Analytical solutions of the Dirac and the Klein-Gordon equations in plasma induced by high-intensity laser

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In this paper we obtain analytical solutions of the Dirac and the Klein-Gordon equations coupled to a strong electromagnetic wave in the presence of a plasma environment. These are a generalization of the familiar Volkov solutions. The contribution of the nonzero photon effective mass to the scalar and fermion wave functions, conserved quantities, and effective mass is demonstrated. The wave functions exhibit differences from Volkov solutions for nowadays available laser intensity.

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I. INTRODUCTION

In the near-future ultraintense lasers with irradiances $I_L \sim 10^{25}$ W/cm^2 are expected to be available [1–5]. Such intensities will allow laboratory exploration of a plethora of physical phenomena, among them QED in strong fields [5,6], Schwinger vacuum decay [7], and Unruh radiation [8,9]. The above is additional to the conventional applications of ultraintense lasers: fast ignition [10,11], ion acceleration [12], high harmonis generation [13], and relativisitc shock waves [14]. Our understanding of these phenomena relies on the basic theoretical description of the interaction of an electromagnetic field with an electron. This interaction does not obey classical electrodynamics if the electron proper acceleration in its rest frame is comparable to the Schwinger acceleration $a_S \equiv mc^3/h$, where m is the electron mass, c is the speed of light, and h is the Planck constant. An electron experiencing Schwinger acceleration gains, by definition, an energy equal to its rest mass over a distance of one Compton wavelength $\lambda_C = h/mc$. Equivalently, the dynamics has a quantum nature if

$$\chi \equiv \frac{h}{mc^3}\sqrt{\left(\frac{du}{ds}\right)^2} > 1, \tag{1}$$

where s is the proper time of the electron and u^μ is the proper velocity. Moreover, the motion is nonlinear in the electromagnetic field amplitude if

$$\xi \equiv \frac{e}{m}\sqrt{-A^2} > 1, \tag{2}$$

where A_μ is the vector potential and e is the electric charge. In practical units the normalized vector potential is given by $\xi^2 = \lambda_L^2 I_L/(1.37 \times 10^{18}\ \mathrm{W}\ \mu\mathrm{m}^2\ \mathrm{cm}^2)$, where λ_L is the laser wavelength. For the laser intensities mentioned above, both conditions are satisfied, calling for a nonperturbative QED formalism. The essence of the nonperturbative attitude (also known as the "Furry picture" [15]) is that, instead of treating the laser background perturbatively, we include it in the free Hamiltonian. Therefore, nonperturbative calculations of QED processes in the presence of a laser ("laser assisted") are carried out [16–20] by replacing the free electron wave function appearing in the quantum calculation with the familiar Volkov solution [21] for an electron interacting with an electromagnetic plane wave.

Laser-assisted QED processes in the nonperturbative regime involve the absorption of many laser photons. The most dominant processes are the nonlinear Compton scattering, where an electron absorbs many laser photons and emits a γ photon, and the nonlinear Breit-Wheeler scattering, which involves a γ photon decaying into an electron-positron pair under the influence of the laser field [5]. These two processes are responsible for the QED cascades expected to play a key role [22] in the dynamics of the plasma created during the laser-matter interaction. In recent years, a considerable effort is invested in order to extend the nonperturbative attitude to realistic laser configurations, e.g., by taking into account the finite temporal width of the laser pulse [23–25]. An additional issue to be treated is the plasma effect on the laser dispersion relation. This problem was considered in Ref. [26] under the assumption of a low density.

In this work, we consider the problem of a particle in the presence of an electromagnetic wave propagating in a plasma environment. We start with a scalar charged particle described by the Klein-Gordon equation, due to its relative simplicity. It will serve as a reference to the Dirac equation, for which we derive an approximate solution utilizing Floquet theorem and solid-state analogy. The new wave function can be substituted in the perturbative procedure in order to calculate the electron energy loss due to photon emission as well as other QED processes. The new solutions are of great relevance if the laser interacts with dense matter or in the case of counter-propagating beams in vacuum (i.e., a rotating electric field).

