Experimental investigation of quantum Simpson's paradox

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The well-known Simpson's paradox, or Yule-Simpson (YS) effect, is often encountered in social-science and medical-science statistics. It occurs when the correlations present in different groups are reversed if the groups are combined. Simpson's paradox also exists in quantum measurements. In this Brief Report, we experimentally realized two analogous effects: the quantum-classical YS effect and the quantum-quantum YS effect in the quantum-dot system. We also compared the probability of obtaining those two effects under identical quantum measurements and found that the quantum-quantum YS effect is more likely to occur than the quantum-classical YS effect.

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Simpson's paradox, also known as the Yule-Simpson (YS) effect [1,2], is well known in statistics, and it has been described in a few introductory statistics books [3,4]. There are several real-life examples in social-science statistics and medical-science statistics [5–7]. In statistics, it is often described visually by this thought experiment: A therapy may appear suitable for both men and women separately but unsuitable for people when the genders are considered together [8]. The practical consequences are so counterintuitive that this phenomenon has a considerable impact in decision making [9]. Therefore, a study of the YS effect is helpful in addressing similar problems in real life. Simpson's paradox has received considerable attention from researchers, including in such fields as social science [10,11], medical science [12–14], geology [15], and game theories [16–18].

In the quantum domain, there also exist analogous effects that suggest an opposite trend when previously partitioned groups are combined [8,19,20]. Recently, mathematical analysis of Simpson's paradox with respect to quantum measurements was proposed [8], and this phenomenon was named quantum Simpson's paradox. It can be realized using light sources which have long coherent length, such as weak coherent light, single photons, etc. Semiconductor quantum dots (QDs) are often mentioned as being artificial atoms because of the complete three-dimensional confinement of carriers and the strong photon antibunching observed in single-exciton emissions [21,22]. Therefore, QDs have been extensively investigated as single-photon sources [23,24]. In addition, each photon from QDs corresponds to an event for statistic research. So QDs is a good photon source for researching quantum Simpson's paradox.

In this Brief Report, we discuss our empirical observations of two types of quantum Simpson's paradoxes: the quantum-classical (QC) YS effect and the quantum-quantum (QQ) YS effect. Briefly, we first prepared two initial states $|\psi_1\rangle, |\psi_2\rangle$ and conducted two quantum measurements, namely, A and B, such that we obtained p_1, p_2 measured by A and q_1, q_2 measured by B. Next, we mixed these two initial states to produce two mixed states, which we then characterized, respectively, with the two former quantum measurements; the results were denoted as p and q. We obtained the QC YS effect if the measurement results satisfied $p_1 > q_1$ and $p_2 > q_2$ but $p < q$. Similarly, the QQ YS effect was obtained when the system was prepared using superposition states instead of mixed states, satisfying the reversal of the measurement result. Then, we compared the QC and QQ YS effects in the same situation and found that the QQ YS effect was more likely to occur than the QC YS effect. Finally, we present two typical conditions where the QQ YS effect may be encountered but the QC YS effect will never occur.

In our experiment, we prepared two initial states $|\psi_1\rangle, |\psi_2\rangle$ using single photons emitted from an InAs/GaAs QD, $|\psi_j\rangle = \cos\theta_j|H\rangle + \sin\theta_j|V\rangle$, $j = 1,2$, where $|H\rangle$, $|V\rangle$ are two polarization states. Quantum measurement can be written as $\Pi_M = |\psi_{\varphi_M}\rangle\langle\psi_{\varphi_M}|$, where $|\psi_{\varphi_M}\rangle = \cos\varphi_M|H\rangle + \sin\varphi_M|V\rangle$, $M = A,B$. In our assumption, the results of measurements on two initial states should satisfy $p_j = \langle\psi_j|\Pi_A|\psi_j\rangle > q_j = \langle\psi_j|\Pi_B|\psi_j\rangle$, $j = 1,2$ [8]. So the parameters in initial states and measurements should satisfy

$$|\cos(\theta_j - \varphi_A)| > |\cos(\theta_j - \varphi_B)|, \quad j = 1,2. \tag{1}$$

For the QC situation, we prepared the system in mixed states ϱ_γ, where $\varrho_\gamma = \cos^2\gamma|\psi_1\rangle\langle\psi_1| + \sin^2\gamma|\psi_2\rangle\langle\psi_2|$, $\gamma = \alpha,\beta$. We observed the QC YS effect if $p < q$. Here $p = \cos^2\alpha\, p_1 + \sin^2\alpha\ p_2$ and $q = \cos^2\beta q_1 + \sin^2\beta\ q_2$ [8].

