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Uncertainty principle in a cavity at finite temperature

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We employ a dressed-state approach to perform a study on the behavior of the uncertainty principle for a system in a heated cavity. We find, in a small cavity for a given temperature, an oscillatory behavior of the momentum-coordinate product $(\Delta p)(\Delta q)$, which attains periodically finite absolute minimum (maximum) values, no matter how large the elapsed time is. This behavior is in sharp contrast to what happens in free space, where the product $(\Delta p)(\Delta q)$ tends asymptotically, for each temperature, to a constant value, independent of time.

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Introduction. An account of the subject of an interacting particle-environment system can be found in Refs. [1–5], in which the environment is considered an infinite set of noninteractng oscillators. Here we consider a similar model, treated with a different approach. Let us therefore begin with some information about the method we employ: From a general point of view, apart from computer calculations in lattice field theory, currently, the most used method to treat the physics of interacting particles is perturbation theory, in which the starting point is bare fields (particles) interacting by means of gauge fields. Actually, as a matter of principle, the idea of bare particles associated with bare matter fields and of a gauge particle mediating the interaction among them is in fact an artifact of perturbation theory and, strictly speaking, is physically meaningless. A charged physical particle is always coupled to the gauge field; it is always "dressed" by a cloud of quanta of the gauge field (photons in the case of electrodynamics). Exactly the same type of argument applies *mutatis mutandis* to a particle-environment system, in which case we may speak of a "dressing" of the particle by the thermal bath, the particle being dressed by a cloud of quanta of the environment. This should be true in general for any system in which a material particle is coupled to a field, no matter the specific nature of the field (environment) or of the interaction involved. We give a treatment to this kind of system using some dressed (or renormalized) coordinates. In terms of these new coordinates dressed states are defined which allow us to divide the coupled system into two parts, the dressed particle and the dressed environment, which makes working directly with the concepts of bare particles, a bare environment, and the interaction between them unnecessary. A detailed exposition of our formalism and of its meaning for both zero and finite temperature can be found in Refs. [6–10].

About the physical situation we deal with, on general grounds, very precise investigations have been done on the fundamentals of quantum physics, in particular on the validity of the Heisenberg uncertainty relation. In [11], it is reported that a great deal of effort is being made to minimize external noise factors, such as thermal fluctuations and electricity oscillations in experiments, in order to verify the relation, in the spirit of zero-temperature quantum physics. However, changes in the uncertainty principle induced by temperature are an idea already explored in the literature, in particular for open systems. In [12], the authors study, with a thermal-field-dynamics formalism, the relation between the sum of information-theoretic entropies in quantum mechanics with measurements of the position and momentum of a particle surrounded by a thermal environment. It is found that this quantity cannot be made arbitrarily small but has a universal lower bound dependent on the temperature. They also show that the Heisenberg uncertainty relation at finite temperature can be derived in this context. In [13,14] the uncertainty relation for a quantum open system consisting of a Brownian particle interacting with an Ohmic bath of quantum oscillators at finite temperature was obtained. These authors claim that this allows us to get some insight into the physical mechanisms involved in the environment-induced decoherence process. Also, modifications of the uncertainty principle have been proposed in, for instance, a cosmological context. As remarked in [15], in quantum gravity a generalized position-momentum uncertainty principle seems to be needed. The authors of Ref. [15] investigate a possible connection between the generalized uncertainty principle and changes in the area-entropy black-hole formula and the black-hole evaporation process.

In this Brief Report, we study the behavior of the particle-environment system contained in a cavity of arbitrary size under the influence of a heated environment. The environment is composed of an infinite number of oscillators and is assumed to be at a given temperature, realized by taking an appropriate thermal distribution for its modes. This generalizes previous works for zero temperature (for instance, Refs. [17–19]) for both inhibition of spontaneous decay in cavities and the Brownian motion. We study the time-dependent mean value for the dressed oscillator position operator taken in a dressed coherent state. We find that in the case of both the environment at zero temperature and the heated environment, these mean values are independent of the temperature and are given by the same expression. On the other hand, the mean-square error for both the particle position and momentum are dependent on the temperature. From them we get the time- and temperature-dependent Heisenberg uncertainty relation. We then investigate how heating affects the uncertainty principle in a cavity of arbitrary size. This is particularly interesting in a small cavity, where the result is not a trivially expected one.