The paper is organized as follows. Section II includes the exact solution of the Klein-Gordon equation in the presence of an electromagnetic field in a plasma environment. In Sec.III we describe in detail the approximate solution of the Dirac equation for the same problem. In Sec.IV we obtain the conserved quantities of the problem and the particle effective mass. Finally, we discuss our results and conclude in Sec.V.

II. KLEIN-GORDON EQUATION SOLUTION

Let us start with the Klein-Gordon equation for a charged scalar with mass M coupled to a classical electromagnetic

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field:

$$[(i\partial - eA)^2 - M^2]\Phi = 0. \tag{3}$$

From now on natural units ($\hbar = c = 1$) are used. Note that $\partial \cdot (A\Phi) = A \cdot \partial\Phi + (\partial \cdot A)\Phi$. We adopt the Lorentz gauge ($\partial \cdot A = 0$) and therefore the last term vanishes:

$$[-\partial^2 - 2ie(A \cdot \partial) + e^2A^2 - M^2]\Phi = 0. \tag{4}$$

The center dot stands for Lorentz contraction. $k \equiv (\omega_L,0,0,k_z)$ is the wave number of the laser and the vector potential A_μ depends only on the quantity $\phi \equiv k \cdot x$. We are interested in a laser wave propagating through dense plasma. It is well known that in plasma environment the photon acquires effective mass due to the screening [27]. The general dispersion relation is

$$k^2 = \omega_L^2 - \vec{\mathbf{k}}^2 = m_{\text{ph}}^2. \tag{5}$$

The effective mass m_{ph} is treated later in the paper. Unless we are dealing with a few-cycle pulse, it is acceptable to assume a plane monochromatic wave, so that the vector potential may be written as

$$A(\phi) = a(\phi)(\epsilon e^{i\phi} + \epsilon^* e^{-i\phi}), \tag{6}$$

where $a(\phi)$ is an envelope that slowly goes to 0 as $\phi \to \pm\infty$. From now on we treat $a(\phi)$ as a constant, similarly to the standard Volkov derivation [28]. If the polarization is circular, we have

$$\epsilon = (e_1 - ie_2)/2, \tag{7}$$

where the unit vectors are $e_1 = (0,1,0,0)$ and $e_2 = (0,0,1,0)$. The polarization vector obeys $\epsilon^2 = \epsilon^{*2} = 0$ as well as $\epsilon \cdot \epsilon^* = -\frac{1}{2}$. It can be easily verified that $A^2 = -a^2$.

Before we dive into the derivation, let us recall that in the interaction of a single particle with an electromagnetic wave the quantum processes are governed by four parameters [29]: ξ and χ, defined above, as well as the electromagnetic-field invariants

$$\mathcal{F} \equiv -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{8}$$

and

$$\mathcal{G} = \tfrac{1}{4}\epsilon_{\alpha\beta\mu\nu}F^{\alpha\beta}F^{\mu.\nu} \tag{9}$$

Einstein convention is used to summarize over identical Greek indices from 0 to 3. The electromagnetic-field tensor is related to the vector potential by $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$, and $\epsilon_{\alpha\beta\mu\nu}$ is the Levi Civita tensor. In terms of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ it acquires the familiar form

$$\mathcal{F} = -\tfrac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2), \tag{10}$$

$$\mathcal{G} = -\mathbf{E} \cdot \mathbf{B}. \tag{11}$$

If the electromagnetic wave is propagating in vacuum, both invariants are 0. However, if plasma is present, then $\mathcal{G}$ remains 0 and $\mathcal{F} = -a^2{m_{\text{ph}}}^2/2$. Hence, the QED cross sections obtained with the solutions derived in this paper will depend upon an additional independent variable compared to the Volkov solutions, which ignore the plasma effect.

