatorname{Tr}_{\mathcal{H}_B} |\psi\rangle\langle\psi|. \tag{3.3}$$

We assume that $\langle\psi|\psi\rangle = 1$. The vector $|\psi\rangle$ is entangled if and only if

$$S(\rho_A) = -\rho_A \ln \rho_A \neq 0. \tag{3.4}$$

The physical meanings of the partial trace and the reduced density matrix are as follows. Suppose that we observe only the subalgebra

$$\mathcal{A}_0 = \{\alpha_0 \in \mathcal{A} \mid \alpha_0 = K_A \otimes \mathbb{1}_B\}, \tag{3.5}$$

where K_A is an observable acting on $\mathcal{H}_A$ and $\mathbb{1}_B$ is the identity operator on $\mathcal{H}_B$. The algebra $\mathcal{A}$ of all observables on $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ contains $\mathcal{A}_0$ as a subalgebra:

$$\mathcal{A}_0 \subset \mathcal{A}. \tag{3.6}$$

Now, for restricted observations of just $\mathcal{A}_0$,

$$\operatorname{Tr}_{\mathcal{H}}(\rho\alpha_0) = \operatorname{Tr}_{\mathcal{H}_A}(\rho_A K_A), \quad \alpha_0 = K_A \otimes \mathbb{1}_B \in \mathcal{A}_0. \tag{3.7}$$

Thus, ρ_A is said to be the *restriction* of ρ to $\mathcal{A}_0$.

Let ω be the state for the density matrix $\rho = |\psi\rangle\langle\psi|$:

$$\omega(\alpha) = \operatorname{Tr}(\rho\alpha), \quad \alpha \in \mathcal{A}. \tag{3.8}$$

In the same way, let ω_A be the state for the density matrix ρ_A:

$$\omega_A(\alpha_0) = \operatorname{Tr}_{\mathcal{H}_A}(\rho_A K_A), \quad \alpha_0 = K_A \otimes \mathbb{1}_B \in \mathcal{A}_0. \tag{3.9}$$

Then ω_A is said to be the *restriction* of the state ω on $\mathcal{A}$ to $\mathcal{A}_0$:

$$\omega_A = \omega|_{\mathcal{A}_0}. \tag{3.10}$$

Thus the partial trace in this case maps a density matrix ρ and a state ω on $\mathcal{A}$ to their restrictions ρ_A, ω_A on $\mathcal{A}_0$.

But there are many cases where the partial trace is not amenable to such a restriction. A well-discussed example is that of identical fermions [6,9,11,12]. Denoting antisymmetrization by $\wedge$,

$$|\psi\rangle \wedge |\chi\rangle = \frac{1}{\sqrt{2}}(|\psi\rangle \otimes |\chi\rangle - |\chi\rangle \otimes |\psi\rangle), \tag{3.11}$$

a generic N-particle vector of identical fermions is a linear combination of vectors of the form

$$|\psi\rangle = |\psi_1\rangle \wedge |\psi_2\rangle \wedge \cdots \wedge |\psi_N\rangle. \tag{3.12}$$

It exists in the N-fold antisymmetric product $\mathcal{H}$ of the one-particle Hilbert space $\mathcal{H}^{(1)}$:

$$\mathcal{H} = \bigwedge^N \mathcal{H}^{(1)}, \quad |\psi\rangle \in \mathcal{H}. \tag{3.13}$$

The algebra $\mathcal{A}$ of *observables* must necessarily leave $\mathcal{H}$ invariant. That means that observables must be permutation invariant. An operator such as $K_1 \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$ ($\mathbb{1}$ is the identity on $\mathcal{H}^{(1)}$ and $K_1 \neq \mathbb{1}$) is not permutation invariant and, therefore, is not an observable. Hence now *partial traces do not correspond to restrictions to subalgebras of observables on* $\mathcal{H}$.

But the restriction of a state ω on $\mathcal{A}$ to a subalgebra $\mathcal{A}_0$ is always sensible. What we need is a criterion to select $\mathcal{A}_0$ appropriately for a physical question.

For example, the algebra $\mathcal{A}_0$ appropriate for single-particle observables is generated by

$$K \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \ + \ \mathbb{1} \otimes K \otimes \cdots \otimes \mathbb{1} \ + \ \cdots + \mathbb{1}$$
$$\otimes \cdots \otimes \mathbb{1} \otimes K, \tag{3.14}$$

where K is an observable on $\mathcal{H}^{(1)}$. Choices such as (3.14) are dictated by a coproduct on the single-particle algebra. We will return to this point later.

For such reasons, as declared in the Introduction, *entanglement is a property characterizing a triple* $(\mathcal{A}, \mathcal{A}_0, \omega)$. *We avoid the use of the partial trace.*

IV. EXAMPLES

A. The algebra $M_2(\mathbb{C})$

The choice $\mathcal{A} = M_2(\mathbb{C})$ of 2×2 complex matrices is a simple nontrivial example to illustrate the GNS construction. It is discussed already in Appendix 3 of Landi [27]. We will recall this example here and, in addition, use it to illustrate the entropy calculation.

The algebra $\mathcal{A}$ acts on $\mathbb{C}^2$. Let

$$\{|i\rangle : \ i = 1, 2, \ \langle i|j\rangle = \delta_{ij}\} \tag{4.1}$$

be an orthonormal basis of $\mathbb{C}^2$. Then the matrix units

$$e_{ij} = |i\rangle\langle j| \tag{4.2}$$

span $M_2(\mathbb{C})$. Note that

$$e_{ij}e_{kl} = \delta_{jk}e_{il}. \tag{4.3}$$

An element α of $\mathcal{A}$ can be expanded in this basis as

$$\alpha = \sum_{i,j} \alpha_{ij} e_{ij}. \tag{4.4}$$

Following [2], for the state ω we choose

$$\omega(\alpha) = \lambda\alpha_{11} + (1-\lambda)\alpha_{22}, \quad 0 \leqslant \lambda \leqslant 1. \tag{4.5}$$

The null space $\mathcal{N}_\omega$ is determined by the condition

$$\omega(\alpha^*\alpha) = 0. \tag{4.6}$$

For our choice (4.5) for ω we obtain, making use of (4.3),

$$\omega(\alpha^*\alpha) = \lambda(|\alpha_{11}|^2 + |\alpha_{21}|^2) + (1-\lambda)(|\alpha_{12}|^2 + |\alpha_{22}|^2). \tag{4.7}$$

The solution depends on λ. We consider three cases.

Case 1. $\lambda = 0$.

In this case, $\alpha \in \mathcal{N}_\omega$ if $\alpha_{12} = \alpha_{22} = 0$. So

$$\mathcal{N}_\omega = \left\{ \begin{pmatrix} \alpha_{11} & 0 \\ \alpha_{21} & 0 \end{pmatrix} : \ \alpha_{11}, \alpha_{21} \in \mathbb{C} \right\} \cong \mathbb{C}^2. \tag{4.8}$$

Since $\hat{\mathcal{A}} \cong \mathbb{C}^4$, we obtain

$$\mathcal{H}_\omega = \hat{\mathcal{A}}/\hat{\mathcal{N}}_\omega \cong \mathbb{C}^2, \tag{4.9}$$

with basis

$$\{|[e_{k2}]\rangle\}_{k=1,2}. \tag{4.10}$$

The representation π_ω of $\mathcal{A}$ on $\mathcal{H}_\omega$ is given by

$$\pi_\omega(e_{ij})|[e_{k2}]\rangle = \delta_{jk}|[e_{i2}]\rangle. \tag{4.11}$$

It is isomorphic to the defining representation on $\mathbb{C}^2$. Therefore it is irreducible. So we conclude that ρ_ω is a rank-1 projector and has vanishing entropy:

$$S(\rho_\omega) = 0. \tag{4.12}$$

Case 2. $\lambda = 1$.

This is similar to the case $\lambda = 0$. The null space $\hat{\mathcal{N}}_\omega \cong \mathbb{C}^2$ is spanned by

$$|e_{12}\rangle, \ |e_{22}\rangle. \tag{4.13}$$

and $\mathcal{H}_\omega = \hat{\mathcal{A}}/\hat{\mathcal{N}}_\omega$ has basis

$$\{|[e_{k1}]\rangle\}_{k=1,2}. \tag{4.14}$$

The representation π_ω is irreducible, isomorphic to the $\lambda = 0$ representation, and carries zero entropy.

Case 3. $0 < \lambda < 1$.

There are no nonzero null vectors in this case:

$$\mathcal{N}_{\omega_\lambda} = \{0\}. \tag{4.15}$$

Hence

$$\mathcal{H}_\omega = \hat{\mathcal{A}}/\hat{\mathcal{N}}_\omega \cong \mathbb{C}^4. \tag{4.16}$$

It has basis

$$\{|[e_{ij}]\rangle\}_{i,j=1,2}. \tag{4.17}$$

The representation π_ω is given by

$$\pi_\omega(e_{ij})|[e_{kl}]\rangle = |[e_{ij}e_{kl}]\rangle = \delta_{jk}|[e_{il}]\rangle. \tag{4.18}$$

This representation is reducible into two two-dimensional irreducible ones: $\mathcal{H}_\omega = \mathbb{C}^2 \oplus \mathbb{C}^2$. The first $\mathbb{C}^2$ has basis

$$\{e_{a1}\}_{a=1,2} \tag{4.19}$$

and the second

$$\{e_{a2}\}_{a=1,2}. \tag{4.20}$$

We must next express $|[\mathbb{1}_\mathcal{A}]\rangle$ in terms of its components in these subspaces. That is easy:

$$|[\mathbb{1}_\mathcal{A}]\rangle = |[e_{11}]\rangle + |[e_{22}]\rangle. \tag{4.21}$$

It follows that ω_λ is not pure and can be expressed in terms of the following density matrix:

$$\rho_{\omega_\lambda} = |[e_{11}]\rangle\langle[e_{11}]| + |[e_{22}]\rangle\langle[e_{22}]|. \tag{4.22}$$

Since

$$\langle[e_{11}]|[e_{11}]\rangle = \omega(e_{11}^*e_{11}) = \omega(e_{11}) = \lambda, \\ \langle[e_{22}]|[e_{22}]\rangle = \omega(e_{22}^*e_{22}) = \omega(e_{22}) = 1-\lambda, \tag{4.23}$$

$$\rho_{\omega_\lambda} = \lambda\rho_{11} + (1-\lambda)\rho_{22}, \tag{4.24}$$

where ρ_{11} and ρ_{22} are the rank-1 density matrices

$$\rho_{11} = \frac{1}{\lambda}|[e_{11}]\rangle\langle[e_{11}]|, \\ \rho_{22} = \frac{1}{1-\lambda}|[e_{22}]\rangle\langle[e_{22}]|. \tag{4.25}$$

We can read off the entropy of ρ_{ω_λ} to be

$$S(\rho_{\omega_\lambda}) = -\lambda \ln \lambda - (1-\lambda)\ln(1-\lambda). \tag{4.26}$$

Remark. This example shows that the irreducible representations of dimension 2 occur with multiplicity 2. This is a general feature: for any representation π of a C^*-algebra $\mathcal{A}$ with a *cyclic and separating vector*[3] [1], an irreducible representation of dimension d occurs with multiplicity d. This is as in regular representations of compact groups and follows from Tomita-Takesaki theory [1].

