Quantum metrology with SU(1,1) coherent states in the presence of nonlinear phase shifts

K. Berrada
Al Imam Mohammad Ibn Saud Islamic University (IMSIU), College of Science, Department of Physics, Riyadh, Saudi Arabia, The Abdus Salam International Centre for Theoretical Physics, Strada Costiera 11, Miramare-Trieste, Italy, and Laboratoire de Physique Théorique, Faculté des Sciences, Université Mohammed V-Agdal, Avenue Ibn Battouta, Boîte Postale 1014, Agdal Rabat, Morocco

(Received 7 February 2013; published 11 July 2013)

We present an improved phase estimation scheme employing entangled SU(1,1) coherent states in comparison to NOON and entangled coherent states under perfect and lossy conditions for a fixed mean photon number. The study is also devoted to the phase enhancement of the quantum states resulting from a generalized nonlinearity of the phase shifts, both without and with losses. Furthermore, we show that these states give the smallest variance in the phase parameter for a large number of photons in a different order of nonlinearity. Finally, the phase sensitivity of this interferometric setting with parity detection is discussed.

DOI: 10.1103/PhysRevA.88.013817 PACS number(s): 42.50.St, 03.67.−a, 03.65.Ta, 42.50.Dv

I. INTRODUCTION

Quantum optical metrology deals with the estimation of an unknown phase by exploiting the quantum nature of the input state under consideration [1–4]. The ultimate goal of quantum metrology is to achieve strong sensitivity of certain quantum states to small variations of external parameters, which opens up great opportunities to increase the resolution of interferometric measurements [5,6]. Experimental, quantum-enhanced metrology promises many technological advances, since an optimally designed quantum measurement procedure outperforms any classical method [7]. In the usual classical setting, for a given number of input photons, N, the best precision that can be obtained using these states of light scales as $1/\sqrt{N}$, which is usually referred to as standard quantum limit or shot-noise limit [8]. By exploiting the quantum signature of highly nonclassical entangled states, it has been shown that the uncertainty can be improved to a scaling $1/N$ known as the Heisenberg limit [5,9]; an enhancement of a factor of $1/\sqrt{N}$ depends on the nature of the input states and the detection strategy of the output measurement. These bounds are obtained by an application of the Cramer-Rao inequality (CRB) in terms of the quantum Fisher information (QFI) for a number of repeated independent trials [5,10]. Unfortunately, maximally entangled states of light which potentially lead to Heisenberg limited sensitivity are very sensitive to photon-loss noise, which is considered the most destructive noise in quantum-enhanced optical interferometry, where the benefit from highly entangled states deteriorates quickly even if only a small amount of noise is detected by the system [11]. Recently, quantum metrology has included very modest entangled resources using so-called BAT state, is formed by passing a dual Fock state, $|\psi\rangle = |N\rangle|N\rangle$ through a 50:50 beam splitter and other states that have potentially better performance against loss compared to NOON states, possessing the same mean particle number, for the realistic scenarios of loss with current technology [12–14]. Recently, several development schemes have shown the potential advantages of nonlinearities [15–18] and the significance of the query complexity for quantum metrology [17], deducing that the same phase operation is essential for the appropriate resource count in different quantum states. These theoretical studies have explained the role of nonlinearity to improve the Heisenberg limit in linear systems and presently leading to the super-Heisenberg limit [19]. The nonlinear phase operations can be achieved experimentally in several ways [20,21]. More recently, it was shown that entangled coherent states (ECSs) can allow the Heisenberg limit and can beat NOON states for small values of photon number with both linear and nonlinear optical elements [18,22,23]. These works left many questions unanswered. Are there any input states that improve the phase enhancement to NOON and ECSs for different orders of the nonlinearity and the asymptotic scaling behavior with larger photon number of the phase sensitivity? In this paper we answer these questions for two-mode optical interferometry, both without and with losses. We use a particular sort of entangled states, called entangled SU(1,1) coherent states (CSs), for phase estimation as shown in Fig. 1. Entangled SU(1,1) CSs are an equal superposition of an SU(1,1) CS in mode a with vacuum in mode b, and vacuum in mode a with a SU(1,1) CS in mode b. Lie algebras of the SU(1,1) groups are widely used related to the squeezing properties of the radiation field which is still a central topic in quantum optics. On the one hand, various experiments have been performed in which squeezed light can be generated. In particular, Wódkiewicz and Eberly [24] discussed the role of SU(1,1) CSs associated with Lie algebra of the group SU(1,1) in connection with variance reduction (squeezing), and the coherence-preserving Hamiltonians associated with SU(1,1) CSs were studied by Gerry [25]. This kind of SU(1,1) CSs is a special case of two-photon coherent states discussed by Yuen [26]. This fact was used by Gerry [25], who applied the SU(1,1) formulation of the two-photon coherent states to the problem of the interaction of squeezed light with a nonlinear-absorbing medium modeled as an anharmonic oscillator. Recently, a practical way for three-boson realization of SU(1,1) and characterizing all squeezed states of this type as SU(1,1) CSs was developed [27]. Such states can be generated by certain multimode linear quantum optical networks comprising one two-mode squeezer and several passive optical elements. Another family of the SU(1,1) CSs can be implemented and naturally emerged from the quantum motion in various potentials such as infinite-well and trigonometric Pöschl-Teller potentials. Indeed, the Pöschl-Teller potentials share with their infinite-well limit the nice property of being analytically integrable. The reason behind this can be understood within a group-theoretical context: These potentials possess an underlying dynamical algebra, namely SU(1,1) and the discrete series representation of the latter [28]. On the other hand, the general form for the entangled SU(1,1) CSs incorporates entangled harmonic oscillator coherent states in the formalism. Entangled SU(1,1) CSs might be created in Fock space using an ideal Kerr nonlinearity by an appropriate arrangement of three nonlinear media elements or by a Hamiltonian evolution which is a generalization of K_z^2 nonlinear evolution from a multipartite system including the SU(1,1) Schrödinger cat–like states [29–31]. Such evolutions are extremely sensitive to environmentally induced decoherence. Other methods for generating entangled coherent states could be considered, but the nonlinear evolution considered illustrated one of the possible approaches to producing these entangled states [29,30]. We were surprised to find that these states lead to a lower precision than the NOON and ECSs [22] for the interesting region of metrology with a large number of photons for different orders of the nonlinearity under perfect and lossy conditions. In the lossy regime, both arms of the interferometer are subject to photon losses which can be modeled by fictitious beam splitters (BSs) inserted at arbitrary locations in both channels. Furthermore, we discuss the phase sensitivity saturation of the SU(1,1) interferometric setting with a realistic measurement approach on one mode of the output state.