In the following derivation, the effective photon mass m_{ph} can either be calculated self-consistently (as will be published in a separate paper) or be taken as an external input from a kinetic model, such as a particle-in-cell code. Nowadays, particle-in-cell simulations for ultraintense laser plasma interactions model QED processes (such as nonlinear Compton scattering and nonlinear Breit-Wheeler) with rates calculated using Volkov wave functions [22]. As a result, they depend only on the parameters χ and ξ evaluated at the particle position. The solutions exhibited here can be utilized for rate calculation depending on an additional parameter, $\mathcal{F} = -a^2m_{\text{ph}}^2/2$.

Since A_μ depends only on ϕ, we seek a solution for (4) in the form

$$\Phi = e^{-ipx}G(\phi). \tag{12}$$

p_μ is a constant four-vector, which is the momentum of the particle in the limit $k \cdot x \to \pm\infty$, where the electromagnetic field is absent. Hence we have $p^2 = M^2$. Substituting (12) into (4) we get a second-order linear ordinary differential equation,

$$-m_{\text{ph}}^2G'' + 2i(k \cdot p)G' + [-2e(p \cdot A) + e^2A^2]G = 0, \tag{13}$$

where the tag symbol represents the derivative with respect to ϕ. For a plane wave traveling in vacuum we have $m_{\text{ph}}^2 = 0$, so that we are left with a first-order equation. Its solution is the familiar Volkov wave function

$$\Phi = e^{iS}, \tag{14}$$

where S is the classical action of a particle in an electromagnetic field:

$$S \equiv -p \cdot x - \int_0^\phi \left[\frac{e}{k \cdot p}(p \cdot A) - \frac{e^2}{2k \cdot p}A^2\right] d\phi' \tag{15}$$

We now assume that $A \cdot p = 0$ since the plasma is initially at rest; i.e., the particle velocities are nonrelativistic, leading to $p \approx (M,0,0,0)$. This condition also holds for a particle beam counter-propagating with respect to the laser beam. Therefore we have

$$-m_{\text{ph}}^2G'' + 2i(k \cdot p)G' - e^2a^2G = 0. \tag{16}$$

The resultant equation has constant coefficients and hence its solution is trivial: $G = e^{i\nu\phi}$. We substitute it into (16) and get an equation for ν:

$$m_{\text{ph}}^2\nu^2 - 2(k \cdot p)\nu - e^2a^2 = 0. \tag{17}$$

So ν is given by

$$\nu = \frac{k \cdot p}{{m_{\text{ph}}}^2}\left(1 - \sqrt{1 + \left(\frac{eam_{\text{ph}}}{k \cdot p}\right)^2}\right). \tag{18}$$

The second solution of the quadratical equation is not physical because it does not go to 0 in the limit $k \cdot x \to \pm\infty$. The final solution is

$$\Phi = e^{-i(p-\nu k)\cdot x}. \tag{19}$$

In the next section we show that the solution of the Dirac equation yields the same value for ν.

III. DIRAC EQUATION SOLUTION

Having examined the simple case of Klein-Gordon, we proceed to the Dirac equation for an electron coupled to a classical electromagnetic field:

$$[i\not{\partial} - e\not{A} - m]\,\psi = 0. \tag{20}$$

$\not{A}$ stands for $\gamma_\mu A^\mu = \gamma \cdot A$, where γ_μ are the Dirac matrices and ψ is the four-component wave function. Now we transform to the second-order Dirac equation by multiplying (20) with the operator $i\not{\partial} - e\not{A} + m$:

$$[-\partial^2 - 2ie(A\cdot\partial) + e^2A^2 - m^2 - ie\not{k}\not{A}']\psi_s = 0. \tag{21}$$