Sorkin has pointed out to us [25] that if $d > 1$, the splitting of $\mathcal{H}_\omega$ into irreducible subspaces such as $\mathcal{H} = \mathbb{C}^2 \oplus \mathbb{C}^2$ before

———

[3]A vector state $|\Omega\rangle \in \mathcal{H}$ is cyclic when $\pi(\mathcal{A})|\Omega\rangle$ is dense in $\mathcal{H}$. A state $|\Omega\rangle \in \mathcal{H}$ is separating when the map $\mathcal{A} \to \pi(\mathcal{A})|\Omega\rangle$ is injective.

(4.19) is not unique. For example, we could have chosen another pair of $\mathbb{C}^2$'s with basis

$$\sum_\alpha \xi_\alpha |[e_{a\alpha}]\rangle, \quad \sum_\alpha \eta_\alpha |[e_{a\alpha}]\rangle, \quad \xi^\dagger \xi, \ \eta^\dagger \eta \neq 0, \ \xi^\dagger \eta = 0,$$
(4.27)

and recalculated ρ_{ω_λ} and its entropy. They depend on ξ, η. This feature is generic for $d > 1$. The entropy (4.26) is the least one. Further discussion of such issues will be given elsewhere [26].

B. A $\mathbb{C}^2 \otimes \mathbb{C}^2$ example

The matrices

$$\sigma_\mu : \quad \sigma_0 = \mathbb{1}_2, \ \sigma_i = \text{Pauli matrices}, \qquad (4.28)$$

form a basis for $M_2(\mathbb{C})$. So the algebra $\mathcal{A} = M_4(\mathbb{C})$ on $\mathbb{C}^2 \otimes \mathbb{C}^2$ is generated by

$$\sigma_\mu \otimes \mathbb{1}_2, \quad \mathbb{1}_2 \otimes \sigma_\mu, \quad \mu = 0, \dots, 3. \qquad (4.29)$$

For the state $\omega \equiv \omega_\theta$, let us choose

$$\rho_{\omega_\theta} = |\psi_\theta\rangle\langle\psi_\theta|, \quad |\psi_\theta\rangle = \cos\theta |+-\rangle - \sin\theta |-+\rangle. \quad (4.30)$$

Here

$$|+-\rangle = |+\rangle \otimes |-\rangle, \quad \text{etc.,} \qquad (4.31)$$

with

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let us now choose, for the subalgebra $\mathcal{A}_0$, the subalgebra generated by the ("local") operators

$$\{\sigma_\mu \otimes \mathbb{1}_2\}_{\mu=0,\dots,3}. \qquad (4.32)$$

In this context, entanglement can be understood in terms of correlations between measurements performed by two observers A and B having access to observables corresponding only to $\mathcal{A}_0$ (in the case of, say, A) and observables corresponding only to the commutant of $\mathcal{A}_0$ (in the case of B).

For A, the state ω_θ becomes the restriction of ω_θ to $\mathcal{A}_0$:

$$\omega_{\theta,0} = \omega_\theta|_{\mathcal{A}_0}. \qquad (4.33)$$

Notice that, *in this case*, the result of the restriction coincides with the one obtained by partial trace. In fact, since every element of $\mathcal{A}_0$ is of the form $\tilde{\alpha} = \alpha \otimes \mathbb{1}_2$ [for some $\alpha \in M_2(\mathbb{C})$], we obtain

$$\begin{aligned} \omega_{\theta,0}(\tilde{\alpha}) &= \langle\psi_\theta|(\alpha \otimes \mathbb{1}_2)|\psi_\theta\rangle \\ &= \cos^2\theta \langle+|\alpha|+\rangle + \sin^2\theta \langle-|\alpha|-\rangle \\ &= \cos^2\theta \, \alpha_{11} + \sin^2\theta \, \alpha_{22}. \end{aligned} \qquad (4.34)$$

Taking into account the fact that $\mathcal{A}_0 \cong M_2(\mathbb{C})$, and comparing Eqs. (4.34) and (4.5), we see that the entropy obtained from the GNS construction is given by (4.26) upon replacing λ by $\cos^2\theta$:

$$S(\theta) = -\cos^2\theta \ln \cos^2\theta - \sin^2\theta \ln \sin^2\theta. \qquad (4.35)$$

Clearly, $S(\theta)$ corresponds to the entanglement of the vector state $|\psi_\theta\rangle$. In particular, for $\theta = \pi/4$ the state is the (maximally

entangled) Bell vector state

$$|\psi_{\theta=\pi/4}\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle), \qquad (4.36)$$

for which (4.35) reduces to $\ln 2$.

Now, in order to illustrate how to deal with cases where $\mathcal{A}_0$ does not act on just one factor of a bipartite system, we consider the Bell state (4.36) together with different choices for $\mathcal{A}_0$. We focus on choices for which the partial trace has no meaning. For instance, we now consider the following three choices:

   (a) $\mathcal{A}_\pm$, generated by $\{\sigma_\mu \otimes (\frac{1\pm\sigma_3}{2})\}_\mu$;

   (b) $\mathcal{A}_+ \oplus \mathcal{A}_-$.

(Note that $\mathcal{A}_+ \mathcal{A}_- = \{0\}$.)

Case 1. $\mathcal{A}_0 = \mathcal{A}_+$. The null space $\widehat{\mathcal{N}}_\omega^+ \subset \hat{\mathcal{A}}_+$ is determined by the equation

$$\rho_\omega(\alpha^*\alpha) = \langle\psi_{\theta=\pi/4}|\alpha^*\alpha|\psi_{\theta=\pi/4}\rangle = 0 \quad \text{for} \quad \alpha \in \mathcal{A}_+,$$
(4.37)

or

$$\alpha|\psi_{\theta=\pi/4}\rangle = 0. \qquad (4.38)$$

Hence

$$\widehat{\mathcal{N}}_\omega^+ = \left\{ \left| \left| \begin{pmatrix} \alpha_{11} & 0 \\ \alpha_{21} & 0 \end{pmatrix} \otimes \left(\frac{1+\sigma_3}{2} \right) \right\rangle : \alpha_{i1} \in \mathbb{C} \right\} \cong \mathbb{C}^2.$$
(4.39)

The quotient space $\hat{\mathcal{A}}/\widehat{\mathcal{N}}_\omega^+ \cong \mathbb{C}^2$ is spanned by

$$\left\{ \left| \left[\begin{pmatrix} 0 & \alpha_{12} \\ 0 & \alpha_{22} \end{pmatrix} \otimes \left(\frac{1+\sigma_3}{2} \right) \right] \right\rangle : \alpha_{i2} \in \mathbb{C} \right\}. \qquad (4.40)$$

It transforms irreducibly under $\mathcal{A}^+$. So ρ_ω remains pure with zero entropy.

Case 2. $\mathcal{A}_0 = \mathcal{A}_-$. This is similar to case 1. The null space is

$$\widehat{\mathcal{N}}_\omega^- = \left\{ \left| \left| \begin{pmatrix} 0 & \alpha_{12} \\ 0 & \alpha_{22} \end{pmatrix} \otimes \left(\frac{1-\sigma_3}{2} \right) \right\rangle : \alpha_{i2} \in \mathbb{C} \right\} \cong \mathbb{C}^2,$$
(4.41)

the quotient space being

$$\begin{aligned} \hat{\mathcal{A}}^-/\widehat{\mathcal{N}}_\omega^- &= \left\{ \left| \left[\begin{pmatrix} \alpha_{11} & 0 \\ \alpha_{21} & 0 \end{pmatrix} \otimes \left(\frac{1-\sigma_3}{2} \right) \right] \right\rangle : \alpha_{i1} \in \mathbb{C} \right\} \\ &\cong \mathbb{C}^2. \end{aligned} \qquad (4.42)$$

It transforms irreducibly under $\mathcal{A}^-$. So ρ_ω remains pure with zero entropy.