The present paper is organized as follows. In Sec. II, we present the phase sensitivity of the two-mode state in the context of SU(1,1) Lie algebra using QFI for generalized phase shifters under perfect conditions. Furthermore, we discuss the phase sensitivity of this interferometric setting with purity detection. In Sec. III, we investigate the effect of photon losses on the phase enhancement behavior. Finally, we summarize our work in Sec. IV.

II. OPTIMAL PHASE ESTIMATION USING SU(1,1) CSs

The phase optimization is related to QFI by the CRB for the output states as

$$\delta\phi \geqslant \frac{1}{\sqrt{F_Q}}, \tag{1}$$

where F_Q is the QFI [32].

We focus on input states as the entangled SU(1,1) CSs in the context of the optical fields using the Holstein-Primakoff realization (HPR). The SU(1,1) CSs are given in terms of a set of single-mode Bose annihilation and creation operators that are associated with the HPR form of the SU(1,1) Lie algebra. This HPR is given by the operators

$$\begin{aligned} &\hat{K}_+ = \hat{a}^\dagger(2k+\hat{a}^\dagger\hat{a})^{1/2}, \quad \hat{K}_- = (2k+\hat{a}^\dagger\hat{a})^{1/2}\hat{a},\\ &\hat{K}_z = \hat{a}^\dagger\hat{a}+k, \end{aligned} \tag{2}$$