The s subscript was added to ψ to emphasize that it solves the second-order Dirac equation. The relation between ψ and ψ_s is discussed at the end of this section. The vector potential and the dispersion relation are given by (6) and (5), correspondingly. Similarly to the previous section, we seek a solution for (21) in the form

$$\psi_s = e^{-ipx}F(\phi). \tag{22}$$

p_μ is a constant four-vector which is the momentum of the electron in the limit $k \cdot x \to \pm\infty$, where the electromagnetic field is absent. Hence we have $p^2 = m^2$. Substituting (22) into (21) we get a second-order linear ordinary differential equation:

$$-m_{\text{ph}}^2F'' + 2i(k\cdot p)F' + [-2e(p\cdot A) + e^2A^2 - ie\not{k}\not{A}']F = 0. \tag{23}$$

For a plane wave traveling in vacuum we have $m_{\text{ph}}^2 = 0$, so that we are left with a first-order equation. Its solution is the familiar Volkov wave function,

$$\psi = \left(1 + \frac{e}{2k\cdot p}\not{k}\not{A}\right)e^{iS}u_p, \tag{24}$$

where the classical action S is given in Eq. (15). Substituting the explicit expression for the vector potential, (6), into Eq. (23) we get

$$-{m_{\text{ph}}}^2F'' + 2ie(k\cdot p)F' + (-e^2a^2 + \Xi e^{i\phi} + \zeta e^{-i\phi})F = 0, \tag{25}$$

where we have defined

$$\Xi = ae\not{k}\not{\epsilon} - 2ea(p\cdot\epsilon) \tag{26}$$

and

$$\zeta = -ae\not{k}\not{\epsilon}^* - 2ea(p\cdot\epsilon^*). \tag{27}$$

Note that Ξ,ζ are 4×4 matrices, while the other coefficients in Eq. (25) are scalars. As in Sec. II we assume that $A\cdot p = 0$ since the plasma is initially at rest; i.e. the electron velocities are nonrelativistic, leading to $p \approx (m,0,0,0)$. This condition also holds for an electron beam counter-propagating with respect to the laser beam. Using the Lorentz gauge ($k\cdot A = 0$) one can prove that $\Xi^2 = \zeta^2 = 0$. Similarly,

$$\Xi\zeta = -e^2a^2m_{\text{ph}}^2D_1 \tag{28}$$

and

$$\zeta\Xi = -e^2a^2m_{\text{ph}}^2D_2, \tag{29}$$

where

$$D_1 \equiv \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad D_2 \equiv \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{30}$$

Equation (25) can be solved in a similar way to the Mathieu equation. The coefficients are periodic in ϕ and therefore we can utilize the Floquet theorem,

$$F = P(\phi)e^{i\nu\phi}, \tag{31}$$

where P is some periodic function of ϕ. Plugging (31) into (25) we see that P obeys the equation

$$-m_{\text{ph}}^2P'' + \mu P' + [\delta + \Xi e^{i\phi} + \zeta e^{-i\phi}]P = 0. \tag{32}$$

The scalars μ and δ are defined as

$$\mu \equiv 2i\left(k\cdot p - \nu m_{\text{ph}}^2\right) \tag{33}$$

and

$$\delta \equiv m_{\text{ph}}^2\nu^2 - 2\nu k\cdot p - e^2a^2. \tag{34}$$

Since P is periodic we write it as a Fourier series,

$$P = \sum_{n=-\infty}^{\infty}\eta_n e^{in\phi}, \tag{35}$$

where η_n are constant bispinors. Substituting (35) into (32), the following recursive relations can be achieved:

$$\rho_n\eta_n + \Xi\eta_{n-1} + \zeta\eta_{n+1} = 0, \tag{36}$$

with $\rho_n \equiv m_{\text{ph}}^2n^2 + i\mu n + \delta$. Substituting the definitions of μ,δ, we have