The model. Our approach to the problem makes use of the notion of dressed thermal states [8], in the context of a model already employed in the literature, of atoms (or, more

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generally, material particles) in the harmonic approximation coupled to an environment modeled by an infinite set of pointlike harmonic oscillators (the field modes). The dressed thermal state approach is an extension of the dressed (zero-temperature) formalism already used earlier [16–19].

We consider a bare particle (atom, molecule, etc.) approximated by a harmonic oscillator described by the bare coordinate and momentum, q_0 and p_0, respectively, having bare frequency ω_0, linearly coupled to a set of N other harmonic oscillators (the environment) described by bare coordinate and momenta, q_k and p_k, respectively, with frequencies ω_k, $k = 1,2,\ldots,N$. The limit $N \to \infty$ will be understood. The whole system is supposed to reside inside a perfectly reflecting spherical cavity of radius R in thermal equilibrium with the environment at a temperature $T = \beta^{-1}$. The system is described by the Hamiltonian

$$H = \frac{1}{2}\left[p_0^2 + \omega_0^2 q_0^2 + \sum_{k=1}^{N}\left(p_k^2 + \omega_k^2 q_k^2\right)\right] - q_0 \sum_{k=1}^{N} c_k q_k. \quad (1)$$

The Hamiltonian (1) is transformed to the principal axis by means of a point transformation, $q_\mu = \sum_{r=0}^{N} t_\mu^r Q_r$, $p_\mu = \sum_{r=0}^{N} t_\mu^r P_r$, where $\mu = (0,\{k\})$, $k = 1,2,\ldots,N$, $r = 0,\ldots,N$, performed by an orthonormal matrix $T = (t_\mu^r)$. The subscript r refers to the normal modes. In terms of normal momenta and coordinates, the transformed Hamiltonian reads $H = \frac{1}{2}\sum_{r=0}^{N}(P_r^2 + \Omega_r^2 Q_r^2)$, where the Ω_r's are the normal frequencies corresponding to the collective *stable* oscillation modes of the coupled system. Using the coordinate transformation in the equations of motion and explicitly making use of the normalization of the matrix (t_μ^r), $\sum_{\mu=0}^{N}(t_\mu^r)^2 = 1$, we get the matrix elements (t_μ^r) [16].

We take $c_k = \eta(\omega_k)^u$, where η is a constant independent of k. In this case the environment is classified according to $u > 1, u = 1$, or $u < 1$ as *supra-Ohmic*, *Ohmic*, or *sub-Ohmic*, respectively [2,3]; we take, as in [16], $\eta = 2\sqrt{g\Delta\omega/\pi}$, where $\Delta\omega$ is the interval between two neighboring field frequencies and g is a fixed constant characterizing the strength of the coupling particle and environment. Restricting ourselves to an Ohmic environment, we get an equation for the $N+1$ eigenfrequencies Ω_r, corresponding to the $N+1$ normal collective modes [16]. In this case the eigenfrequencies equation contains a divergence for $N \to \infty$, and a renormalization procedure is needed. This leads to the *renormalized* frequency [16] (this renormalization procedure was pioneered in [20]),

$$\bar{\omega}^2 = \omega_0^2 - \delta\omega^2 = \lim_{N\to\infty}\left(\omega_0^2 - N\eta^2\right), \quad (2)$$

where we have defined the counterterm $\delta\omega^2 = N\eta^2$.

We introduce dressed and renormalized coordinates q_0' and $\{q_k'\}$ for the dressed atom and the dressed field, respectively, defined by

$$\sqrt{\bar{\omega}_\mu}\, q_\mu' = \sum_r t_\mu^r \sqrt{\Omega_r}\, Q_r, \quad (3)$$

where $\bar{\omega}_\mu = \{\bar{\omega}, \omega_k\}$. In terms of these, we define thermal dressed states, precisely described in [7,8].

It is worthwhile to note that our renormalized coordinates are objects different from both the bare coordinates q and the normal coordinates Q. Also, our dressed states, although being collective objects, should not be confused with the eigenstates of the system [21]. In terms of our renormalized coordinates and dressed states, we can find a natural division of the system into the dressed (physically observed) particle and the dressed environment. The dressed particle will contain automatically all the effects of the environment on it.