satisfying the commutation relations

$$[\hat{K}_+,\hat{K}_-] = -2\hat{K}_z, \quad [\hat{K}_\pm,\hat{K}_z] = \mp\hat{K}_\pm, \tag{3}$$

where it is further assumed that $\hat{a}$ and $\hat{a}^\dagger$ satisfy the Bose algebra $[\hat{a},\hat{a}^\dagger] = \mathbb{1}$. The parameter k is the so-called Bargmann index related to the eigenvalue of the SU(1,1) Casimir operator $\hat{K}_z^2 - (\hat{K}_+\hat{K}_- + \hat{K}_-\hat{K}_+)/2$ given by $k(k-1)$. For the relevant representations, the positive discrete series, the Bargman index takes on the values $k = \frac{1}{2},1,\frac{3}{2},\ldots$. The corresponding bases $\{|k,n\rangle\}$ satisfy the eigenvalue problem $\hat{K}_z|k,n\rangle = (n+k)|k,n\rangle$. By comparing with Eq. (2) it is clear that, for the HPR, the states $|k,n\rangle$ are related to the Bose number states $|n\rangle$ according to $|k,n\rangle \sim |n\rangle$. That is, the bases are independent of the value of the Bargmann index and so are identical for all k, being simply the entire infinite-dimensional Hilbert space of the number states $|n\rangle$ for $n = 0,1,2,\ldots$.

Now, we aim to find the input state that allows performing phase estimation with the best precision possible, i.e., yielding the highest value of the QFI. In particular, we consider the pure two-mode input state, superposition of macroscopic SU(1,1) CSs which can be understood as a superposition of NOON states, given by

$$\left|\Psi_E^{\text{int}}\right\rangle_{ab} = \mathcal{N}_{\xi,k}[|\xi,k\rangle_a|0\rangle_b + |0\rangle_a|\xi,k\rangle_b], \tag{4}$$

where the normalization factor $\mathcal{N}_{\xi,k}$ is dependent on the construction of coherent states associated to SU(1,1) Lie algebra.

Since the group SU(1,1) is noncompact and its unitary representations must be dimensionally infinite, then there are two principal kinds of coherent states to be considered within the positive discrete series. The first kind is generated from an extremal state by an element of the respective groups via a displacement operator which generates the Perelomov coherent states (PCSs) from the vacuum [33]

$$\begin{aligned} |\xi_P,k\rangle_i &= \exp(\mu\hat{K}_+ - \mu\hat{K}_-)|k,0\rangle_i\\ &= \frac{1}{\sqrt{{}_1F_0(2k;|\xi_P|^2)}}\sum_{n=0}^{\infty}\left[\frac{(2k)_n}{n!}\right]^{\frac{1}{2}}\xi_P^n|n\rangle_i, \end{aligned} \tag{5}$$

where $\mu = (\vartheta/2)\exp(-i\varphi)$, $\xi_P = e^{-i\varphi}\tanh(\vartheta/2)$, and where $(2k)_n$ is a Pochammer symbol: $(2k)_n = \Gamma(n+2k)/\Gamma(2k)$. The angle φ is azimuthal with $0 \leqslant \varphi \leqslant 2\pi$ but ϑ is a hyperbolic angle with $0 \leqslant \vartheta < \infty$. The function ${}_1F_0(2k;|\xi_P|^2)$ is a hypergeometric function. The parameter ξ_P is restricted to within the unit circle in the complex plane; $0 \leqslant |\xi_P| < 1$. The average photon number of the PCS is

$$\langle\hat{n}\rangle_P = \frac{k|\xi_p|^2}{1-|\xi_p|^2} = k(\cosh\vartheta - 1). \tag{6}$$

