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Entanglement detection via quantum Fisher information

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We propose an entanglement criterion in terms of quantum Fisher information, which complements the criterion based on variance and local uncertainty relations [H. F. Hofmann and S. Takeuchi, Phys. Rev. A **68**, 032103 (2003)]. We illustrate the significance of this criterion by showing that it can reveal entanglement undetectable by the variance method. The dual relation between the criterion based on quantum Fisher information and that based on variance are highlighted. The combination of these two criteria leads to a refined method for entanglement detection, which is stronger than when either one is used alone.

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I. INTRODUCTION

A fundamental issue in quantum information theory is the characterization or detection of entanglement [1–3]. Although for pure states, this issue is well understood, the situation is extremely complicated for mixed states. Even though much progress has been made in the last two decades, we still do not have an efficient method to tell whether a given quantum state is entangled or not.

There are a variety of special approaches to entanglement detection, among them the Peres-Horodecki positive partial transpose (PPT) criterion [4], the reduction criterion [5], the majorization criterion [6], and the computable cross norm (or realignment) criterion [7], etc., are measurement (observable) independent; and various methods based on Bell-type inequalities [8], entanglement witnesses [9], Bloch representations of density operators [10], local orthogonal observables [11], and local uncertainty relations [12–15], etc., are measurement (observable) dependent. The latter category of criteria are often expressed in terms of inequalities satisfied by separable states, which implies that any state violating these inequalities must be entangled. The effectiveness of these criteria relies heavily on certain notions of information content of quantum states and choice of observables.

In this work, by use of quantum Fisher information [16–23], which is a significant quantity with deep information content and an intrinsic link with variance, we propose an alternative entanglement criterion complementing the criterion based on variance. The basic idea is to exploit the different manifestations, for separable states and entangled states, of the relation between local and global information content of composite states. We illustrate the power of the criterion via several examples.

The work is arranged as follows. In Sec. II, we review some basic properties of quantum Fisher information, and highlight its dual relation with variance. In Sec. III, we derive an entanglement criterion based on quantum Fisher information, demonstrate its power in detecting entanglement, and compare it with the method based on variance. Finally, we conclude with a summary and discussion in Sec. IV.

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II. QUANTUM FISHER INFORMATION

Give a quantum state ρ (density operator) and an observable A on a system Hilbert space H, the variance

$$V(\rho,A) = \mathrm{tr}\rho A^2 - (\mathrm{tr}\rho A)^2$$

is usually interpreted as the uncertainty of A in the state ρ. Taking an alternative viewpoint, and for the expedition of comparison with quantum Fisher information, we may also regard the variance as a quantity of certain "information content" of ρ with respect to the observable A, although it seems not to be a very "information-theoretic" quantity [21]. In this context, a more information-oriented quantity is quantum Fisher information defined as [16–18]

$$F(\rho,A) = \tfrac{1}{4}\mathrm{tr}\rho L^2,$$

where L is the symmetric logarithmic derivative determined by

$$i[\rho,A] = \tfrac{1}{2}(L\rho + \rho L),$$

and the square bracket denotes commutator between operators.

Quantum Fisher information reduces to variance for pure states, namely, $F(\rho,A) = V(\rho,A)$ for any pure state ρ. In general, if ρ is mixed, then

$$0 \leqslant F(\rho,A) \leqslant V(\rho,A).$$

Furthermore, variance and quantum Fisher information are dual to each other in the sense that

$$V(|\phi_j\rangle\langle\phi_j|,A) = F(|\phi_j\rangle\langle\phi_j|,A),$$

and [24,25],

$$V(\rho,A) = \max \sum_j p_j V(|\phi_j\rangle\langle\phi_j|,A),$$

$$F(\rho,A) = \min \sum_j p_j F(|\phi_j\rangle\langle\phi_j|,A),$$

where the max and min are over all pure state ensemble decompositions $\{p_j,|\phi_j\rangle\}$ of ρ, that is, $\rho = \sum_j p_j|\phi_j\rangle\langle\phi_j|$, with $p_j \geqslant 0$, $\sum_j p_j = 1$, and $|\phi_j\rangle$ pure states.

Quantum Fisher information plays an extremely important role in quantum detection and quantum estimation [16,17], in particular in quantum metrology [26], since it places a fundamental limit to the accuracy of quantum estimation. It has the following remarkable information-theoretic properties

[19–25], which are crucial ingredients of our approach to entanglement detection.