Case 3. $\mathcal{A}_0 = \mathcal{A}_+ \oplus \mathcal{A}_-$. The null space $\widehat{\mathcal{N}}_\omega$ is the direct sum

$$\widehat{\mathcal{N}}_\omega = \widehat{\mathcal{N}}_\omega^+ \oplus \widehat{\mathcal{N}}_\omega^-, \qquad (4.43)$$

while the quotient space is

$$\begin{aligned} (\hat{\mathcal{A}}_+ \oplus \hat{\mathcal{A}}_-)/\widehat{\mathcal{N}}_\omega = \Bigg\{ \Bigg| \Bigg[&\begin{pmatrix} 0 & \alpha_{12} \\ 0 & \alpha_{22} \end{pmatrix} \otimes \left(\frac{1+\sigma_3}{2} \right) \\ &+ \begin{pmatrix} \alpha_{11} & 0 \\ \alpha_{21} & 0 \end{pmatrix} \otimes \left(\frac{1-\sigma_3}{2} \right) \Bigg] \Bigg\rangle \Bigg\}. \end{aligned}$$
(4.44)

This representation is the direct sum of two irreducible representations given by the two terms in (4.44). We must now restrict

$$|\mathbb{1}_{\mathcal{A}}\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (4.45)$$

into its components in $(\hat{\mathcal{A}}_+ \oplus \hat{\mathcal{A}}_-)/\hat{\mathcal{N}}_\omega$.

Its component in $\hat{\mathcal{A}}_+/\hat{\mathcal{N}}_\omega^+$ is

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} := \mathbb{1}^+, \qquad (4.46)$$

while that in $\hat{\mathcal{A}}_-/\hat{\mathcal{N}}_\omega^-$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} := \mathbb{1}^-. \qquad (4.47)$$

The squared norm of $|[\mathbb{1}^\pm]\rangle$ is $1/2$:

$$\langle [\mathbb{1}^\pm]|[\mathbb{1}^\pm]\rangle = \text{Tr}[\rho_\omega(\mathbb{1}^\pm)^* \mathbb{1}^\pm] = \frac{1}{2}. \qquad (4.48)$$

Hence

$$\begin{aligned} \rho_\omega|_{(\mathcal{A}_+\oplus\mathcal{A}_-)} = \tfrac{1}{2}(\sqrt{2}|[\mathbb{1}^+]\rangle\langle[\mathbb{1}^+]|\sqrt{2} \\ + \sqrt{2}|[\mathbb{1}^-]\rangle\langle[\mathbb{1}^-]|\sqrt{2}), \end{aligned} \qquad (4.49)$$

giving $S(\rho_\omega|_{(\mathcal{A}_+\oplus\mathcal{A}_-)}) = \ln 2$ for the entropy.

C. Role of Hopf algebras

In elementary quantum physics, one starts with a Hilbert space $\mathcal{H}_{A_i}$ which typically carries the representation of the algebra $\mathcal{A}_{A_i}$ of single-particle observables for particle A_i. In the second-quantized version, there is an isomorphism of $\mathcal{A}_{A_i}$ into the full algebra $\mathcal{A}_{A_1,A_2,\ldots,A_k}$ of observables on

$$\mathcal{H}_{A_1,A_2,\ldots,A_k} = \mathcal{H}_{A_1} \otimes \cdots \otimes \mathcal{H}_{A_k}, \qquad (4.50)$$

if the particles $A_1, A_2, \ldots, A_k$ are nonidentical.

We must identify the A_1-particle observables in the k-particle Hilbert space to be able to observe properties of A_1, say, in $\mathcal{H}_{A_1,A_2,\ldots,A_k}$. They are given by the isomorphism Δ^k for nonidentical particles defined by

$$\Delta^k(\alpha_{A_1}) = \alpha_{A_1} \otimes \mathbb{1}_{A_2} \otimes \cdots \otimes \mathbb{1}_{A_k}, \quad \alpha_{A_1} \in \mathcal{A}_{A_1}, \qquad (4.51)$$

as we saw earlier.

When the particles are identical so that $A_1 = A_2 = \cdots = A_k$ and are fermions or bosons, such an isomorphism still exists: they are the totally symmetrized versions of (4.51) as we also saw [cf. (3.14)]. This is, in fact, the simplest choice of a coproduct

$$\Delta \equiv \Delta^1, \qquad (4.52)$$

given by

$$\begin{aligned} \Delta^k(\alpha) := \alpha \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} + \mathbb{1} \otimes \alpha \otimes \cdots \otimes \mathbb{1} + \cdots \\ \cdots + \mathbb{1} \otimes \cdots \otimes \alpha. \end{aligned} \qquad (4.53)$$

When the particles are identical, but fulfill braid-group statistics, there still exists Δ^k as we will show below. As a special case, the same holds for parastatistics, its expression then being the same as for bosons and fermions.

The importance of Δ^k is as follows: *it identifies the single-particle observables in the k-particle Hilbert space.*

The choice of Δ^k typically comes from the statistics group. It can differ if the latter differs. If the statistics group is the braid group or its quotient, the permutation group, then $(\mathcal{A}_{A_i}, \Delta^k)$ defines a "quasitriangular Hopf algebra" [21]. The isomorphism $\Delta^1 \equiv \Delta$ then defines a "coproduct" from which Δ^k can be deduced.

More general possibilities than the braid group and accordingly more general single-particle algebras than the above can also be contemplated [28–30].

There is conceptually no problem in restricting a state ω on $\mathcal{A}_{A_1,A_2,\ldots,A_k}$ to the subalgebra $\Delta^k(\mathcal{A}_{A_i})$ and comparing the entropies of $\omega|_{\mathcal{A}_{A_1,A_2,\ldots,A_k}}$ and $\omega|_{\Delta^k(\mathcal{A}_{A_i})}$.

We may also wish to study a group of k particles with an algebra $\mathcal{A}^{(k)}$ in an n-particle Hilbert space, for any $n > k$. If $\mathcal{A}^{(k)}$ is Hopf, then we can find its isomorphic algebra at the n-particle level, often more than one. The ambiguity in its choice has to be resolved by the context. Entropy considerations can again be pushed through.

D. Identical particles

In this section we illustrate the use of the GNS construction for the evaluation of entanglement entropy in systems of identical particles, making use of the coproduct in order to identify subalgebras of one-particle observables. The general setting for the three examples we consider below is the following. We consider a one-particle Hilbert space $\mathcal{H}^{(1)} \cong \mathbb{C}^d$. In this case, the full one-particle observable algebra is given by the group algebra of $U(d)$, $\mathbb{C}U(d)$. The two-particle Hilbert space is then given by the subspace of $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(1)}$ consisting of either *symmetric* (bosonic statistics) or *antisymmetric* (fermionic statistics) tensors. The coproduct is a homomorphism $\Delta : \mathbb{C}U(d) \to \mathbb{C}U(d) \otimes \mathbb{C}U(d)$ that allows us to map one-particle observables from the one-particle sector to the two-particle sector. The map Δ is not fixed *a priori*. Here we consider the standard choice

$$\Delta(g) = g \otimes g, \qquad (4.54)$$

for $g \in U(d)$, linearly extended to all of $\mathbb{C}U(d)$. This coproduct is the exponentiated form of (4.53) (see also below). The crucial property is *coassociativity*:

$$(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta. \qquad (4.55)$$

This property allows us, starting from $\mathbb{C}U(d)$ and via the coproduct, to construct observables acting at the k-particle level, for any k. In the next section, other choices of the coproduct will be used in order to apply our ideas to examples with braid-group statistics.

We always consider algebras $\mathcal{A}$ with unity and subalgebras $\mathcal{A}_0$ which contain this unity. The physical reason for including the unity of $\mathcal{A}$ in $\mathcal{A}_0$ will become apparent in Sec. VI.

Now, if we perform measurements where only a restricted set of one-particle observables is considered, we may study the entanglement of a given two-particle state $|\psi\rangle$ that arises from the corresponding restriction. For example, if from the d available "levels" in $\mathcal{H}^{(1)} = \mathbb{C}^d$ we consider only d' of them $(d' < d)$, the algebra of one-particle observables will be reduced to the algebra generated by $\mathbb{C}U(d')$ and unity. Its dimension is $d'^2 + 1$. The two-particle Hilbert space will then decompose into irreducible representations of this algebra,

this being directly reflected in the entanglement structure of the state $|\psi\rangle$.

1. Two fermions, $\mathcal{H}^{(1)} = \mathbb{C}^4$

As a first example, we consider the case $d = 4$, $d' = 2$. The two-fermion space $\Lambda^2 \mathcal{H}^{(1)}$ is six dimensional, as can be seen from the decomposition $4 \otimes 4 = 10 \oplus 6$ of $\mathbb{C}^4 \otimes \mathbb{C}^4$ into symmetric and antisymmetric tensors. Let us consider a particular orthonormal basis $\{|e_1\rangle, |e_2\rangle, |e_3\rangle, |e_4\rangle\}$ for $\mathcal{H}^{(1)}$. A basis for the two-fermion space $\Lambda^2 \mathcal{H}^{(1)} \cong \mathbb{C}^6$ is then given by $\{|e_i\rangle \wedge |e_j\rangle\}_{1 \leqslant i < j \leqslant 4}$.

Now we will assume that *only* one-particle observables containing $\mathbb{1}_4$ and those causing transitions between the states $|e_1\rangle$ and $|e_2\rangle$ are considered. Then, the relevant algebra of observables is isomorphic to $\mathbb{C}U(2) \otimes \mathbb{1}_4$. This example is very relevant physically. For example, we can consider two electrons located in a two-well potential modeling a couple of nearby quantum dots. This is an example that has also been discussed in [31], in the context of entanglement of identical particles. In this case, our choice of subalgebra corresponds to local observations on one of the dots. Thus, the correlations obtained in such a setup will go beyond the ones coming from indistinguishability. The corresponding entanglement entropy will reflect this fact.