A cavity of arbitrary size at finite temperature. To study the behavior of the system in a cavity of arbitrary size, we write the initial physical state in terms of dressed coordinates. We assume that, initially, the system is described by the density operator, $\hat{\rho}(0) = \hat{\rho}_0 \otimes \hat{\rho}_\beta'$, where $\hat{\rho}_0$ is the density operator associated with the oscillator q_0', which can be, in general, in a pure or mixed state. Also, $\hat{\rho}_\beta'$ is the dressed density operator associated with the dressed field modes. We assume thermal equilibrium for these dressed modes; thus

$$\hat{\rho}_\beta' = \frac{\bigotimes_k e^{-\beta \hat{H}_k'}}{\mathrm{Tr}\bigotimes_k e^{-\beta \hat{H}_k'}}, \quad \hat{H}_k' = \frac{1}{2}\hat{p}_k'^2 + \frac{1}{2}\omega_k^2 \hat{q}_k'^2 . \quad (4)$$

The time evolution of the density operator is given by the Liouville–von Neumann equation, whose solution, in the case of an entropic evolution, is given by $\hat{\rho}(t) = e^{-\frac{i}{\hbar}\hat{H}t}\hat{\rho}(0)e^{\frac{i}{\hbar}\hat{H}t}$. Then, the time evolution of the average thermal expectation value of an operator is given by

$$\langle \hat{A}\rangle(t) = \mathrm{Tr}[\hat{A}e^{-\frac{i}{\hbar}\hat{H}t}\hat{\rho}(0)e^{\frac{i}{\hbar}\hat{H}t}] = \mathrm{Tr}[\hat{A}(t)\hat{\rho}(0)], \quad (5)$$

where the cyclic property of the trace has been used; above, $\hat{A}(t) = e^{\frac{i}{\hbar}\hat{H}t}\hat{A}e^{-\frac{i}{\hbar}\hat{H}t}$ is the time-dependent operator $\hat{A}$ in the Heisenberg representation.

Dressed coherent states in a heated environment. Let us consider a Brownian particle embedded in a heated environment as described above. In our notation, we speak of a dressed Brownian particle, and we use the dressed-state formalism. We assume, as usual, that, initially, the particle and the environment are decoupled and that the coupling is turned on suddenly at some given time, which we choose at $t = 0$. In our formalism, we define $|\lambda\rangle$ as a dressed coherent state given by

$$|\lambda, n_1', n_2', \ldots; t = 0\rangle = e^{-|\lambda|^2/2}\sum_{n_0'=0}^{\infty}\frac{\lambda^{n_0'}}{\sqrt{n_0'!}}|n_0' n_1', \ldots\rangle , \quad (6)$$

where n_0' stands for the occupation number of the dressed particle and $n_1', n_2', \ldots$ are the occupation numbers of the field modes. For zero temperature we have $n_1' = n_2', \ldots = 0$. For finite temperature, we will perform computations taking $\hat{\rho}_0 = |\lambda\rangle\langle\lambda|$. This means that, at the initial time, the dressed particle oscillator is in a pure coherent state. Keeping this in mind, we consider the quantity $\langle \hat{q}_0'\rangle(t)$, which we denote by $q_0'(t)$, $q_0'(t) = \mathrm{Tr}[\hat{q}_0'(t)\hat{\rho}(0)]$. In order to evaluate the above expression we first compute $\hat{q}_0'(t)$. Using the relation between the dressed coordinates and the normal coordinates, Eq. (3), the expression for $\hat{H}$ in terms of the normal coordinates, and the Baker-Campbell-Hausdorff formulas, we get, after some steps of calculation,

$$q_0'(t) = \sqrt{\frac{\hbar}{2\bar{w}_0}}[\lambda f_{00}(t) + \lambda^* f_{00}^*(t)], \quad (7)$$

where $f_{00}(t)$ is one of the quantities $f_{\mu\nu}(t) = \sum_s t_\mu^s t_\nu^s e^{-i\Omega_s t}$ [17].