For the QQ situation, any superposition state can be expressed as $|\psi_\gamma\rangle = \frac{1}{\sqrt{N_\gamma}}[\cos\gamma|\psi_1\rangle + e^{-i\phi_\gamma}\sin\gamma|\psi_2\rangle]$, where $\sqrt{N_\gamma}$ is the normalization [8]. We used five variable parameters in each measurement: θ_1, θ_2, α, ϕ_γ, and φ_M, describing the initial states $|\psi_1\rangle$, $|\psi_2\rangle$, the superposition state $|\psi_\gamma\rangle$, and the measurement M. For simplicity, we set the value of ϕ_γ to zero in the experiment. This reduced the complexity of the experiment without changing the quantum properties of the superposition state. Accordingly, the state of the system became $|\psi_\gamma\rangle = \frac{1}{\sqrt{N_\gamma}}[\cos\gamma|\psi_1\rangle + \sin\gamma|\psi_2\rangle]$, and $N_\gamma = 1 + \cos(\theta_1 - \theta_2)\sin 2\gamma$, $\gamma = \alpha,\beta$. The QQ YS effect

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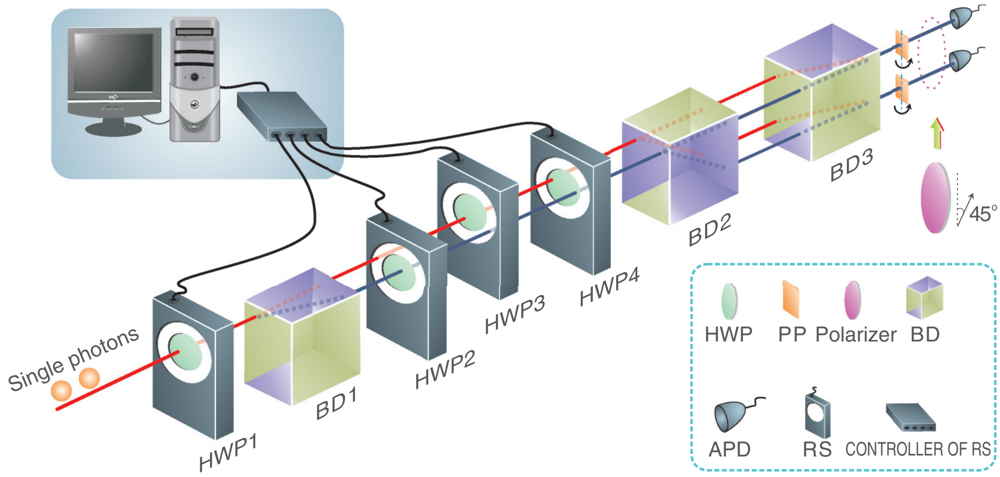

FIG. 1. (Color online) Experimental setup, showing the three stages of the experiment. Note that the parameters γ, θ_1, θ_2, and φ_M in the main text were set to variable values by modulating HWP1–HWP4, respectively, which were installed in four motorized rotation stages (RS) and computer controlled.

subsequently occurred if appropriate parameters were chosen to satisfy $P < Q$, where

$$P = \frac{p + \cos(\theta_1 - \varphi_A)\cos(\theta_2 - \varphi_A)\sin 2\alpha}{1 + \cos(\theta_1 - \theta_2)\sin 2\alpha},$$
$$Q = \frac{q + \cos(\theta_1 - \varphi_B)\cos(\theta_2 - \varphi_B)\sin 2\beta}{1 + \cos(\theta_1 - \theta_2)\sin 2\beta}. \quad (2)$$