(1) Additivity:

$$F(\rho^a \otimes \rho^b, A \otimes \mathbf{1}^b + \mathbf{1}^a \otimes B) = F(\rho^a, A) + F(\rho^b, B),$$

where ρ^a and ρ^b are quantum states, A and B are observables, and $\mathbf{1}^a$ and $\mathbf{1}^b$ stand for the identity operators, for the subsystems a and b, respectively. The above equation means that the information content of a composite system comprising two uncorrelated subsystems is the sum of the information content of the subsystems.

(2) Convexity:

$$F\left(\sum_j \lambda_j \rho_j, A\right) \leqslant \sum_j \lambda_j F(\rho_j, A),$$

where $\sum_j \lambda_j = 1, \lambda_j \geqslant 0$ and ρ_j are quantum states. The above inequality means that if several different quantum systems are mixed, the information content of the resulting system is not larger than the average information content of the component systems. Intuitively, by mixing, one erases the identity information of the component systems.

If we know the spectral decomposition

$$\rho = \sum_k \lambda_k |k\rangle\langle k|,$$

with $\{|k\rangle\}$ an orthonormal base for the system Hilbert space H (we append some zero λ_k if necessary), then quantum Fisher information can be evaluated as [18,20]

$$F(\rho, A) = \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{2(\lambda_k + \lambda_l)} |\langle k|A|l\rangle|^2.$$

Moreover, if $\{A_\mu\}$ is a complete set of orthonormal observables (with respect to the Hilbert-Schmidt inner product of operators), that is, $\mathrm{tr} A_\mu A_\nu = \delta_{\mu\nu}$ and $\{A_\mu\}$ constitutes a base for the real Hilbert space of all observables (Hermitian operators) on H, then

$$\sum_\mu F(\rho, A_\mu) = m - \sum_{k,l} \frac{2\lambda_k \lambda_l}{\lambda_k + \lambda_l} \tag{1}$$

is independent of the choice of the orthonormal observable base $\{A_\mu\}$. Here $m = \dim H$. In particular, we have

$$\sum_\mu F(\rho, A_\mu) \leqslant m - 1. \tag{2}$$

To establish the above relations, we first show that if $\{A'_r\}$ is another orthonormal observable base, then

$$\sum_\mu F(\rho, A_\mu) = \sum_r F(\rho, A'_r).$$

Noting that both $\{A_\mu\}$ and $\{A'_r\}$ are orthonormal observable bases, we may write

$$A'_r = \sum_{\mu=1}^{m^2} a_{r\mu} A_\mu, \quad r = 1,2,\ldots,m^2,$$

with $\{a_{r\mu}\}$ a real orthogonal matrix, that is,

$$\sum_{r=1}^{m^2} a_{r\mu} a_{r\nu} = \delta_{\mu\nu}, \quad \mu,\nu = 1,2,\ldots,m^2.$$

Consequently,

$$\begin{aligned}
&\sum_r F(\rho, A'_r) \\
&= \sum_r \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{2(\lambda_k + \lambda_l)} \left| \langle k| \sum_{\mu=1}^{m^2} a_{r\mu} A_\mu |l\rangle \right|^2 \\
&= \sum_r \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{2(\lambda_k + \lambda_l)} \langle k| \sum_{\mu=1}^{m^2} a_{r\mu} A_\mu |l\rangle \langle l| \sum_{\nu=1}^{m^2} a_{r\nu} A_\nu |k\rangle \\
&= \sum_r \left(\sum_{\mu,\nu=1}^{m^2} a_{r\mu} a_{r\nu} \right) \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{2(\lambda_k + \lambda_l)} \langle k|A_\mu|l\rangle\langle l|A_\nu|k\rangle \\
&= \sum_{\mu,\nu=1}^{m^2} \left(\sum_r a_{r\mu} a_{r\nu} \right) \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{2(\lambda_k + \lambda_l)} \langle k|A_\mu|l\rangle\langle l|A_\nu|k\rangle \\
&= \sum_\mu \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{2(\lambda_k + \lambda_l)} |\langle k|A_\mu|l\rangle|^2 \\
&= \sum_\mu F(\rho, A_\mu).
\end{aligned}$$