Using the coproduct, we can obtain the image of this algebra acting on the two-particle sector. For our choice of the subalgebra, the relevant observables are generated by operators of the form $M_{ij} = |e_i\rangle\langle e_j|$, with $1 \leqslant i, j \leqslant 2$, and $\mathbb{1}_4$. In this context, it turns out to be useful to work with the "infinitesimal" version of (4.54), that is, if L is an element of the *Lie algebra* of $U(d)$, we set

$$\Delta(L) = L \otimes \mathbb{1} + \mathbb{1} \otimes L. \tag{4.56}$$

It is also convenient to label the basis vectors of $\Lambda^2 \mathcal{H}^{(1)}$ in the following way:

$$
\begin{aligned}
|a\rangle &= |e_1\rangle \wedge |e_2\rangle \equiv \frac{1}{\sqrt{2}}(|e_1\rangle \otimes |e_2\rangle - |e_2\rangle \otimes |e_1\rangle), \\
|\alpha_1\rangle &= |e_1\rangle \wedge |e_3\rangle, \quad |\alpha_2\rangle = |e_2\rangle \wedge |e_3\rangle, \\
|\beta_1\rangle &= |e_1\rangle \wedge |e_4\rangle, \quad |\beta_2\rangle = |e_2\rangle \wedge |e_4\rangle, \\
|b\rangle &= |e_3\rangle \wedge |e_4\rangle.
\end{aligned}
\tag{4.57}
$$

From (4.56) it is easy to obtain explicit expressions for the matrix representations of the relevant one-particle observables. As an illustration, we compute

$$
\begin{aligned}
\Delta(M_{12})|\alpha_2\rangle &= \Delta(M_{12})|e_2\rangle \wedge |e_3\rangle \\
&= (M_{12} \otimes \mathbb{1} + \mathbb{1} \otimes M_{12})|e_2\rangle \wedge |e_3\rangle \\
&= \frac{1}{\sqrt{2}}(|e_1\rangle \otimes |e_3\rangle - |e_3\rangle \otimes |e_1\rangle) = |\alpha_1\rangle.
\end{aligned}
\tag{4.58}
$$

The four matrices $A_{ij} \equiv \Delta(M_{ij})$ (for $i, j = 1, 2$) turn out to be block diagonal in the chosen basis:

$$
\begin{aligned}
A_{11} &= \mathrm{diag}\{1, e_{11}, e_{11}, 0\}, \\
A_{22} &= \mathrm{diag}\{1, e_{22}, e_{22}, 0\}, \\
A_{12} &= \mathrm{diag}\{0, e_{12}, e_{12}, 0\}, \\
A_{21} &= \mathrm{diag}\{0, e_{21}, e_{21}, 0\},
\end{aligned}
\tag{4.59}
$$

where e_{ij} denote the standard matrix units on $M_2(\mathbb{C})$, i.e., $e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and so on. To this must be added the unit matrix $\mathbb{1}_6$.

The algebra $\mathcal{A}_0$ will be generated by exponentials of these matrices and their products. It has a basis consisting of the five matrices $A_{11}, A_{22}, A_{12}, A_{21}$, and $\mathbb{1}_6$. This same example can be worked using creation and annihilation operators. In that case, as explained in [2], the algebra corresponding to observables involving only the identity and transitions between $|e_1\rangle$ and $|e_2\rangle$ is six dimensional. However, one of the basis elements corresponds to a two-particle observable, being the product of the number operators of particles 1 and 2. The one-particle observable algebra is therefore the one obtained after taking the quotient by the ideal generated by that basis element. Here we are using coproducts, and only one-particle observables appear upon application of the homomorphism Δ.

We now consider a θ-dependent state vector, given by

$$|\psi_\theta\rangle = \cos\theta|\beta_1\rangle + \sin\theta|\alpha_2\rangle. \tag{4.60}$$

As mentioned above, at the two-particle level the full observable algebra $\mathcal{A}$ is given by $M_6(\mathbb{C})$. The subalgebra of one particles we have chosen is the $\mathcal{A}_0$ constructed above. We proceed to the construction of the GNS representation corresponding to different values of θ when the state $|\psi_\theta\rangle$ is restricted to $\mathcal{A}_0$.

Case 1. $0 < \theta < \frac{\pi}{2}$. When $0 < \theta < \pi/2$ we can easily check that the only nonzero elements $\alpha \in \mathcal{A}_0$ for which

$$\omega_\theta(\alpha^*\alpha) \equiv \langle\psi_\theta|\alpha^*\alpha|\psi_\theta\rangle = 0 \tag{4.61}$$

are linear combinations of $B = \mathrm{diag}\{1, 0_2, 0_2, 0\}$ and $\mathbb{1}_6 - A_{11} - A_{22}$, that is, the null space $\mathcal{N}_\theta$ is generated by these elements. The GNS Hilbert space $\mathcal{H}_\theta$ is thus four dimensional.

Let π_θ denote the corresponding GNS representation of $\mathcal{A}_0$ on $\mathcal{H}_\theta$. A convenient basis for $\mathcal{H}_\theta$ is given by $\{|[A_{ij}]\rangle\}_{i,j=1,2}$. A straightforward computation shows that the subspace spanned by $|[A_{12}]\rangle$ and $|[A_{22}]\rangle$, as well as the subspace spanned by $|[A_{11}]\rangle$ and $|[A_{21}]\rangle$, is irreducible. The two representations are isomorphic.

The corresponding projections P_1 and P_2 can then be used in order to obtain the components of $|[\mathbb{1}_6]\rangle$ in each irreducible subspace. From

$$|[\mathbb{1}_6]\rangle = |[A_{11} + A_{22}]\rangle \tag{4.62}$$

we obtain

$$P_1|[\mathbb{1}_6]\rangle = |[A_{11}]\rangle, \quad P_2|[\mathbb{1}_6]\rangle = |[A_{22}]\rangle. \tag{4.63}$$

Using (4.60), we compute

$$\|P_1|[\mathbb{1}_6]\rangle\|^2 = \cos^2\theta, \quad \|P_2|[\mathbb{1}_6]\rangle\|^2 = \sin^2\theta. \tag{4.64}$$

A density matrix acting on the GNS space of the restricted state can be obtained as explained above and its entropy can be computed. The result is

$$S(\theta) = -\cos^2\theta \ln \cos^2\theta - \sin^2\theta \ln \sin^2\theta. \tag{4.65}$$

Case 2. $\theta = 0$. In this case we have

$$|\psi_0\rangle = |\beta_1\rangle. \tag{4.66}$$

So the space of null vectors is given by the four-dimensional space

$$\mathcal{N}_{0,0} = \text{Span}\{|B\rangle, |[\mathbb{1}_6 - A_{11}]\rangle, |[A_{22}]\rangle, |[A_{12}]\rangle\}. \quad (4.67)$$

This means that $\mathcal{H}_{\theta=0} \cong \mathbb{C}^2$. Hence, the representation is irreducible so that the corresponding entropy vanishes.

The situation is completely equivalent for the case $\theta = \frac{\pi}{2}$. Thus, $\mathcal{H}_\theta$ decomposes into irreducible subspaces according to the following pattern:

$$\mathcal{H}_\theta \cong \begin{cases} \mathbb{C}^2, & \theta = 0, \pi/2, \\ \mathbb{C}^4 \cong \mathbb{C}^2 \oplus \mathbb{C}^2, & \theta \in (0, \pi/2). \end{cases} \quad (4.68)$$

The significant aspect of this example is the fact that for the values of θ for which the Slater rank of $|\psi_\theta\rangle$ is 1, namely, $\theta = 0$ and $\frac{\pi}{2}$, we obtain exactly zero for the entropy. In previous treatments of entanglement for identical particles, the minimum value for the von Neumann entropy of the reduced density matrix (obtained by partial trace) has been found to be $\ln 2$. This has been a source of embarrassment: it seems to suggest that different entanglement criteria have to be adopted, depending on whether one is dealing with nonidentical particles, or with bosons or fermions. For a critical review of previous attempts at a solution to this problem, see [3]. We have shown here that, by replacing the notion of the *partial trace* by the more general one of *restriction to a subalgebra*, all cases can be treated on an equal footing.

2. Two fermions, $\mathcal{H}^{(1)} = \mathbb{C}^3$

As a second example, we consider the case $d = 3$, with two choices for d': $d' = 2, 3$. In this case, the two-fermion space is given by $\mathcal{H}^{(2)} = \Lambda^2 \mathbb{C}^3 \subset \mathcal{H}^{(1)} \otimes \mathcal{H}^{(1)}$. Let $\{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$ denote an orthonormal basis for $\mathcal{H}^{(1)}$. Then

$$\{|f^k\rangle := \varepsilon^{ijk} |e_i \wedge e_j\rangle\}_{1 \leqslant k \leqslant 3} \quad (4.69)$$

provides an orthonormal basis for $\mathcal{H}^{(2)}$. This choice of basis is particularly useful if we take into account that it provides a basis for the [SU(3)] representation $\bar{3}$ obtained from the decomposition $3 \otimes 3 = 6 \oplus \bar{3}$, corresponding to symmetrization and antisymmetrization of tensors in $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(1)}$. The $|f^i\rangle$ span the $\bar{3}$ representation.

Here, the representation 3 stands for the defining $U(3)$ representation on $\mathcal{H}^{(1)}$ $[U^{(1)}(g) = g]$. This means that, at the two-fermion level, one-particle observables are given by the action of $\mathbb{C}U(3)$ on $\mathcal{H}^{(2)}$. This action is obtained from the restriction of operators of the form $\widehat{\alpha} = \int_{U(d)} d\mu(g) \alpha(g) U^{(1)}(g) \otimes U^{(1)}(g)$ to $\bar{3}$, regarded as a subspace of 3×3.

It follows that the algebra $\mathcal{A}$ of observables *for the two-fermion system* is generated by $|f^i\rangle\langle f^j|$ $(i, j = 1, 2, 3)$. Hence, $\mathcal{A} \cong M_3(\mathbb{C})$.

Below we consider two different choices for the subalgebra $\mathcal{A}_0$. This will make clear that the notion of *entanglement* is very sensitive not only to the choice of state, but also to the choice of the observable algebra.