To realize the calculation above, we added two labels $|1\rangle$, $|2\rangle$ to the initial states $|\psi_1\rangle$ and $|\psi_2\rangle$, respectively, and prepared the photon in state

$$|\psi\rangle = \cos\gamma|\psi_1\rangle|1\rangle + \sin\gamma|\psi_2\rangle|2\rangle, \quad (3)$$

where $\cos^2\gamma$ and $\sin^2\gamma$ are the probabilities of the photon existing in states $|\psi_1\rangle$ and $|\psi_2\rangle$, respectively. Now, the classical mixture of these two initial states was derived by simply tracing out the states of the labels. This mixture was expressed by the density operator ϱ_γ, $\varrho_\gamma = \cos^2\gamma|\psi_1\rangle\langle\psi_1| + \sin^2\gamma|\psi_2\rangle\langle\psi_2|$. On the other hand, it is also possible to postselect only the $|+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$ component of the label states to erase the man-made labels in Eq. (3). In this manner, it became possible to derive the quantum superposition of these two initial states as $|\psi_\gamma\rangle = \cos\gamma|\psi_1\rangle + \sin\gamma|\psi_2\rangle$.

The experimental setup, shown in Fig. 1, was divided into three parts. The first part (not shown here) involved the generation of single photons [25]. A self-assembled InAs/GaAs QD sample was placed in a 7 K cryostat. Single photons are generated by the sample when excited by a focused He-Ne laser and separated by a grating [26]. Then we filtered out **H** polarized photons using a polarized beam splitter (PBS). The second part involved the preparation of the state shown in Eq. (3). The polarization states were rotated by a half-wave plate (HWP1) and split into two paths by beam displacer 1 (BD1), with different probabilities. Next, the photons in path 1 were prepared in $|\psi_1\rangle$ by HWP2, while photons in path 2 were prepared in $|\psi_2\rangle$ by HWP3. The most important part of the experimental setup is the third part, which was controlled by HWP4, BD2, BD3, two phase plates (PP), and two single-photon avalanche photodiodes (APDs). This involved the preparation of the mixed state (in the QC situation) or the superposition state (in the QQ situation), followed by the quantum measurements. Typically, when the system is prepared in a classical-mixture state, the photon numbers can be directly counted using two APDs, on the basis of which it is possible to calculate the probability of the measurement. For the quantum superposition state, we inserted a 45° polarizer into the position indicated by a dashed circle, on the basis of which we postselected the $|+\rangle$ term. Next, we tilted two PPs to simultaneously set the phase of the two paths to zero. Finally, we counted and calculated the probability of the measurements we conducted.

The first step of our experiment involved generating single photons. The microphotoluminescence spectra of the QDs is shown in the inset of Fig. 2. We used a grating to obtain the exciton emission (peak X) with a wavelength at

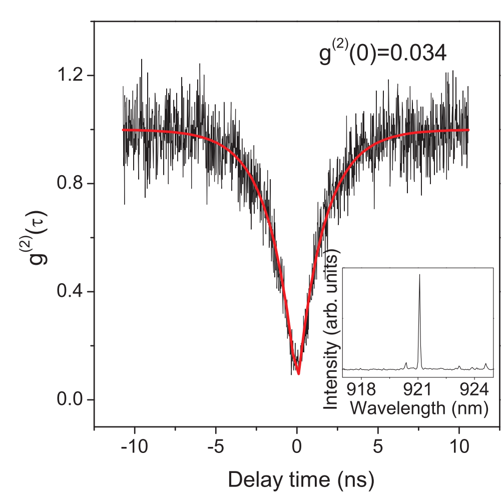

FIG. 2. (Color online) The HBT result of the QD sample that we used in the experiment. By fitting the results [red (gray) curve], we derived a second-order correlation function and obtained $g^2(0) = 0.034$. The inset shows the microphotoluminescence spectra for the QDs studied in this Brief Report.