$$\rho_n = m_{\text{ph}}^2(\nu+n)^2 - 2(k\cdot p)(\nu+n) - e^2a^2. \tag{37}$$

Taking the sum in Eq. (35) from $-N$ to N (for $N \gg 1$), the finite set of equations is described by the matrix equation,

$$\Lambda\eta = 0, \tag{38}$$

where we have defined

$$\Lambda \equiv \begin{pmatrix} \rho_{-N}I_4 & \zeta & & 0\\ \Xi & \rho_{-N+1}I_4 & \zeta & \\ & \ddots & \ddots & \ddots \\ 0 & & \Xi & \rho_N I_4 \end{pmatrix} \tag{39}$$

and

$$\eta \equiv \begin{pmatrix} \eta_{-N}\\ \vdots \\ \eta_N \end{pmatrix}. \tag{40}$$

I_4 is the 4×4 unit matrix and the dimensions of Λ are $4\times(2N+1)$. It is interesting to point out that the procedure described in Eqs. (31)–(40) is similar to the band structure calculation in solid-state problems, where the vector potential is analogous to the periodic crystal potential and νk_μ resembles the quasimomentum.

For a given value of ν, at most one of the coefficients ρ_i can vanish; i.e, $\rho_i \neq 0$ for $i \neq l$, where l is some unknown index. Since ν is defined up to an integer, the coefficient indices can be shifted by any integer number. Hence, we can choose $l = 0$ without loss of generality. In order to calculate the bispinors η_i, we would like to express them in terms of η_0. Let us look at the equation corresponding to the first row of the matrix equation (38). Since ρ_{-N} is diagonal, η_{-N} can be expressed in terms of η_{-N+1}:

$$\eta_{-N} = -\rho_{-N}{}^{-1}\zeta\eta_{-N+1}. \tag{41}$$

Substituting (41) into the second row in Eq. (38) yields

$$\eta_{-N+1} = -\left(-\frac{1}{\rho_{-N}}\Xi\zeta + I_4\rho_{-N+1}\right)^{-1}\zeta\eta_{-N+2}. \tag{42}$$

We now use the matrix identity [where α_1 and α_2 are constant scalars and D_1 is given in Eq. (30)]

$$(\alpha_1 I_4 + \alpha_2 D_1)^{-1} = \frac{1}{\alpha_1}I_4 - \frac{\alpha_2}{\alpha_1(\alpha_1+\alpha_2)}D_1 \tag{43}$$

and the relations (28) and $\zeta^2 = 0$ in order to obtain

$$\eta_{-N+1} = -\frac{1}{\rho_{-N+1}}\zeta\eta_{-N+2}. \tag{44}$$

Following this procedure we get, in the general case,

$$\eta_{-i} = -\frac{1}{\rho_{-i}}\zeta\eta_{-i+1}. \tag{45}$$

For $n > 0$ we use the same method, beginning at $n = N$ and going downward. It is possible because identity (43) is satisfied also for D_2 :

$$\eta_i = -\frac{1}{\rho_i}\Xi\eta_{i-1}. \tag{46}$$

Due to the relation $\Xi^2 = \zeta^2 = 0$ we get a truncation $\eta_i = 0$ for $i \neq 0, \pm 1$. We are left with

$$\begin{pmatrix} \rho_{-1}I_4 & \zeta & 0 \\ \Xi & \rho_0 I_4 & \zeta \\ 0 & \Xi & \rho_1 I_4 \end{pmatrix}\begin{pmatrix}\eta_{-1}\\ \eta_0 \\ \eta_1\end{pmatrix} = 0. \tag{47}$$

If we express η_1, η_{-1} in terms of η_0 we get

$$\left(\rho_0 I_4 - \frac{1}{\rho_{-1}}\Xi\zeta - \frac{1}{\rho_1}\zeta\Xi\right)\eta_0 = 0. \tag{48}$$