Choice 1 for $\mathcal{A}_0$. Here we consider $\mathcal{A}_0$ to be the full algebra $\mathcal{A}$. That is, $\mathcal{A}_0$ is chosen to be the full algebra of one-particle observables acting on $\mathcal{H}^{(2)}$. Now, we pick any two-fermion *pure* state $\omega : \mathcal{A} \to \mathbb{C}$. Being a pure state on $\mathcal{A} \cong \mathcal{A}_0$, the GNS representation corresponding to the pair $(\mathcal{A}_0, \omega)$ is irreducible. This is equivalent to the statement that $\bar{3}$ is an irreducible $SU(3)$

representation. This in turn corresponds to the well-known fact that, for $\mathcal{H}^{(1)} = \mathbb{C}^3$, all two-fermion vector states have Slater rank 1 (cf. [9]). These states are therefore to be considered as nonentangled states. Notice, however, that if we use partial trace to compute the von Neumann entropy we get a result different from zero. In contrast, computing the von Neumann entropy via the GNS construction automatically gives zero in this case (because of irreducibility), this being in accordance with the fact that all states for $d = 3$ are nonentangled (as long as $\mathcal{A}_0 = \mathcal{A}$).

Choice 2 for $\mathcal{A}_0$. Now we let $\mathcal{A}_0$ be the subalgebra of $\mathcal{A}$ consisting of all one-particle observables that involve *only* the one-particle states $|e_1\rangle$ and $|e_2\rangle$ and unity $\mathbb{1}_{\mathcal{A}} = \mathbb{1}_3$. It can be easily checked that this subalgebra is generated by the operators $M^{ij} := |f^i\rangle\langle f^j|$ $(i, j = 1, 2)$, as well as $\mathbb{1}_{\mathcal{A}}$. This is, therefore, a five-dimensional matrix algebra.

Consider the following two-fermion vector state:

$$|\psi_\theta\rangle = \cos\theta |f^1\rangle + \sin\theta |f^3\rangle, \quad (4.70)$$

and let $\omega_\theta : \mathcal{A} \to \mathbb{C}$ denote the corresponding state:

$$\omega_\theta(\alpha) = \langle \psi_\theta | \alpha | \psi_\theta \rangle, \quad \forall \alpha \in \mathcal{A}. \quad (4.71)$$

Consider now the restriction of ω_θ to the subalgebra $\mathcal{A}_0$:

$$\omega_{\theta,0} = \omega_\theta |_{\mathcal{A}_0}. \quad (4.72)$$

We proceed to perform the GNS construction corresponding to the pair $\mathcal{A}_0, \omega_{\theta,0}$, assuming that $0 < \theta < \frac{\pi}{2}$. By direct computation we check that both M^{12} and M^{22} are null vectors:

$$|[M^{12}]\rangle = |[M^{22}]\rangle = 0.$$

In this range of values for θ these are all the linearly independent null vectors. This can be seen from

$$\langle \psi_\theta | \alpha^* \alpha | \psi_\theta \rangle = 0 \Rightarrow \alpha | \psi_\theta \rangle = 0 \Rightarrow \alpha = \sum c_i M^{i2},$$
$$c_i \in \mathbb{C}. \quad (4.73)$$

This means that the null space $\mathcal{N}_{\theta,0}$ is two dimensional and that, therefore, $\mathcal{H}_\theta = \hat{\mathcal{A}}_0 / \mathcal{N}_{\theta,0}$ is isomorphic to $\mathbb{C}^3$, with basis $\{|[M^{11}]\rangle, |[M^{21}]\rangle, |[E^3]\rangle\}$, where $E^3 := \mathbb{1}_{\mathcal{A}} - M^{11} - M^{22}$.

Noticing that $\alpha_0 \in \mathcal{A}_0$ implies $\alpha_0 E^3 = 0$, we can check that the GNS space has the following decomposition in terms of irreducible representations: $\mathcal{H}_\theta = \mathbb{C}^2 \oplus \mathbb{C}^1$.

Denoting by P_1 and P_2 the corresponding projections and using the fact that $[M^{11} + M^{22}] = [\mathbb{1}_2]$, we obtain

$$P_1 |[\mathbb{1}_{\mathcal{A}}]\rangle = |[M^{11}]\rangle, P_2 |[\mathbb{1}_{\mathcal{A}}]\rangle = |[E^3]\rangle. \quad (4.74)$$

Using the inner product of $\mathcal{H}_\theta$ to compute $|\mu_i|^2 = \||P_i|[\mathbb{1}_{\mathcal{A}}]\rangle\|^2$, we obtain

$$|\mu_1|^2 = \cos^2\theta, \quad |\mu_2|^2 = \sin^2\theta. \quad (4.75)$$

Hence,

$$\omega_{\theta,0} = \cos^2\theta \left(\frac{1}{\cos^2\theta} |[M^{11}]\rangle\langle[M^{11}]| \right)$$
$$+ \sin^2\theta \left(\frac{1}{\sin^2\theta} |[E^3]\rangle\langle[E^3]| \right). \quad (4.76)$$

The result for the entropy as a function of θ is therefore

$$S(\theta) = -\cos^2\theta \ln\cos^2\theta - \sin^2\theta \ln\sin^2\theta. \quad (4.77)$$

The cases $\theta = 0$ and $\theta = \pi/2$ differ from the above mainly in the dimension of the GNS Hilbert space. Nevertheless, the result for the entropy remains the same. It is given by the same formula (4.77), extended now to the values $\theta = 0$ and $\theta = \pi/2$. This means that the entropy for $\theta = 0$ or $\theta = \pi/2$ is zero, and corresponds to the fact that, for these values of θ, the GNS Hilbert space is irreducible. Indeed, the result for the GNS space in the range $\theta \in [0, \pi/2]$ is

$$\mathcal{H}_\theta \cong \begin{cases} \mathbb{C}^2, & \theta = 0, \\ \mathbb{C}^3 \cong \mathbb{C}^2 \oplus \mathbb{C}, & \theta \in (0, \pi/2), \\ \mathbb{C}, & \theta = \pi/2. \end{cases} \quad (4.78)$$

This result should be contrasted against the fact that the $\bar{3}$ representation, when regarded as a representation space for SU(2), splits as $2 \oplus 1$.

3. Two bosons, $\mathcal{H}^{(1)} = \mathbb{C}^3$

This is the bosonic counterpart of the previous example. Let $\{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$ denote an orthonormal basis for $\mathcal{H}^{(1)} = \mathbb{C}^3$. Recalling the decomposition $3 \otimes 3 = 6 \oplus \bar{3}$ of the previous example, we see that the six-dimensional space corresponds to the two-particle space $\mathcal{H}^{(2)}$, i.e., the space of symmetric tensors in $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(1)}$. The algebra $\mathcal{A}$ of observables for this two-boson system is then isomorphic to $M_6(\mathbb{C})$. As a basis for $\mathcal{H}^{(2)}$ we choose the vectors $\{|e_i \vee e_j\rangle\}_{i,j \in \{1,2,3\}}$, where

$$|e_i \vee e_j\rangle \equiv \begin{cases} \frac{1}{\sqrt{2}}(|e_i\rangle \otimes |e_j\rangle + |e_j\rangle \otimes |e_i\rangle), & i \neq j, \\ |e_i\rangle \otimes |e_i\rangle, & i = j. \end{cases} \quad (4.79)$$

They form an orthonormal basis.

Now we consider the (pure) state $\omega_{(\theta,\phi)} : \mathcal{A} \to \mathbb{C}$ that corresponds to

$$|\psi_{(\theta,\phi)}\rangle = \sin\theta\cos\phi|e_1 \vee e_2\rangle + \sin\theta\sin\phi|e_1 \vee e_3\rangle$$
$$+ \cos\theta|e_3 \vee e_3\rangle. \quad (4.80)$$

We are interested in the restriction of $\omega_{(\theta,\phi)}$ to the subalgebra $\mathcal{A}_0$ of one-particle observables which besides $\mathbb{1}_6$, pertains *only* to the one-particle vectors $|e_1\rangle$ and $|e_2\rangle$. Proceeding in the same way as in the previous examples, we recognize that the 6 representation, when regarded as a representation space for SU(2) acting nontrivially on $|e_1\rangle$ and $|e_2\rangle$, splits as $6 = 3 \oplus 2 \oplus 1$. The basis vectors for these three invariant subspaces are given below:

$$3 : |1\rangle = |e_1 \vee e_1\rangle, \quad |0\rangle = |e_1 \vee e_2\rangle, \quad |-1\rangle = |e_2 \vee e_2\rangle,$$
$$2 : |1/2\rangle = |e_1 \vee e_3\rangle, \quad |-1/2\rangle = |e_2 \vee e_3\rangle, \quad (4.81)$$
$$1 : |\tilde{0}\rangle = |e_3 \vee e_3\rangle.$$

The one-particle observables on $\mathcal{H}^{(2)}$ are obtained from the operators $|e_i\rangle\langle e_j|$ (with $i,j = 1,2$), as well as from the unit operator on $\mathcal{H}^{(1)}$, by means of the coproduct. Thus, the subalgebra $\mathcal{A}_0$ is generated by operators of the form $|u\rangle\langle v|$, with both $|u\rangle$ and $|v\rangle$ belonging to the *same* irreducible component of $\mathcal{H}^{(2)}$. (Note that the image of unity on $\mathcal{H}^{(1)}$ under the coproduct Δ is $\mathbb{1}_{\mathcal{A}}$. Hence by taking combinations of images of the above $\mathcal{H}^{(1)}$ observables under Δ, we see that $\mathcal{A}_0$ contains $|\tilde{0}\rangle\langle\tilde{0}|$.) In other words, $\mathcal{A}_0$ is given by block-diagonal matrices, with each block corresponding to one of the

irreducible components in the decomposition $6 = 3 \oplus 2 \oplus 1$. The dimension of $\mathcal{A}_0$ is therefore $3^2 + 2^2 + 1^2 = 14$.