921.093 nm. Then we performed a Hanbury Brown–Twiss (HBT) experiment. We fitted the experimental data with the expression of $a - b\exp(-|t|/c)$ and $g^{(2)}(0) = (a-b)/a$. As Fig. 2 shows, the red (gray) solid line is the fitting result, taking $a = 1 \pm 0.004$, $b = 0.966 \pm 0.012$, and $c = 1.661 \pm 0.040$. Therefore we obtained $g^{(2)}(0) = 0.034$. If we ignore the influences of background and dark counts, photons in the exciton signal can be considered single photons, and each photon corresponds to a single event in statistical research. The bandwidth of the exciton emission is $\Delta\lambda = 0.085$ nm, so the coherent length of those photons is $l = \lambda^2/\Delta\lambda = 9.98$ mm. Single photons emitted from QDs may not be indistinguishable because of photon decoherence, which will broaden the bandwidth of the exciton emission. However, the coherent length of the photons is much larger than their wavelength, so that the broadening does not affect the feasibility of this experiment. In the following, we will present two kinds of YS effects in the QD system and compare them under identical values of the variable parameters.

We prepared single photons collected in the first step in two initial states $|\psi_1\rangle$, $|\psi_2\rangle$ by modulating HWP2 and HWP3. The probabilities of photons existing in these two states were determined by the angle between the crystal axis of HWP1 and the **H** polarization direction of the photons. Then we prepared the system in classical mixed states or quantum superposition states and measured them with measurements A and B, which were determined by HWP4.

Considering the complexity of the experiment, we simply set the values of all the parameters mentioned (α, β, θ_1, θ_2, φ_A, φ_B) to be $\frac{l}{n}2\pi + \pi/4$, $l,n \in N$, $l \leqslant n$. Further selection was conducted to satisfy the constraints $p_j > q_j$, $j = 1,2$. Next, we conducted quantum measurements on both the mixture state and superposition state under the same parameters. The results are presented in Fig 3, where the ratio P/Q is shown as a function of the ratio p/q. The region $p/q \in (0,1) \times P/Q \in (0,1)$ corresponds to the joint occurrence of the QC and QQ YS effects. When the occurrence of the effects are considered separately, the QC YS effect occurs alone in the region $p/q \in (0,1) \times P/Q \in (1,\infty)$, while the QQ YS effect occurs in the region $p/q \in (1,\infty) \times P/Q \in (0,1)$. As Fig. 3 shows, when the QQ YS effect occurs, the QC effect may or may not occur; that is, the occurrences of the QC and QQ YS effects are independent of each other. We can also easily conclude that the QQ YS effect is more likely to occur than the QC YS effect among all the parameter values.

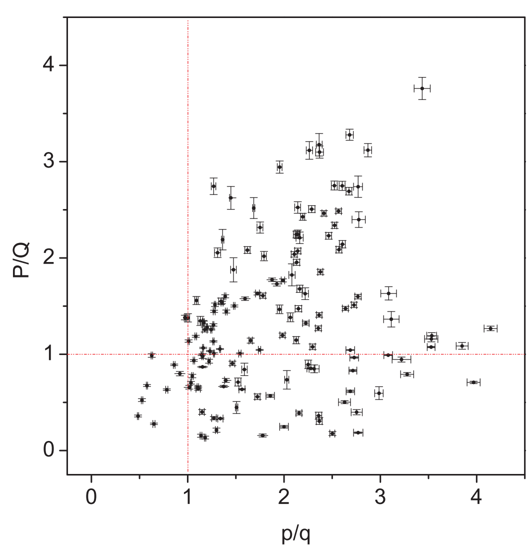

FIG. 3. (Color online) The value of quantum measurements for both the mixture state and the superposition state. The results are presented as the ratio P/Q as a function of p/q. In the experiment, the values of all variable parameters were set to $\frac{l}{n}2\pi + \pi/4, l,n \in N$, $l \leqslant n$, and $n = 7$ here.