In order to have a nontrivial solution, we require that the determinant vanishes; i.e.,

$$\det\left[\rho_0 I_4 + e^2a^2m_{\text{ph}}^2\left(\frac{1}{\rho_{-1}}D_1 + \frac{1}{\rho_1}D_2\right)\right] = 0, \tag{49}$$

where relations (28) and (29) were used. The solution of the above equation gives us ν. Regarding the structure of D_1, D_2, Eq. (49) reduces to two scalar equations:

$$\rho_0\rho_1 + e^2a^2m_{\text{ph}}^2 = 0, \tag{50}$$

$$\rho_0\rho_{-1} + e^2a^2m_{\text{ph}}^2 = 0. \tag{51}$$

We assume $e^2a^2 \gg m_{\text{ph}}^2$, or equivalently $(\xi m/m_{\text{ph}})^2 \gg 1$, which is a very good approximation for an optical high-intensity laser ($m_{\text{ph}} < \omega \approx 1$ eV). Therefore, the second term in Eq. (49) is negligible with respect to the constant part of $\rho_1\rho_0$. Consequently, Eq. (50) reduces to $\rho_0 = 0$. A similar argument holds for (51). It allows us to find ν

$$\nu = \frac{k\cdot p}{m_{\text{ph}}^2}\left(1 - \sqrt{1 + \left(\frac{eam_{\text{ph}}}{k\cdot p}\right)^2}\right). \tag{52}$$

The formula for ν is identical to the case of Klein-Gordon (18). The second solution of the quadratical equation $\rho_0 = 0$ is not physical because it does not go to 0 in the limit $k\cdot x \to \pm\infty$. Subsituting expression (52) for ν into (37) we get

$$\rho_n = -2n(k\cdot p)\sqrt{1 + \left(\frac{eam_{\text{ph}}}{k\cdot p}\right)^2} + m_{\text{ph}}^2 n^2. \tag{53}$$

Having attained closed formulas for ν and η_i, the solution of the second-order Dirac equation, ψ_s, can be expressed in a Volkov-like form:

$$\psi_s = e^{-i(p-\nu k)\cdot x}\left(1 - \frac{1}{\rho_1}\Xi e^{i\phi} - \frac{1}{\rho_{-1}}\zeta e^{-i\phi}\right)\eta_0. \tag{54}$$

Under the assumption $ea \gg m_{\text{ph}}$, the coefficient ρ_1 may be approximated by

$$\rho_1 = -\rho_{-1} = -2(k\cdot p)\sqrt{1 + \left(\frac{eam_{\text{ph}}}{k\cdot p}\right)^2}. \tag{55}$$

Accordingly, the wave function, (54), simplifies to

$$\psi_s = e^{-i(p-\nu k)\cdot x}\left(1 - \frac{e}{\rho_1}\not{k}\not{A}\right)\eta_0. \tag{56}$$

However, the second-order Dirac equation includes a redundant solution which does not satisfy the original Dirac equation. Fortunately, the solution of the first-order Dirac equation is related to the solution of the second-order one by the simple transformation $\psi = (i\not{\partial} - e\not{A} + m)\psi_s$ [28]. Therefore, the final solution takes the form

$$\psi = \left[1 - \frac{\not{k}}{2m}\left(\nu - \frac{e^2a^2}{\rho_1}\right) - \frac{e}{\rho_1}\not{k}\not{A} - \frac{e}{4m}\left(1 + \frac{2k\cdot p}{\rho_1}\right)\not{A} - \frac{iem_{\text{ph}}^2}{2m\rho_1}\not{A}'\right]e^{-i(p-\nu k)\cdot x}\eta_0. \tag{57}$$

Now we have to determine η_0. From Eqs. (49)–(51) we deduce that the degeneracy order of Λ is 4 (when the term $e^2a^2m_{\text{ph}}^2$ is neglected). For this reason we have 4 degrees of freedom, meaning that η_0 can be arbitrarily chosen. However, for $k\cdot x \to \pm\infty$ the solution has to go to the free particle wave function, implying $\eta_0 = u_p$.