Note that the above expression is independent of the temperature and coincides with the one obtained previously for the zero-temperature case [17]. This is because $\hat{\rho}_\beta$ has even parity in the dressed momentum and position operators. For the same reason, we find an entirely similar formula for $p_0'(t)$. The situation is different for the quantity $q_0'^2(t) = \langle \hat{q}_0'^2 \rangle(t)$. After performing computations similar to those above, we get

$$q_0'^2(t,\beta) = \frac{\hbar}{2\bar{\omega}}\left\{[\lambda f_{00}(t) + \lambda^* f_{00}^*(t)]^2 + 2\sum_k |f_{0k}(t)|^2 n_k'(\beta) + 1\right\}. \tag{8}$$

Then from Eqs. (7) and (8) we obtain for the mean-square error

$$(\Delta q_0')^2(t,\beta) = \langle \hat{q}_0'^2 \rangle(t,\beta) - [\langle \hat{q}_0' \rangle(t)]^2 = \frac{\hbar}{2\bar{\omega}} + \frac{\hbar}{\bar{\omega}}\sum_k |f_{0k}(t)|^2 n_k'(\beta), \tag{9}$$

where $n_k'(\beta)$ is given by the Bose-Einstein distribution, $n_k'(\beta) = 1/(e^{\hbar\beta\omega_k} - 1)$ [8].

Analogously, we obtain the momentum mean-square error,

$$(\Delta p_0)^2(t,\beta) = p_0'^2(t,\beta) - [p_0'(t)]^2 = \frac{\hbar\bar{\omega}}{2} + \hbar\bar{\omega}\sum_k |f_{0k}(t)|^2 n_k'(\beta). \tag{10}$$

From Eqs. (9) and (10) we obtain the time- and temperature-dependent Heisenberg relation,

$$\Delta q_0'(t,\beta)\Delta p_0'(t,\beta) = \frac{\hbar}{2} + \hbar\sum_k |f_{0k}(t)|^2 n_k'(\beta). \tag{11}$$

Time behavior in a cavity with a heated environment. Let us consider the time evolution of the uncertainty relation $\Delta q_0'(t,\beta)\Delta p_0'(t,\beta)$ given in Eq. (11) in a finite (small) cavity, characterized by the dimensionless parameter $\delta = gR/\pi c$, and take a coupling regime defined by a relation between g and the emission frequency $\bar{\omega}$, $g = \alpha\bar{\omega}$. For instance, if we consider $\delta = 0.1$, $\alpha = 0.2$, and $\bar{\omega} \approx 10^{14}$/s (in the visible red), this corresponds to a cavity radius $R \sim 10^{-6}$ m. We measure the uncertainty relation in units of $\hbar$ and call it for simplicity $\Delta q_0'(t,\beta)\Delta p_0'(t,\beta) = \Delta(t,\beta)$. Then calculations can be performed in a similar way as in Ref. [8], and we obtain (there is no confusion between the variable t describing time and the matrix elements t_μ^r)

$$\Delta(t,\beta) = \frac{1}{2} + \sum_{k=1}^{\infty}\frac{1}{e^{(\hbar\beta g/\delta)k} - 1}\left[\left(t_0^0\right)^2\left(t_k^0\right)^2 + 2\sum_{l=1}^{\infty} t_0^0 t_0^l t_k^0 t_k^l \cos(\Omega_0 - \Omega_l)t + \sum_{l,n=1}^{\infty} t_0^l t_0^n t_k^l t_k^n \cos(\Omega_l - \Omega_n)t\right]. \tag{12}$$

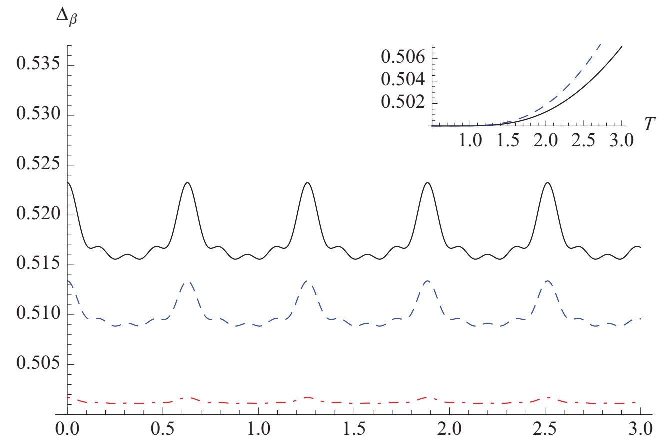

FIG. 1. (Color online) Time evolution of the thermal-dependent uncertainty relation $\Delta_\beta(t)$ for three different values of the temperature, $T = 3.85$ ($\beta = 0.26$; solid line), $T = 3.57$ ($\beta = 0.28$; dashed line), and $T = 1.96$ ($\beta = 0.51$; dot-dashed line). We take $g = 1.0$, $\bar{\omega} = 5.0$, and $\delta = 0.1$; the scale for the temperature, the vertical axis, and time is in units such that $k_B = \hbar = c = 1$; the inset shows the temperature dependence of two neighboring minimum (solid line) and maximum (dashed line) values of $\Delta_\beta(t)$ for two times, $t \approx 2.3$ and $t \approx 2.5$, where they occur.