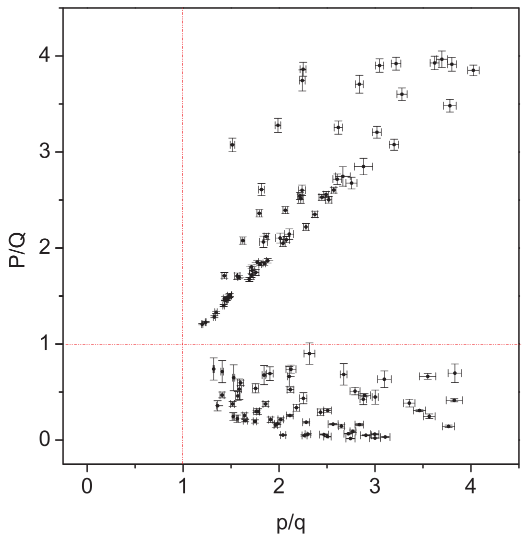

FIG. 4. (Color online) Experimental comparison of the quantum-quantum YS effect with the quantum-classical YS effect when the mixing parameters α and β are fixed with an equal value of $\pi/4$. Values of other parameters were set to $\frac{l}{n}2\pi + \pi/4$, $l,n \in N$, $l \leqslant n$, and $n = 21$ here.

Next, we present two typical situations where the QQ YS effect may occur but the QC YS effect never does. For the first situation, α, β are set to an equal value: $\alpha = \beta = \pi/4$. In this case, $\cos^2\alpha = \sin^2\alpha = \cos^2\beta = \sin^2\beta = \frac{1}{2}$. Therefore, p is always larger than q under the given constraints. However, the QQ YS effect may occur when appropriate parameters are chosen, as Fig. 4 shows. There are some dots in the region $p/q \in (1,\infty) \times P/Q \in (0,1)$, which denotes the occurrence of the QQ YS effect, but there are no dots in the region $p/q \in (0,1) \times P/Q \in (1,\infty)$. The second situation occurs when $p_1 = p_2 > q_1 = q_2$. If the system is prepared in a mixed state, $p = p_1$, $q = q_1$, which means the QC YS effect will never occur. But under certain predetermined parameters, the

TABLE I. An example for the quantum-quantum YS effect when $p_1 = p_2 > q_1 = q_2$. In this case, the quantum-classical YS effect will never occur, but the quantum-quantum YS effect may occur when appropriate parameters are chosen. Here, n was set to be 20, and $\alpha = \frac{7\pi}{20}$, $\beta = \frac{9\pi}{20}$, $\theta_1 = \frac{\pi}{5}$, $\theta_2 = \frac{4\pi}{5}$, $\varphi_A = 0$, $\varphi_B = \frac{\pi}{2}$.

		Π_A	Π_B		
$\cos\alpha$	$\lvert\psi_1\rangle$	$p_1 = 0.635$	$> q_1 = 0.359$	$\lvert\psi_1\rangle$	$\cos\beta$
$\sin\alpha$	$\lvert\psi_2\rangle$	$p_2 = 0.626$	$> q_2 = 0.329$	$\lvert\psi_2\rangle$	$\sin\beta$
	$\lvert\psi_\alpha\rangle$	$P = 0.169$	$< Q = 0.500$	$\lvert\psi_\beta\rangle$	

QQ YS effect will take place. An example of this is shown in Table I.

Here we should point out that there is actually no real paradox in Simpson's paradox, including the classical [8,13,27], QC, and QQ cases. In the classical and QC situations, the YS effect arises from the redistribution of the samples when the original two groups of samples are combined with different weights. In the QQ situation, the redistribution of samples is not the only reason that causes the paradox any longer. In some special cases, the YS effect does not appear in the QC case but occurs in the QQ case with the same weighting of initial states, as shown in Fig 4. The quantum interference plays a very important role here, which constitutes another reason for the YS effect in the quantum mechanics frame and makes the quantum YS effect strikingly differ from its classical version.

In summary, Simpson's paradox is a representative paradoxical effect that is mostly observed in social science, medical science, and game theory. In this work, we have presented an experimental realization of this effect in a QD system prepared in either a classical mixed state or a quantum superposition state. Moreover, we also found that the QC and QQ YS effects can occur independently. In addition, we observed that the QQ YS effect is more likely to occur than the QC one. Finally, we discussed two typical situations where only the QQ YS effect occurs.

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