For $m_{\text{ph}} = 0$ we have $\rho_1 = -2(k\cdot p)$, and additionally, (52) reduces to $\nu = -\frac{e^2a^2}{2(k\cdot p)}$. Consequently, our solution recovers the Volkov wave function [see Eq. (24)] if the photon effective mass vanishes. The only assumption we have made is $ea \gg m_{\text{ph}}$ and hence our solution is applicable over the entire intensity range relevant for high-intensity lasers. One can verify that our final solution, (57), with ν given by (52), solves the Dirac equation (23). Obviously, the expression for the wave function exhibits significant deviation from the Volkov expression if $(eam_{\text{ph}}/k\cdot p)^2 > 1$ is satisfied. Interestingly, this result contradicts the common assumption that as long as $\mathcal{F}/E_s^2 \ll 0$, where $E_s = m^2/e$ is the Schwinger critical field, the particle wave function is approximated by the Volkov solution [5,29].

IV. CONSERVED QUANTITIES

According to quantum mechanics, the temporal dynamics of an observable O are determined by the commutation relation of its correspondent operator $\hat{O}$ with the Hamiltonian

$$\frac{d}{dt}\langle\hat{O}\rangle = i\langle[H,\hat{O}]\rangle + \left\langle\frac{\partial\hat{O}}{\partial t}\right\rangle. \tag{58}$$

The operators discussed below have no explicit time dependence, meaning that the second term is identically 0.

It is well known [29] that in the Volkov case, the operators $i\partial_x$, $i\partial_y$, and $i(k\cdot\partial)$ commute with the Dirac Hamiltonian corresponding to Eq. (20),

$$H_D = \gamma_0[\boldsymbol{\gamma}\cdot(-i\nabla - e\boldsymbol{A}) + m] + eA_0, \tag{59}$$

where $\boldsymbol{\gamma}$ stands for $(\gamma_1,\gamma_2,\gamma_3)$. For the operators $i\partial_x, i\partial_y$ this statement is straightforward since A_μ does not depend on x,y. Let us prove it explicitly for $(k\cdot\partial)$. Since $(k\cdot\partial)$ commutes with the nabla operator, the commutation relation reads

$$[k\cdot\partial, H_D] = [k\cdot\partial, eA_0] - [k\cdot\partial, e\gamma_0\boldsymbol{\gamma}\cdot\boldsymbol{A}]. \tag{60}$$

Consider the first term in Eq. (60). According to the differentiation chain rule,

$$(k\cdot\partial)A_0 = A_0(k\cdot\partial) + [(k\cdot\partial)A_0]. \tag{61}$$

Because of $[(k\cdot\partial)A_0] = m_{\text{ph}}^2 A_0 = 0$, we get

$$[(k\cdot\partial), eA_0] = e(k\cdot\partial)A_0 - eA_0(k\cdot\partial) = 0. \tag{62}$$

Using the same procedure one can show that the second term in Eq. (60) vanishes as well, and hence $[(k\cdot\partial), H_D] = 0$.

However, if $m_{\text{ph}} \neq 0$ the above derivation is no longer valid. In order to construct an alternative conserved operator, we define

$$\tilde{k} \equiv (k_z, 0, 0, \omega_L). \tag{63}$$

Since $\tilde{k}\cdot k = 0$, one can verify that the operator $\tilde{k}\cdot\partial$ does commute with H_D. The action of the operators $i\partial_x, i\partial_y, i\tilde{k}\cdot\partial$ on the wave function given by (57) yields $p_x, p_y, \tilde{k}\cdot p$, respectively. Therefore, p_μ is a conserved quantity. As a result, the orthogonality of the wave functions, which are characterized by p_μ, is assured. Note that if $m_{\text{ph}} = 0$, then $\tilde{k} = k$ and the third conserved quantity is the same as in the Volkov case.