Consequently, $\sum_\mu F(\rho, A_\mu)$ is independent of the choice of the orthonormal observable base $\{A_\mu\}$, which means that we can evaluate it in terms of any orthonormal observable base. In particular, we may take

$$\{A_\mu\} = \{E_k, E^+_{k,l}, E^-_{k,l}\} \tag{3}$$

with

$$\begin{aligned}
E_k &= |k\rangle\langle k|, \quad k = 1,2,\ldots,m, \\
E^+_{k,l} &= \frac{1}{\sqrt{2}}(|k\rangle\langle l| + |l\rangle\langle k|), \quad k < l,\ k,l = 1,2,\ldots,m, \\
E^-_{k,l} &= \frac{i}{\sqrt{2}}(|k\rangle\langle l| - |l\rangle\langle k|), \quad k < l,\ k,l = 1,2,\ldots,m,
\end{aligned}$$

then by straightforward calculations, we have

$$F(\rho, E_k) = 0, \quad F(\rho, E^+_{k,l}) = \frac{(\lambda_k - \lambda_l)^2}{2(\lambda_k + \lambda_l)},$$
$$F(\rho, E^-_{k,l}) = \frac{(\lambda_k - \lambda_l)^2}{2(\lambda_k + \lambda_l)},$$

from which Eq. (1) follows.

Inequality (2) follows readily from Eq. (1) and

$$\sum_{k,l} \frac{2\lambda_k\lambda_l}{\lambda_k + \lambda_l} = 1 + \sum_{k \neq l} \frac{2\lambda_k\lambda_l}{\lambda_k + \lambda_l} \geqslant 1.$$

III. ENTANGLEMENT CRITERION

With the above preparation, we now consider entanglement detection via quantum Fisher information. For a bipartite quantum system $H^a \otimes H^b$, let $\{A_\mu\}$ and $\{B_\mu\}$ be any local observables (not necessarily orthonormal at present) for H^a and H^b, respectively. The Hofmann-Takeuchi entanglement criterion based on local uncertainty relations states that if [12]

$$\sum_\mu V(\rho^a, A_\mu) \geqslant V^a, \quad \sum_\mu V(\rho^b, B_\mu) \geqslant V^b \tag{4}$$

for *any* local quantum states ρ^a and ρ^b pertaining to subsystems a and b, respectively, then for any separable state ρ^{ab} on $H^a \otimes H^b$, it holds that

$$\sum_\mu V(\rho^{ab}, A_\mu \otimes \mathbf{1}^b + \mathbf{1}^a \otimes B_\mu) \geqslant V^a + V^b. \tag{5}$$

Therefore, if a quantum state violates the above inequality, then it must be entangled.

This criterion has the intuitive interpretation that a global state, if separable, will inherit the uncertainty relations of the local subsystems. The power of this criterion lies in the flexibility to choose observables A_μ and B_μ. As demonstrated by Gühne *et al.* [13], this criterion is strictly stronger than the computable cross norm criterion (realignment criterion).

Now, since quantum Fisher information is a more information-theoretic quantity than variance, one is naturally led to replace variance by quantum Fisher information to devise some entanglement criteria [24]. The precise formulation is as follows. As opposed to inequality (4), if

$$\sum_\mu F(\rho^a, A_\mu) \leqslant F^a, \quad \sum_\mu F(\rho^b, B_\mu) \leqslant F^b \tag{6}$$

for any local quantum states ρ^a and ρ^b pertaining to subsystems a and b, respectively, then for any separable state ρ^{ab} on $H^a \otimes H^b$, it holds, in contrast to inequality (5), that

$$\sum_\mu F(\rho^{ab}, A_\mu \otimes \mathbf{1}^b + \mathbf{1}^a \otimes B_\mu) \leqslant F^a + F^b, \tag{7}$$

from which we obtain an entanglement criterion based on quantum Fisher information: Whenever ρ^{ab} violates inequality (7), it must be entangled.

To establish the above criterion, note that any separable state can be expressed as

$$\rho^{ab} = \sum_j \lambda_j \rho_j^a \otimes \rho_j^b,$$

where $\sum_j \lambda_j = 1, \lambda_j \geqslant 0$, ρ^a and ρ^b are local quantum states for subsystems a and b, respectively. Now by the additivity and convexity of quantum Fisher information, we have

$$\begin{aligned}
&\sum_\mu F(\rho^{ab}, A_\mu \otimes \mathbf{1}^b + \mathbf{1}^a \otimes B_\mu) \\
&\leqslant \sum_\mu \sum_j \lambda_j F\big(\rho_j^a \otimes \rho_j^b, A_\mu \otimes \mathbf{1}^b + \mathbf{1}^a \otimes B_\mu\big) \\
&= \sum_j \lambda_j \sum_\mu \big[F\big(\rho_j^a, A_\mu\big) + F\big(\rho_j^b, B_\mu\big)\big] \\
&\leqslant \sum_j \lambda_j (F^a + F^b) \\
&\leqslant F^a + F^b.
\end{aligned}$$

The above entanglement criterion via inequality (7) has a clear and intuitive informational interpretation: The information content of a *separable* state of the global system, as quantified by quantum Fisher information, is bounded above by the sum of local information content.