The construction of the GNS representation corresponding to each particular value of the parameters θ and ϕ is performed following the same procedure as in the previous examples. Let us introduce the notation $B_{u,v} \equiv |u\rangle\langle v|$, for any pair $|u\rangle, |v\rangle$ in (4.81). Then, from (4.80) we see that as long as the (θ,ϕ) coefficients are all different from zero, those elements of $\mathcal{A}_0$ of the form $B_{j,\pm 1}$ ($j = 0, \pm 1$) and $B_{\sigma,-1/2}$ ($\sigma = \pm 1/2$) generate the null vectors. That these generate all the null vectors follows from the fact that (4.80) contains one basis element for every irreducible component, so that no further linear relation can arise that lead to null vectors. So in this case we have

$$\mathcal{H}_{(\theta,\phi)} := \hat{\mathcal{A}}_0/\mathcal{N}_{(\theta,\phi),0} = \mathbb{C}^6,$$
$$\mathcal{N}_{(\theta,\phi),0} = \text{nullspace}. \quad (4.82)$$

In terms of irreducible subspaces, one can readily see that $\mathbb{C}^6$ decomposes according to $\mathbb{C}^6 = \mathbb{C}^3 \oplus \mathbb{C}^2 \oplus \mathbb{C}^1$.

In general, we can read off the decomposition of $\mathcal{H}_{(\theta,\phi)}$ into irreducible subspaces from (4.80), depending on which of its coefficients vanish. For example, if only the first one vanishes,

$$\mathcal{H}_{(\theta,\phi)} = \mathbb{C}^2 \oplus \mathbb{C}^1. \quad (4.83)$$

It is interesting to consider the entropy as a function of (θ,ϕ). For the case in which all (θ,ϕ) coefficients are nonzero, we have

$$|[\mathbb{1}_{\mathcal{A}}]\rangle = |[B_{1,1}]\rangle + |[B_{1/2,1/2}]\rangle + |[B_{\tilde{0},\tilde{0}}]\rangle, \quad (4.84)$$

from which the entropy is readily computed as before. The result is

$$S(\theta,\phi) = -\sin^2\theta[\cos^2\phi\ln(\sin\theta\cos\phi)^2$$
$$+ \sin^2\phi\ln(\sin\theta\sin\phi)^2]$$
$$- \cos^2\theta\ln(\cos\theta)^2. \quad (4.85)$$

The analytic formulas for entropy when one or more of the coefficients in (4.81) vanish can be obtained from (4.85) by taking suitable limits on θ and ϕ.

We can see that the entropy vanishes whenever $|\psi_{(\theta,\phi)}\rangle$ lies in a single irreducible component. This happens precisely at those points of the two-sphere generated by the parameters (θ,ϕ) that correspond to the coordinate axes. There are therefore six points where the entropy vanishes exactly. This is depicted in Fig. 1, where the (θ,ϕ)-sphere has been mapped to the x-y plane through a stereographic projection. The figure shows the entropy as a function of the coordinates of that plane.

From the examples above we see that all Slater rank-1 states will be nonentangled whenever the full algebra of one-particle observables is chosen as subalgebra. But, in addition, for a state that has Slater rank greater than 1, the entanglement will also depend on the choice of subalgebra. This is, in fact, one of the main points of our approach: Entanglement depends not only on the state, but also on the choice of subalgebra.

E. Entanglement for braid-group statistics

Let us now outline how this approach can be used to compute entanglement entropy for systems with braid statistics. For that we need to recollect some facts from [32] about the quantum group $U_q(\text{su}(n))$.
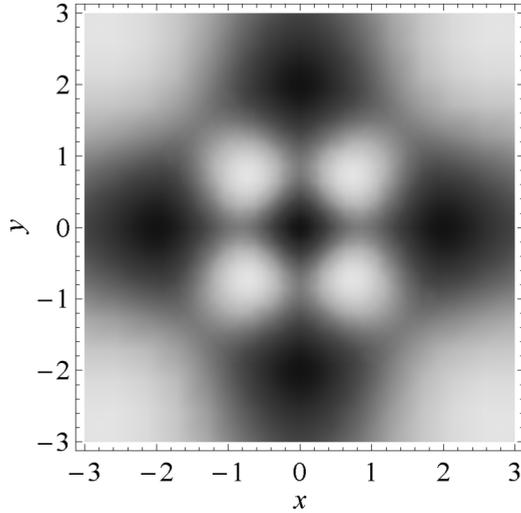
FIG. 1. The entropy equation (4.85) as a function of x and y, the (dimensionless) coordinates of a plane representing the unit (θ,ϕ)-sphere through stereographic projection. Darker regions correspond to lower values of the entropy. Five of the six vanishing points of the entropy can be seen on the picture (black spots). The sixth one, corresponding to the north-pole of the sphere, lies "at infinity" in this representation.

1. Preliminaries: Bosonic realization of $U_q(su(n))$

The q number $[s]_q$ is defined by

$$[s]_q = \frac{q^{s/2} - q^{-s/2}}{q^{1/2} - q^{-1/2}}. \tag{4.86}$$

It satisfies the following properties of importance for us:

(1) $[s+t]_q = q^{-\frac{s}{2}}[t]_q + q^{\frac{t}{2}}[s]_q$.

(2) The Jacobi identity: $[r]_q[s-t]_q + [s]_q[t-r]_q + [t]_q[r-s]_q = 0$.

(3) $[1]_q = 1$. By definition, $[0]_q = 0$.

Let $A, A^\dagger$ be q-deformed oscillators and let $a, a^\dagger$ be the standard undeformed bosonic oscillators. They are related by a dressing transformation:

$$A = a\sqrt{\frac{[N]_q}{N}} = \sqrt{\frac{[N+1]_q}{N+1}}a, \tag{4.87}$$

$$A^\dagger = \sqrt{\frac{[N]_q}{N}}a^\dagger, \tag{4.88}$$

where

$$N = a^\dagger a. \tag{4.89}$$

From this dressing transformation we may find the representation for the q-deformed oscillators $A, A^\dagger$ from the known representation of $a, a^\dagger$. In particular, both sets have the same vacuum $|0\rangle$ such that $A|0\rangle = a|0\rangle = 0$.

From (4.87) and (4.88), we see that

$$A^\dagger A = [N]_q, \quad AA^\dagger = [N+1]_q, \tag{4.90}$$

$$N = a^\dagger a. \tag{4.91}$$

This gives

$$AA^\dagger - q^{1/2}A^\dagger A = q^{-N/2}. \tag{4.92}$$

Also

$$[N,A^\dagger] = A^\dagger, \quad [N,A] = -A;$$
$$[N,a^\dagger] = a^\dagger, \quad [N,a] = -a. \tag{4.93}$$

We may now construct $U_q(su(n))$ using a set of N $A_i, A_i^\dagger$ oscillators with a fix q following the Schwinger procedure. Let λ_a, with $a = 1,\ldots,n^2-1$, be the $n \times n$ Gell-Mann matrices of su(n). Then

$$\Lambda_a = A_i^\dagger (\lambda_a)^{ij} A_j \tag{4.94}$$

are the generators of $U_q(su(n))$. We may organize this set of generators in the Cartan-Chevalley basis, where H_i, with $i = 1,\ldots,n-1$, generates the $U_q(su(n))$ Cartan subalgebra, and $E_{\pm\alpha}$ are ladder operators:

$$E_{ij} = A_i^\dagger A_j, \quad E_{ij}^\dagger = A_j^\dagger A_i \quad \text{for} \quad i < j, \tag{4.95}$$

$$H_l = \tfrac{1}{2}(N_l - N_{l+1}) \quad \text{for} \quad l \leqslant n-1, \tag{4.96}$$

where

$$N_l = a_l^\dagger a_l. \tag{4.97}$$

In the case of $U_q(su(2))$, the Gell-Mann matrices are the 2×2 Pauli matrices so that

$$\Lambda_a = A_i^\dagger (\sigma_a)^{ij} A_j \quad \text{with} \quad a = 1,2,3 \quad \text{and} \quad i = 1,2. \tag{4.98}$$

From this we obtain

$$E_{12} := J_+ = A_1^\dagger A_2, \quad E_{21} := J_- = A_2^\dagger A_1,$$
$$H_1 := J_3 = \frac{N_1 - N_2}{2}, \tag{4.99}$$

which satisfy

$$[J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = [2J_3]_q. \tag{4.100}$$

From now on, we consider only the cases for which q is real and positive, $q > 0$. In this case, $U_q(su(n))$ is a $*$-Hopf-algebra with coproduct

$$\Delta(J_\pm) = q^{-J_3/2} \otimes J_\pm + J_\pm \otimes q^{J_3/2},$$
$$\Delta(J_3) = \mathbb{1} \otimes J_3 + J_3 \otimes \mathbb{1}. \tag{4.101}$$

The unitary irreducible representations of $U_q(su(2))$ are labeled by $j \in \mathbb{Z}^+/2$. For fixed j, $-j \leqslant m \leqslant j$, an orthonormal basis for the carrier vector space is

$$|jm\rangle = \frac{(A_1^\dagger)^{j+m}(A_2^\dagger)^{j-m}}{\sqrt{[j+m]_q![j-m]_q!}}|0\rangle, \tag{4.102}$$

where for $k \in \mathbb{Z}^+$, $[k]_q! = [1]_q[2]_q \cdots [k]_q$. The generators $J_\pm$ and J_3 act in (4.102) as expected:

$$J_\pm|jm\rangle = \sqrt{[j \mp m]_q[j \pm m + 1]_q}|j,m \pm 1\rangle, \tag{4.103}$$

$$J_3|jm\rangle = m|jm\rangle. \tag{4.104}$$

2. The braid group

We illustrate the ideas using the "two-particle" representation of $U_q(su(2))$. For the undeformed oscillators, we can easily see that the vector state $a_1^\dagger|0\rangle \otimes a_2^\dagger|0\rangle$ for example decomposes

in terms of the action of the symmetric group $\mathcal{S}_2$ as

$$
\begin{aligned}
a_1^\dagger|0\rangle \otimes a_2^\dagger|0\rangle &= \tfrac{1}{2}(a_1^\dagger|0\rangle \otimes a_2^\dagger|0\rangle + a_2^\dagger|0\rangle \otimes a_1^\dagger|0\rangle) \\
&\quad + \tfrac{1}{2}(a_1^\dagger|0\rangle \otimes a_2^\dagger|0\rangle - a_2^\dagger|0\rangle \otimes a_1^\dagger|0\rangle) \\
&\equiv |1,2\rangle_S + |1,2\rangle_A, \qquad (4.105)
\end{aligned}
$$

where the indices S and A stand for symmetric and antisymmetric, respectively.