The matrix elements t_μ^r in the above formulas are evaluated in [8],

$$t_k^0 \approx \frac{k g^2\sqrt{2\delta}}{k^2 g^2 - \Omega_0^2\delta^2},\quad t_k^l \approx \frac{2k\delta}{k^2 - (l+\epsilon_l)^2}\frac{1}{l},$$
$$(t_0^k)^2 \approx \frac{2gR}{\pi c k^2} = \frac{2\delta}{k^2},\quad (t_0^0)^2 \approx 1 - \frac{\pi g R}{3c} = 1 - \frac{\pi^2\delta}{3}, \tag{13}$$

where ϵ_l is a small quantity such that $0 < \epsilon_l < 1$. Actually, for a small cavity ($\delta \ll 1$), $\epsilon_l \approx \delta/k$.

Comments. Equation (12) describes the time evolution of the uncertainty relation $\Delta q_0'(t,\beta)\Delta p_0'(t,\beta) \equiv \Delta(t,\beta)$ in a small cavity. A plot of this time evolution is given in Fig. 1 for some representative values of the temperature. The thermal uncertainty function $\Delta(t,\beta)$ is an oscillating function which attains periodically an absolute minimum (maximum) value, Min[$\Delta(t,\beta)$] (Max[$\Delta(t,\beta)$]). Since the periodic character of $\Delta(t,\beta)$ does not involve β, the location of these extrema on the time axis does not depend on the temperature. Indeed, we can see from Fig. 1 that the values of these minima and maxima depend on the temperature, β^{-1}, but appear to be, for each temperature, the same for all values of time where they occur; in other words the values of the absolute extrema appear to be independent of time. The inset in Fig. 1 shows a neighboring absolute minimum and maximum (corresponding to $t \approx 2.3$ and $t \approx 2.5$) as functions of temperature. We find from Fig. 1 that raising the temperature increases the amplitude of oscillation and the mean value of the uncertainty relation and that its lower and upper bounds also grow with temperature.

We infer from Fig. 1 that for a small cavity in all cases an oscillatory behavior is present for $\Delta(t,\beta)$, with the amplitude of the oscillation depending on the temperature T. For larger values of T the amplitude of the oscillation and both its absolute minimum and maximum values are larger than for lower temperatures. This behavior of the uncertainty principle should be contrasted with the case of an arbitrarily large cavity (free space). In this last case, the product $(\Delta p)(\Delta q)$ goes, asymptotically, as $t \to \infty$ for each temperature, to a constant value $\Delta(\beta)$. This asymptotic value depends on the temperature and grows with it but is independent of time. Distinctly, for a small cavity, even for $t \to \infty$, the product $(\Delta p)(\Delta q)$ presents oscillations which have larger and larger amplitudes for higher temperatures.

The result above falls into a general context of the different behaviors of quantum systems confined in cavities, as compared to free space, in both zero and finite temperature. In [17] some of us got the expected result that the probability $P(t)$ that a simple cold atom in free space, excited at $t = 0$ and remaining excited after an elapsed time t, decays monotonically, going to zero as $t \to \infty$, while in a small cavity $P(t)$ has an oscillatory behavior, never reaching zero. For $\bar{\omega} \sim 10^{14}$ (in the visible red), $R \approx 10^{-6}$ m in a weak (of the order of electromagnetic) coupling regime, Min $P(t) \approx 0.98$, which is in agreement with experimental observations [22]. At finite temperature Ref. [8] obtained, for a small cavity, that the occupation number of a simple atom in a heated environment has an oscillatory behavior with time and that its mean value increases with increasing temperature. In [9,10] the behavior of an entangled bipartite system at zero and finite temperature is investigated; taking two measures of entanglement, an oscillatory behavior for a small cavity (entanglement is preserved at all times) results, while it disappears as $t \to \infty$ for free space.

We hope that the result presented in this Brief Report will have some usefulness in nanophysics or in quantum information theory. At this moment we are not able to comment about these aspects; they will be the subject of future studies.

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