The SU(1,1) CSs as defined by Perelomov were previously discussed in connection with squeezed states of a single-mode field. The realization of the SU(1,1) Lie algebra required for these states involves bilinear and quadratic products of the annihilation and creation operators of the single mode. Such states may be produced out of vacuum by a degenerate parametric amplifier whose Hamiltonian is linear to SU(1,1) generators. But there is another, equivalent, form of SU(1,1) CSs that we consider, namely, the Barut-Girardello coherent states (BGCSs) [34], given as a right eigenstate of the SU(1,1) lowering operator. Using the HPR of the operator $\hat{K}_-$ as given by Eq. (2), the corresponding BGCS is

$$\hat{K}_-|\xi_B,k\rangle_i = \xi_b|\xi_B,k\rangle_i \tag{7}$$

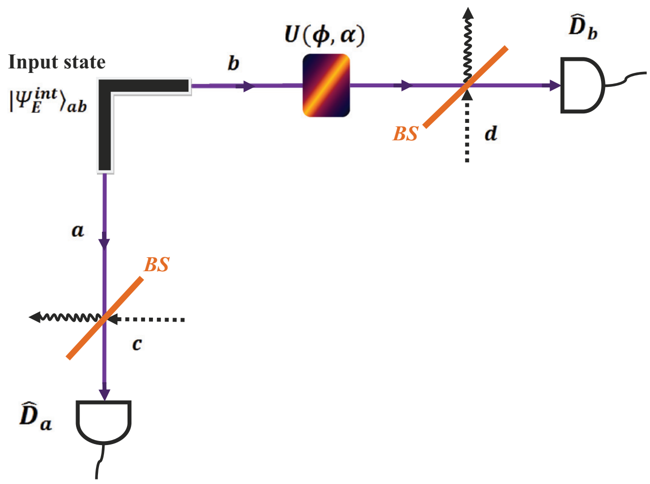


FIG. 1. (Color online) Interferometric phase estimation scheme for the entangled SU(1,1) CSs. Channel b acquires a phase ϕ relative to channel a. After applying a phase shift $\hat{U}(\phi,\alpha)$ in mode b, the parity measurement is performed.

and is given in normalized form as

$$|\xi_B,k\rangle_i = \frac{1}{\sqrt{{}_0F_1(2k;|\xi_B|^2)}}\sum_{n=0}^{\infty}\left[\frac{1}{(2k)_n n!}\right]^{\frac{1}{2}}\xi_B^n|n\rangle_i. \tag{8}$$

The average photon number of the state is

$$\langle\hat{n}\rangle_B = \frac{\mathcal{I}_{2k}(2|\xi_B|)|\xi_B|}{\mathcal{I}_{2k-1}(2|\xi_B|)}, \tag{9}$$

where $\mathcal{I}_l$ is the modified Bessel function of order l.

Let us now consider an interferometer with two arms a and b, as shown in Fig. 1. An initial $|\Psi_E^{\text{int}}\rangle_{ab}$ is prepared in modes a and b and acquires a phase ϕ in channel b relative to channel a by a generalized nonlinear phase-shifter operation, $\hat{U}(\phi,\alpha) = \exp i\phi(\hat{b}^\dagger\hat{b})^\alpha$, where $\hat{b}$ is the annihilation operator in mode b. Nonlinear transformations with "nonlinear phase shifts" can be used in precision interferometric measurements and are very easily implemented in practice since they correspond to propagation in media with nonlinear optical properties. Several experiments have demonstrated that nonlinear phase operations can be realized in various setups. For example, self-Kerr phase modulation ($k = 2$) has been measured as a function of electric field amplitudes in waters, fibers, nitrobenzene, Rydberg states, etc. [35]. Notably, the phase shift dependent on the applied field clearly follows theoretical predictions in the case of a Rydberg electromagnetically induced transparency medium [20]. A well-known example of a nonlinear phase operation is given by the Kerr interaction for $k = 2$ [36]. The predicted advantage applies generally to quantum interferometry and proposed mechanisms to produce metrologically relevant phase shifts from a nonlinear Kerr-like interaction. Thus, a new frontier arises if we consider that the signal can be imprinted in the probe via nonlinear processes. The key point is that nonlinear schemes allow us to improve the scaling and reach larger resolutions than the linear ones. This leads to new quantum limits, new experiments, and eventually new devices.