For the q-deformed case, the symmetric group decomposition has to be changed. This is simple to see.

We start with the highest-weight state, call it $|1,1\rangle$, with respect to $U_q(su(2))$:

$$
|1,1\rangle = A_1^\dagger|0\rangle \otimes A_1^\dagger|0\rangle, \qquad (4.106)
$$

$$
\Delta(J_+)|1,1\rangle = 0, \qquad (4.107)
$$

$$
\Delta(J_3)|1,1\rangle = |1,1\rangle. \qquad (4.108)
$$

Next we lower using J_-:

$$
\begin{aligned}
\Delta(J_-)|1,1\rangle &= [q^{-J_3/2} \otimes J_- + J_- \otimes q^{J_3/2}]A_1^\dagger|0\rangle \otimes A_1^\dagger|0\rangle \\
&= [q^{-1/2}A_1^\dagger|0\rangle \otimes A_2^\dagger|0\rangle + q^{\frac{1}{2}}A_2^\dagger|0\rangle \otimes A_1^\dagger|0\rangle].
\end{aligned}
$$
$$(4.109)$$

It is thus clear that this state vector cannot be decomposed under the symmetric group, since the q powers are breaking the structure of the symmetric tensor product.

Formally, the structure we have just seen carries a representation of the braid group. We will not go into details here; they can be found in [21,32]. In particular, as discussed by Biedenharn and Lohe, the multiparticle q-boson states constructed from the q-deformed oscillators are invariant under the braid group.

3. Example

The Schwinger realization of q-deformed oscillators given above is adapted to treat bosons. So we generalize the $q = 1$ bosonic example above where $\mathcal{H} = \mathbb{C}^3$ and we observe just the algebra generated by the observables mixing 1 and 2 and the unit operator.

Thus we now have three q-deformed oscillators $A_i, A_i^\dagger$ which commute for $i \neq j$. The two-particle q-boson states are spanned by $A_i^\dagger A_j^\dagger|0\rangle$ and are six-dimensional. Following (4.81), we construct the orthonormal basis for the three subspaces invariant under the observables acting on 1 and 2 particles and the unit operator:

$$
\begin{aligned}
3: \ |1\rangle_q &= \frac{(A_1^\dagger)^2}{\sqrt{[2]_q}}|0\rangle, \quad |0\rangle_q = A_1^\dagger A_2^\dagger|0\rangle, \\
|-1\rangle_q &= \frac{(A_2^\dagger)^2}{\sqrt{[2]_q}}|0\rangle, \\
2: \ |1/2\rangle_q &= A_1^\dagger A_3^\dagger|0\rangle, \quad |-1/2\rangle_q = A_2^\dagger A_3^\dagger|0\rangle, \\
1: \ |\tilde{0}\rangle_q &= \frac{1}{\sqrt{[2]_q}}(A_3^\dagger)^2|0\rangle.
\end{aligned}
$$
$$(4.110)$$

We also generalize the vector state (4.80) to the normalized vector state

$$
\begin{aligned}
|\psi_{(\theta,\phi)}\rangle_q &= \Big(\sin\theta\cos\phi\, A_1^\dagger A_2^\dagger + \sin\theta\sin\phi\, A_1^\dagger A_3^\dagger \\
&\quad + \cos\theta\frac{(A_3^\dagger)^2}{\sqrt{[2]_q}}\Big)|0\rangle.
\end{aligned}
$$
$$(4.111)$$

It induces a state $\omega_{(\theta,\phi)}^{(q)}$ on the full algebra of 6×6 matrices.

We now restrict $\omega_{(\theta,\phi)}^{(q)}$ to the observables pertaining to operators acting on 1 and 2 indices and the unit operator. This subalgebra $\mathcal{A}_0(q)$ is spanned by

$$
|i\rangle_q\,{}_q\langle j|, \quad i,j \in [-1,0,1], \quad \text{and} \quad \mathbb{1}_6. \qquad (4.112)
$$

Now, the algebra $\mathcal{A}_0(q)$ generated by (4.112) and the scalar products induced by $\omega_{(\theta,\phi)}^{(q)}$ are all independent of q. The conclusion is that the GNS Hilbert space and its properties are all quite independent of q. That includes entropy as well. These observations can be generalized to more involved situations.

This simple example, to be contrasted with the usual two-fermion system already worked out here, shows once more that our approach allows one to naturally obtain a zero von Neumann entropy for separable systems even in the case of more sophisticated statistics. We think that this sets the stage for a more comprehensive study of systems with braid statistics, like the Kitaev model, that may play a crucial role for developments in quantum computation.

V. TIME EVOLUTION

If a unitary time evolution $U(t)$ of a pure state ω on an algebra $\mathcal{A}$ is given, then the time evolution of its restriction $\omega|_{\mathcal{A}_0} = \omega_0$ is determined by

$$
\omega_0 \rightarrow \omega_0(t) = [U(t)\omega]|_{\mathcal{A}_0}, \quad \omega_0(0) = \omega_0. \qquad (5.1)
$$

The evolution of ω_0 is in general by positive maps. This fact is a consequence of the Stinespring-Choi theorem. That is the case even when $U(t)$ gives a unitary evolution on ω with Hamiltonian H:

$$
U(t)\omega = e^{iHt}\omega e^{-iHt}. \qquad (5.2)
$$

Here ω is a density matrix.

An important point is that *the rank of $\omega_0(t)$ need not be continuous in t even if that of $U(t)\omega$ is continuous in t. It can change discontinuously*. This is shown by the example below. In that example, entropy is periodic in time, not monotone increasing in time, as is thought to be the case in nature.

The case of a fermion with three internal degrees of freedom treated in Sec. IV D is a simple example. The single-particle Hilbert space $\mathcal{H}^{(1)}$ was $\mathbb{C}^3$ with orthonormal basis $\{|e_i\rangle\}_{i=1,2,3}$. The two-particle state space was $\bar{3} = \bigwedge^2\mathcal{H}^{(1)} \equiv \mathcal{H}^{(2)}$, with an orthonormal basis $\{|f^i\rangle = \varepsilon^{ijk}|e_j \wedge e_k\rangle\}_{i=1,2,3}$.

The single-particle algebra acting on $\mathcal{H}^{(2)}$ was $\mathbb{C}U(3) \otimes \mathbb{1}_6$. We chose the pure state

$$
\omega_\theta = |\psi_\theta\rangle\langle\psi_\theta|, \quad |\psi_\theta\rangle = \cos\theta|f^1\rangle + \sin\theta|f^2\rangle, \qquad (5.3)
$$

in the two-particle sector.

The subalgebra $\mathcal{A}_0$ was the image under the coproduct of the single-particle algebra on $\mathcal{H}^{(1)}$ acting just on $|e_1\rangle$ and $|e_2\rangle$.

Our results pertinent for the discussion of time evolution were

$$\omega_{\theta,0} = \cos^2 \theta \rho_\theta^{(1)} + \sin^2 \theta \rho_\theta^{(3)}, \tag{5.4}$$

where

$$\rho_\theta^{(1)} = \frac{1}{\cos^2 \theta} |[M^{11}]\rangle \langle [M^{11}]|, $$
$$\rho_\theta^{(3)} = \frac{1}{\sin^2 \theta} |[E^3]\rangle \langle [M^3]|, \tag{5.5}$$

and

$$\mathcal{H}_\theta^{\text{GNS}} = \begin{cases} \mathbb{C}^2, & \theta = 0, \\ \mathbb{C}^3 = \mathbb{C}^2 \oplus \mathbb{C}, & 0 < \theta < \pi/2, \\ \mathbb{C}, & \theta = \pi/2. \end{cases} \tag{5.6}$$

Thus the rank of $\omega_{\theta,0} = \omega_\theta|_{\mathcal{A}_0}$ jumps from 2 to 1 as θ approaches 0 or $\pi/2$.

Now consider the unitary evolution of $|\psi_\theta\rangle$ and ω_θ under the self-adjoint Hamiltonian

$$H = -i|f^2\rangle\langle f^1| + i|f^1\rangle\langle f^2|. \tag{5.7}$$

It generates rotations in the plane of $\{|f^1\rangle, |f^2\rangle\}$ and hence changes θ:

$$e^{itH}|\psi_\theta\rangle = \cos(\theta+t)|f^1\rangle + \sin(\theta+t)|f^2\rangle. \tag{5.8}$$

The restriction of this evolution to $\omega_{\theta,0}$ is

$$U(t) : \omega_{\theta,0} \to \omega_{\theta+t,0}. \tag{5.9}$$

It is not unitary. It does not even preserve the rank of $\omega_{\theta,0}$: it jumps from 2 to 1 and back as t increases.

We can write the time evolution as positive maps so long as the rank of the density matrix stays constant or decreases. Thus we consider $\omega_{\theta,0}$ for $0 < \theta < \pi/2$. It is of rank 2 expressible in terms of the orthonormal eigenvectors

$$|\chi^{(1)}(\theta)\rangle = \frac{1}{\cos\theta}|[M^{11}]\rangle, \quad |\chi^{(3)}(\theta)\rangle = \frac{1}{\sin\theta}|[M^{21}]\rangle \tag{5.10}$$

and the corresponding eigenvalues

$$\lambda_1(\theta) = \cos^2(\theta), \qquad \lambda_3(\theta) = \sin^2(\theta). \tag{5.11}$$

For $0 < \theta < \pi/2$, we can then write

$$\omega_{\theta',0} = \sum_{a=1}^2 \Lambda_a^\dagger(\theta',\theta)\omega_{\theta,0}\Lambda_a(\theta',\theta), \tag{5.12}$$

$$\Lambda_a(\theta',\theta) = \left(\frac{\lambda_a(\theta')}{\lambda_a(\theta)}\right)^{1/2} |\chi_a(\theta)\rangle\langle\chi_a(\theta')|. \tag{5.13}$$

This makes sense for $\theta = 0$ ($\pi/2$) if the $a = 2$ ($a = 1$) term in (5.12) is understood as zero.