The above arguments hold for the Klein-Gordon Hamiltonian as well. For comfort reasons, we work with the two-component representation of the wave function; i.e., Φ is replaced by $\binom{\theta}{\chi}$, where

$$\theta \equiv \frac{1}{2}\left(\phi + \frac{i}{M}\frac{\partial\phi}{\partial t}\right), \tag{64}$$

$$\chi \equiv \frac{1}{2}\left(\phi - \frac{i}{M}\frac{\partial\phi}{\partial t}\right). \tag{65}$$

The Hamiltonian takes the form [30]

$$H_{\text{KG}} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\frac{(-i\nabla - e\boldsymbol{A})^2}{2M} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M + eA_0. \tag{66}$$

Evidently, the three operators mentioned above commute with H_{KG} [the proof is analogous to (61) and (62)].

Now we would like to obtain the expression for the effective mass of the particle in the presence of an electromagnetic field. For this purpose, we recall that due to the periodicity of the vector potential, the wave function may be cast in the form

$$\psi = e^{-iq\cdot x}\tilde{P}(\phi), \tag{67}$$

where $\tilde{P}$ is some periodic function of ϕ. q_μ is called the quasimomentum since it appears in the four-momentum conservation δ functions associated with the Feynman diagrams of laser-assisted QED processes [29]. As a result, the effective mass is $m^* \equiv \sqrt{q^2}$. One can deduce the quasimomentum related to the wave function, (57), found in the previous section: $q_\mu = p_\mu - \nu k_\mu$. Thus, the effective mass is

$$\frac{m^*}{m} = \frac{1}{m}\sqrt{(p_\mu - \nu k_\mu)^2} = \sqrt{1 + \frac{e^2 a^2}{m^2}}. \tag{68}$$

Though ν given by (52) is not the same as in the Volkov solution [28], formula (68) is identical to the formula obtained with the Volkov wave function. In other words, the effective mass is not affected by the plasma presence. For a scalar particle, ν is the same and therefore its effective mass is also (68) with $m \to M$.

V. CONCLUSIONS

We have derived analytical solutions for the Volkov problem (i.e., a Klein-Gordon or a Dirac particle under the influence of an electromagnetic wave) taking into account the plasma dispersion relation. Since the Volkov solutions [21], a very long time has passed without the introduction of the plasma contribution to the particle wave function. In our opinion, this fact in itself deserves the consideration of the scientific community. The solutions are of great relevance for laser-matter interaction or for a standing wave (i.e., rotating electric field) created by counter-propagating pulses. The assumptions in our derivation are $ea \gg m_{\text{ph}}$ as well as $p\cdot A = 0$. Therefore, it is applicable for any realistic intense laser parameter as long as the matter is initially at rest. We have shown that if $(eam_{\text{ph}}/k\cdot p)^2 > 1$, the differences between the Volkov solutions and our solutions are significant. In the case of an electron initially at rest, this condition is equivalent to $(\xi m_{\text{ph}}/\omega_L)^2 > 1$. For optical lasers and dense plasma (i.e., $m_{\text{ph}} \sim \omega_L$), it corresponds to $I_L > 10^{18}$ W/cm^2, which is already available experimentally. This result contradicts the common assumption in the literature that as long as $\mathcal{F}/E_s^2 \ll 0$, the Volkov wave function is valid [5,29]. Moreover, the new solutions enable us to deduce the effective mass of the particle in the presence of an electromagnetic field. Interestingly, the effective mass is not affected by the plasma environment. The photon effective mass was considered in this paper as an external input (e.g., from a particle-in-cell code). An extension of the model to include the general case of $p\cdot A \neq 0$ as well as self-consistent calculation of m_{ph} will be published soon. To conclude, the new solutions presented in this work may pave the way towards the calculation of laser-assisted QED cross sections in a plasma environment.

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