When the operator $\hat{U}(\phi,\alpha)$ is applied to mode b of the input state, it leads to the following output state:

$$\begin{aligned}\left|\Psi_E^{\text{out}}\right\rangle_{ab} &= (\mathbb{1}\otimes\hat{U}(\phi,\alpha))\left|\Psi_E^{\text{int}}\right\rangle_{ab}\\ &= \mathcal{N}_{\xi,k}[|\xi,k\rangle_a|0\rangle_b + |0\rangle_a|\xi e^{in^{\alpha-1}\phi},k\rangle_b].\end{aligned} \tag{10}$$

The exponent α shows the order of the nonlinearity: $\alpha = 1$ corresponds to a linear phase shift on the input state, $\alpha = 2$ represents a Kerr phase shift, and $\alpha \neq 2$ exhibits a more general nonlinear effect on phase operation.

The QFI for the pure state $|\Psi_E^{\text{out}}\rangle_{ab}$ is given by

$$F_K^\alpha = 4\left[\langle\Phi|\Phi\rangle_{ab} - \left|\left\langle\Phi\middle|\Psi_E^{\text{out}}\right\rangle_{ab}\right|^2\right], \tag{11}$$

where $|\Phi\rangle_{ab} = \partial|\Psi_E^{\text{out}}\rangle_{ab}/\partial\phi$. The subscript $K = P,B,N,C$, corresponding to entangled PCS (EPCS), entangled BGCS (EBGCS), NOON, and ECS, respectively.

Let us first consider the situation with no loss of photons, in which the QFI of the pure states with a nonlinear phase shift of order α is analytically achieved. For the NOON state, we find that $F_N^\alpha = 1/N^{2\alpha}$ with $\delta\phi_N^\alpha \geqslant 1/N^\alpha$, and for EPCS and EBGCS,

$$\begin{aligned}F_P^\alpha = f_{\xi_P,k}&\left[\sum_{n=0}^{\infty}\frac{n^{2\alpha}\Gamma(n+2k)|\xi_P|^{2n}}{n!\Gamma(2k)}\right.\\ &\left.- f_{\xi,k}\left(\sum_{n=0}^{\infty}\frac{n^{\alpha}\Gamma(n+2k)|\xi_P|^{2n}}{n!\Gamma(2k)}\right)^2\right]\end{aligned} \tag{12}$$

and

$$\begin{aligned}F_B^\alpha = g_{\xi_B,k}&\left[\sum_{n=0}^{\infty}\frac{n^{2\alpha}|\xi_B|^{2n}}{n!(n+2k-1)!}\right.\\ &\left.- g_{\xi_B,k}\left(\sum_{n=0}^{\infty}\frac{n^{\alpha}|\xi_B|^{2n}}{n!(n+2k-1)!}\right)^2\right],\end{aligned} \tag{13}$$

respectively, and

$$\delta\phi_{P,B}^\alpha \geqslant \frac{1}{\sqrt{F_{P,B}^\alpha}}, \tag{14}$$

where $f_{\xi_P,k} = 4(1-|\xi_P|)^{2k}\mathcal{N}_{\xi_P,k}^2$, and $g_{\xi_B,k} = 4\mathcal{N}_{\xi_B,k}^2|\xi_B|^{2k-1}/\mathcal{I}_{2k-1}(|\xi_B|)$.

In order to compare the phase uncertainty for the EPCS and EBGCS with NOON and ECS, we take into account the equivalent resource case for the states [37]. We consider the same average photon number for the mode a given by

$$\bar{n}_K = \frac{N}{2} = \left(\mathcal{N}_{\xi_P,k}\right)^2\langle\hat{n}\rangle_P = \left(\mathcal{N}_{\xi_B,k}\right)^2\langle\hat{n}\rangle_B, \tag{15}$$

where the normalization factors $\mathcal{N}_{\xi_P,k}$ and $\mathcal{N}_{\xi_B,k}$ are obtained from Eq. (10).