But positive maps cannot increase the rank of a state. Hence we cannot write evolution starting from $\theta = 0$ or $\pi/2$ in terms of positive maps.

VI. ANOMALIES AND RESTRICTIONS

In this paper, mixed states emerge from restrictions of pure states ω on an algebra $\mathcal{A}$ to a subalgebra $\mathcal{A}_0$. In a series of recent papers [23,24], mixed states were introduced to eliminate anomalies. There it was proposed that anomalies can be eliminated by averaging, say, a pure state ω over the anomalous group.

We will now argue that the averaged state in the second case can also be regarded as the restriction of ω to a subalgebra. Let us focus on parity anomaly caused for example by the QCD θ angle. The discussion is valid for any $\mathbb{Z}_2$ symmetry group though.

Let $\mathcal{A}$ be the algebra of observables with unity $\mathbb{1}$. If P is the parity, then $\mathcal{A}$ has the parity-even subalgebra

$$\mathcal{A}_+ = \{a_+ \in \mathcal{A} : Pa_+P = a_+\} \tag{6.1}$$

and also its complement

$$\mathcal{A}_- = \{a_- \in \mathcal{A} : Pa_-P = -a_-\}, \tag{6.2}$$

which is not an algebra. The unity $\mathbb{1}$ is clearly in $\mathcal{A}_+$. But we assume that $\mathcal{A}_+$ has its own unity (projector) $\mathbb{1}_+$:

$$\mathbb{1}_+ a_+ = a_+\mathbb{1}_+ = a_+, \quad \mathbb{1}_+ a_- = a_-\mathbb{1}_+ = 0. \tag{6.3}$$

There is good physical meaning in assuming that $\mathcal{A}_+$ contains both $\mathbb{1}_+$ and $\mathbb{1}$. The projector $\mathbb{1}_+$ is the unity on the parity-even elements. We need its orthogonal projector as observable to tell us that the state of the system certainly has no component in $\mathcal{A}_+$. This orthogonal projector is $\mathbb{1}_-$. And $\mathbb{1} = \mathbb{1}_+ + \mathbb{1}_-$.

Then

$$\mathbb{1}_- = \mathbb{1} - \mathbb{1}_+ \tag{6.4}$$

is the projector onto $\mathcal{A}_-$:

$$\mathbb{1}_- a_+ = a_+\mathbb{1}_- = 0, \quad \mathbb{1}_- a_- = a_-\mathbb{1}_- = a_-. \tag{6.5}$$

The parity-even subalgebra we consider is $\mathcal{A}_0 = \mathcal{A}_+ \oplus \mathbb{C}\mathbb{1}_-$

Let ω_θ, regarded as a density matrix, be such that

$$P\omega_\theta P = \omega_{-\theta}. \tag{6.6}$$

For instance, a θ QCD state in QCD is of the above kind.

Since

$$\mathbb{1} = \mathbb{1}_+ + \mathbb{1}_-, \tag{6.7}$$

ω_θ splits on restriction to $\mathcal{A}_0$ as

$$\omega_\theta = |[\mathbb{1}_+],_\theta\rangle\langle[\mathbb{1}_+],_\theta| + |[\mathbb{1}_-],_\theta\rangle\langle[\mathbb{1}_-],_\theta|$$
$$= \omega_{\theta,+} + \omega_{\theta,-}. \tag{6.8}$$

Now consider the expectation value $\omega_\theta(\alpha)$ for

$$\alpha = \alpha_+ + \lambda\mathbb{1}_- \in \mathcal{A}_0. \tag{6.9}$$

We have

$$\omega_\theta(\alpha) := \text{Tr}\,\omega_\theta\alpha = \omega_{\theta,+}(\alpha_+) + \omega_{\theta,-}(\lambda\mathbb{1}_-). \tag{6.10}$$

Since

$$P\alpha_+P = \alpha_+, \tag{6.11}$$

$$\omega_{\theta,+}(\alpha_+) = \omega_{-\theta,+}(\alpha_+) = \tfrac{1}{2}[\omega_{\theta,+} + \omega_{-\theta,+}](\alpha_+). \tag{6.12}$$

Similarly, since $P\mathbb{1}_- P = \mathbb{1}_-$,

$$\omega_{\theta,-} = \tfrac{1}{2}[\omega_{\theta,-} + \omega_{-\theta,-}]. \qquad (6.13)$$

Hence

$$\omega_\theta|_{\mathcal{A}_0} = \tfrac{1}{2}(\omega_\theta + \omega_{-\theta})|_{\mathcal{A}_0}. \qquad (6.14)$$

But

$$(\omega_\theta + \omega_{-\theta})(\alpha_-) = \omega_\theta(\alpha_-) + \omega_\theta(P\alpha_- P) = 0, \qquad (6.15)$$

which is true of the right-hand side of (6.13) as well, extended to $\mathcal{A}_-$.

Hence *restriction and averaging give the same answer. They will coincide whenever $\mathcal{A}_0$ has a projector, the analog of $\mathbb{1}_+$ here.* This is clear from the computations above. Note that the normalization of vectors in (6.8) is given by

$$\langle [\mathbb{1}_\pm]_{,\theta} | [\mathbb{1}_\pm]_{,\theta} \rangle = \omega_\theta(\mathbb{1}_\pm). \qquad (6.16)$$

VII. STATE RESTRICTIONS AND QUANTUM OBSERVATIONS

Suppose we study the restriction of a state ω on an algebra $\mathcal{A}$ of observables to a subalgebra $\mathcal{A}_0$. For reasons explained in Sec. VI, we assume that $\mathcal{A}_0$ contains the projector $\mathbb{1}_+$ and its orthogonal projector

$$\mathbb{1}_- = \mathbb{1} - \mathbb{1}_+, \qquad (7.1)$$

$\mathbb{1}$ being the unity of $\mathcal{A}$. The projectors satisfy

$$\mathbb{1}_+ \mathbb{1}_- = 0. \qquad (7.2)$$

Using these projectors, we can decompose $\mathcal{A}_0$ into two parts:

$$\mathcal{A}_0 = \mathcal{A}_0 \mathbb{1}_+ \oplus \mathcal{A}_0 \mathbb{1}_-. \qquad (7.3)$$

Let ω be pure on $\mathcal{A}$. Observe that $\mathcal{A}_0 \mathbb{1}_+$, $\mathcal{A}_0 \mathbb{1}_-$ are both invariant by $\mathcal{A}_0$. Then, in the GNS construction, the restricted state splits into two parts:

$$\omega|_{\mathcal{A}_0} = |[\mathbb{1}_+]\rangle\langle[\mathbb{1}_+]| + |[\mathbb{1}_-]\rangle\langle[\mathbb{1}_-]| \qquad (7.4)$$

and is not pure.

Can we interpret (7.4) as emergent from observations? The answer seems to be yes. If one measures the probability of finding either 1 or 0 for the observable $\mathbb{1}_+$ on an ensemble with state ω, the resultant state $\omega|_{\mathcal{A}_0}$ is after measurement exactly (7.4)

$$\omega \to \omega|_{\mathcal{A}_0} = \mathbb{1}_+ |[\mathbb{1}]\rangle\langle[\mathbb{1}]| \mathbb{1}_+ + \mathbb{1}_- |[\mathbb{1}]\rangle\langle[\mathbb{1}]| \mathbb{1}_-. \qquad (7.5)$$

In this case, we are given the projector $\mathbb{1}_+$ and we can reconstruct $\mathcal{A}_0 \subseteq \mathcal{A}$ as its commutant:

$$\mathcal{A}_0 = \text{commutant of} \quad \mathbb{1}_+ \quad \text{in} \quad \mathcal{A}. \qquad (7.6)$$

Working from this $\mathcal{A}_0$, we then show that the state restricted to (7.6) coincides with (7.5).

VIII. CONCLUSIONS

We have seen in this work that there is a natural formulation of quantum physics dispensing with the use of the Hilbert space as initial data which is well adapted to the study of entanglement and entropy. In this approach, the Hilbert space is an emergent concept. Instead, the initial data are the algebra of observables and their expectation values. From the expectation values one abstracts the notion of a state on the algebra.

In this formulation, the Hilbert space is obtained from the GNS construction that resembles the construction of the regular representation of finite (or compact) groups. Furthermore, one may compute the von Neumann entropy associated with a density matrix that is obtained from a state on the algebra. It should be emphasized that with each state one may associate many distinct density matrices and therefore distinct von Neumann entropies. A discussion of this point is carried out in [26].

A state ω on an algebra $\mathcal{A}$ can be restricted to a subalgebra $\mathcal{A}_0$. The new state $\omega|_{\mathcal{A}_0}$ may not be pure even if ω is. Its entropy is a measure of entanglement of $\mathcal{A}_0$ with $\mathcal{A}$.

This approach to entanglement constitutes a generalization and extension of the approach based on the partial trace and lets us treat identical particles obeying Bose, Fermi, or even braid statistics with ease. Particle identity has posed severe problems in conventional approaches, in the context of entanglement of particles.

We have also shown how time evolution by positive maps for $\omega|_{\mathcal{A}_0}$ emerges when ω evolves unitarily.

We have treated further points concerning quantum anomalies and their elimination by restricting states to subalgebras. In this manner, we can understand the use of mixed states to eliminate anomalies suggested by our previous work [23]. We also discussed how the restriction $\omega|_{\mathcal{A}_0}$ emerges from a standard interpretation of quantum physics from the observation of projectors.

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