In the case $H^a = H^b$ with dimension m, if we take $\{A_\mu\}$ and $\{B_\mu\}$ as complete sets of orthonormal observables for subsystems a and b, respectively, then by inequality (2), we have

$$\sum_\mu F(\rho^a, A_\mu) \leqslant m - 1, \quad \sum_\mu F(\rho^b, B_\mu) \leqslant m - 1,$$

for any states ρ^a and ρ^b of subsystems a and b, respectively. Therefore, by inequality (7), we have

$$\sum_\mu F(\rho^{ab}, A_\mu \otimes \mathbf{1}^b + \mathbf{1}^a \otimes B_\mu) \leqslant 2m - 2, \tag{8}$$

for any separable state ρ^{ab}. Consequently, any state violating the above inequality has to be entangled.

In contrast, for the conventional variance, we have [21]

$$\sum_\mu V(\rho^a, A_\mu) = m - \mathrm{tr}(\rho^a)^2 \geqslant m - 1,$$
$$\sum_\mu V(\rho^b, B_\mu) = m - \mathrm{tr}(\rho^b)^2 \geqslant m - 1,$$

from which we obtain, via inequality (5),

$$\sum_\mu V(\rho^{ab}, A_\mu \otimes \mathbf{1}^b + \mathbf{1}^a \otimes B_\mu) \geqslant 2m - 2 \tag{9}$$

for any separable state ρ^{ab}. Consequently, any state violating this inequality has to be entangled.

Due to the extremal properties of variance and quantum Fisher information, as recently conjectured by Toth and Petz [24], and established by Yu [25], it is amusing to compare criteria (8) and (9): When variance is replaced by quantum Fisher information, the inequality is reversed. Thus they are formally dual to each other and may be combined together to yield a stronger criterion than when either one is used alone. More precisely, for any bipartite $m \times m$ dimensional state ρ^{ab}, and any local orthonormal observable bases $\{A_\mu\}$ and $\{B_\mu\}$, we evaluate

$$F = \sum_\mu F(\rho^{ab}, A_\mu \otimes \mathbf{1}^b + \mathbf{1}^a \otimes B_\mu),$$
$$V = \sum_\mu V(\rho^{ab}, A_\mu \otimes \mathbf{1}^b + \mathbf{1}^a \otimes B_\mu),$$

and partition the interval $[0,\infty)$ into three disjoint subintervals as (note that $F \leqslant V$, and without loss of generality, we assume that F does not vanish)

$$[0,\infty) = [0,F) \cup [F,V] \cup (V,\infty),$$

then we have the following mutually exclusive scenarios, among which one and only one occurs:

(i) If $2m - 2 \in [0,F)$, then entanglement is detected by the criterion based on quantum Fisher information.

(ii) If $2m - 2 \in [F,V]$, the detection is inconclusive in the sense that we cannot tell whether the state is entangled or not.

(iii) If $2m - 2 \in (V,\infty]$, then entanglement is detected by the criterion based on variance.

In particular, when $\rho^{ab} = |\Psi\rangle\langle\Psi|$ is an $m \times m$ dimensional pure state, by noting that for any pure state, quantum Fisher information coincides with variance, we conclude from inequalities (8) and (9) that

$$\sum_\mu V(\rho^{ab}, A_\mu \otimes \mathbf{1}^b + \mathbf{1}^a \otimes B_\mu) = 2m - 2, \quad (10)$$

$$\sum_\mu F(\rho^{ab}, A_\mu \otimes \mathbf{1}^b + \mathbf{1}^a \otimes B_\mu) = 2m - 2. \quad (11)$$

Thus any *pure* state violating the above *equalities* must be entangled. Consequently, any pure state entanglement can always be determined by the combination of the criteria based on variance and quantum Fisher information, although it may not be detected by either criterion, i.e., inequality (8) or (9), alone!

To gain a feeling of how criteria (8) and (9) work, let us illustrate them by some examples. The first example, despite its simplicity and triviality, highlights the merit of combining the criteria based on variance and quantum Fisher information, which has its origin of concavity of variance and convexity of quantum Fisher information.

Example 1. Let $H^a = H^b$ with dimension $m = 2$. For a two-qubit pure state $\rho^{ab} = |\Psi\rangle\langle\Psi|$ with $|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ (Bell state), take the local orthonormal observables as

$$\{A_\mu\} = \{B_\mu\} = \left\{\frac{\mathbf{1}}{\sqrt{2}}, \frac{\sigma_1}{\sqrt{2}}, \frac{\sigma_2}{\sqrt{2}}, \frac{\sigma_3}{\sqrt{2}}\right\},$$

where σ_j are the Pauli matrices, then we have

$$\sum_\mu F(\rho^{ab}, A_\mu \otimes \mathbf{1}^b + \mathbf{1}^a \otimes B_\mu)$$
$$= \sum_\mu V(\rho^{ab}, A_\mu \otimes \mathbf{1}^b + \mathbf{1}^a \otimes B_\mu) = 4 > 2m - 2 = 2.$$

Consequently, the entanglement in ρ^{ab} can be detected by quantum Fisher information and variance since Eqs. (10) and (11) are not satisfied. Similarly, if $|\Psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ (Bell state), then

$$\sum_\mu F(\rho^{ab}, A_\mu \otimes \mathbf{1}^b + \mathbf{1}^a \otimes B_\mu)$$
$$= \sum_\mu V(\rho^{ab}, A_\mu \otimes \mathbf{1}^b + \mathbf{1}^a \otimes B_\mu) = 0 < 2m - 2 = 2.$$

In this case, the entanglement in ρ^{ab} can also be detected by quantum Fisher information and variance.

The following example shows that the entanglement criterion based on quantum Fisher information may be more powerful than that based on variance in some nontrivial cases.

Example 2. Let $H^a = H^b$ with dimension $m = 3$, and $\{|0\rangle, |1\rangle, |2\rangle\}$ be an orthonormal base of H^a (and also of H^b). Consider the 3×3 dimensional state

$$\rho_p^{ab} = (1 - p)\tfrac{\mathbf{1}}{9} + p|\Omega\rangle\langle\Omega| \quad (12)$$

on $H^a \otimes H^b$ with $\mathbf{1}$ the identity operator on $H^a \otimes H^b$, and

$$|\Omega\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle).$$

Let $\{A_\mu\}$ be the local orthonormal observable base for subsystem a defined by Eq. (3), more explicitly,

$$A_1 = |0\rangle\langle 0|, \quad A_2 = |1\rangle\langle 1|, \quad A_3 = |2\rangle\langle 2|,$$

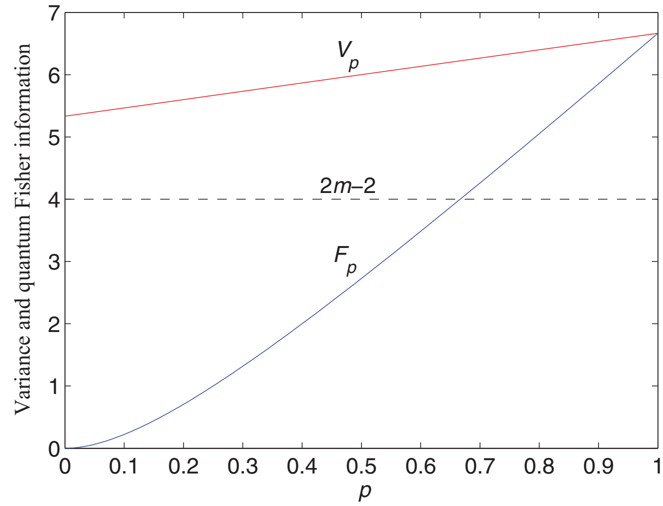

FIG. 1. (Color online) Variance $V_p = \sum_\mu V(\rho_p^{ab}, A_\mu \otimes \mathbf{1}^b + \mathbf{1}^a \otimes B_\mu)$ and quantum Fisher information $F_p = \sum_\mu F(\rho_p^{ab}, A_\mu \otimes \mathbf{1}^b + \mathbf{1}^a \otimes B_\mu)$ versus the parameter p in the state ρ_p^{ab}, as defined by Eq. (12). We see clearly that in this case, the entanglement criterion based on variance is inconclusive in detecting entanglement of ρ_p^{ab} since inequality (9) is not violated for any p; while by the entanglement criterion based on quantum Fisher information, we conclude that ρ_p^{ab} is entangled whenever $p > 2/3$ since inequality (8) is violated for such a p.

$$A_4 = \frac{1}{\sqrt{2}}(|0\rangle\langle 1| + |1\rangle\langle 0|), \quad A_5 = \frac{1}{\sqrt{2}}(|0\rangle\langle 2| + |2\rangle\langle 0|),$$

$$A_6 = \frac{1}{\sqrt{2}}(|1\rangle\langle 2| + |2\rangle\langle 1|), \quad A_7 = \frac{i}{\sqrt{2}}(|0\rangle\langle 1| - |1\rangle\langle 0|),$$

$$A_8 = \frac{i}{\sqrt{2}}(|0\rangle\langle 2| - |2\rangle\langle 0|), \quad A_9 = \frac{i}{\sqrt{2}}(|1\rangle\langle 2| - |2\rangle\langle 1|),$$

and let $\{B_\mu\}$ be similarly defined for subsystem b. When $p = 0.7$, we obtain

$$\sum_\mu F\left(\rho_p^{ab}, A_\mu \otimes \mathbf{1}^b + \mathbf{1}^a \otimes B_\mu\right) = 4.2609 > 2m - 2 = 4,$$

$$\sum_\mu V\left(\rho_p^{ab}, A_\mu \otimes \mathbf{1}^b + \mathbf{1}^a \otimes B_\mu\right) = 6.2667 > 2m - 2 = 4.$$

Thus the entanglement in this mixed state ρ_p^{ab} for $p = 0.7$ can be detected by the criterion based on quantum Fisher information, inequality (8), but cannot be detected by the criterion based on variance, inequality (9). Put it alternatively, by inequality (8), we conclude that the state is entangled; while by inequality (9), we cannot tell if the state is entangled or not.

More generally, we depict the graph of variance and quantum Fisher information versus the parameter p in Fig. 1. We see that when $p \in (2/3, 1]$, the state ρ_p^{ab} violates inequality (8), and thus ρ_p^{ab} with such a p is entangled. On the other hand, we observe that the state ρ_p^{ab} always satisfies inequality (9) for $p \in [0, 1]$, and thus the criterion based on variance cannot detect the entanglement.

It should be noted that, apart from quantum Fisher information, there are many (actually an infinite number of) other versions of quantum Fisher information arising from monotonic metrics which generalize the classical (unique)

Fisher information [19–22]. For example, the Wigner-Yanase skew information [27]

$$I(\rho,A) = -\tfrac{1}{2}\mathrm{tr}[\sqrt{\rho},A]^2$$

is another remarkable information quantity generalizing the classical Fisher information [20]. This quantity has similar properties as that of quantum Fisher information such as additivity and convexity [27,28]. Moreover, we have [20]

$$0 \leqslant I(\rho,A) \leqslant F(\rho,A) \leqslant V(\rho,A). \tag{13}$$

Proceeding analogously as the above derivation of criterion (8), one may employ the skew information to devise the following entanglement criterion [29,30]:

$$\sum_\mu I(\rho^{ab}, A_\mu \otimes \mathbf{1}^b + \mathbf{1}^a \otimes B_\mu) \leqslant 2m - 2.$$

However, this criterion is strictly weaker than that based on quantum Fisher information due to inequality (13) and the extremal property of F recently discovered in Refs. [24,25]. In particular, following example 2, we have for $p = 0.7$,

$$\sum_\mu I\left(\rho_p^{ab}, A_\mu \otimes \mathbf{1}^b + \mathbf{1}^a \otimes B_\mu\right) = 3.0265 < 2m - 2 = 4,$$

and thus the criterion based on the skew information fails to detect the entanglement, which in contrast can be detected by quantum Fisher information.

IV. DISCUSSION

In summary, we have obtained an entanglement criterion based on quantum Fisher information and have made a comparison with that based on variance. This criterion can be used to detect entanglement that cannot be detected by variance in certain cases. The two criteria are dual to each other, and when combined together, gain more advantage in detecting entanglement.

We have only considered the scalar case. In general, quantum Fisher *matrices* are more informative than the scalar quantum Fisher information, which is usually expressed as the trace of the information matrices [31]. Therefore, it is desirable to construct an entanglement criterion by use of quantum Fisher information matrices, and make a comparative study with other detection methods, such as the covariance matrix criterion [32] and PPT. This will be pursued elsewhere.

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