In Fig. 2, the phase sensitivity for EPCS and EBGCS can be compared with respect to N for the NOON state for different orders of the nonlinearity. The dash-dotted black line is for the NOON state, the dotted green line is for ECS, the dashed red ($k = 1$) and solid red ($k = 15$) lines are for EPCS, and the long-dashed blue ($k = 1$) and long dash-dotted blue ($k = 2$) lines are for EBGCS. We can see the values

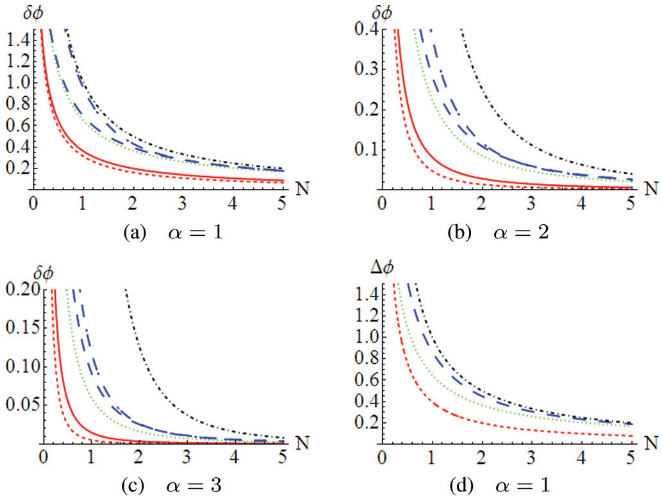


FIG. 2. (Color online) The lower bound on the phase uncertainty for EPCS, EBGCS, NOON, and ECS is plotted as a function of photon number for different orders of the nonlinearity, (a) $\alpha = 1$, (b) $\alpha = 2$, and (c) $\alpha = 3$. The dashed red ($k = 1$) and solid red ($k = 15$) lines are for EPCS; the long dashed blue ($k = 1$) and long dash-dotted blue ($k = 2$) lines are for EBGCS; the dash-dotted black line is for the NOON state; and the dotted green line is for ECS [22]. The values of optimal phase estimation are given for the different kinds of the states satisfying the inequality $\delta\phi_P^\alpha < \delta\phi_C^\alpha \leqslant \delta\phi_B^\alpha \leqslant \delta\phi_N^\alpha$ for any N and α for small values of the Bargmann index k. (d) $\alpha = 1$, showing the optimal phase estimation of the states after performing the parity measurement.

of optimal phase estimation for the different kinds of the states satisfying the inequality $\delta\phi_P^\alpha < \delta\phi_C^\alpha \leqslant \delta\phi_B^\alpha \leqslant \delta\phi_N^\alpha$ for any N and α. In this context, the phase variation decreases with increasing nonlinear phase order, which illustrates that the entangled SU(1,1) CSs outperform the phase enhancement achieved by NOON and ECSs. For large N, $\delta\phi_B^\alpha \approx \delta\phi_C^\alpha \approx \delta\phi_N^\alpha$ and the EGBCS becomes approximately equivalent to the NOON state, being dominated by the NOON amplitude at $N = \langle\hat{n}\rangle_B$, whereas for small values of N the optimal phase for EBGCS is shown to be less than that for NOON. On the other hand, the phase estimation $\delta\phi_P^\alpha$ for EPCS is still smaller than other optimal phases, even for large values of photon number N, due to its superposition property of the input states, where the state $|\Psi_E^{\text{int}}\rangle_{12}$ contains an infinite number of NOON states including N values exceeding $\langle\hat{n}\rangle_P$. Then, the superposition property provides an advantage for the EPCS at smaller Bargmann index values with large ranges of $\overline{n}_P$. In both cases of SU(1,1) CSs, the decreasing Bargmann index k gives the system better sensitivity to different orders of nonlinearity. For a more detailed example, consider $\alpha = 2$ and take $N = 4$ for the NOON state with $\overline{n}_N = 2$, $\overline{n}_C \approx 1.964$ for the ECS [22], $\langle\hat{n}\rangle_B = 4$ with $\overline{n}_B \approx 1.994$ for the EGBCS, and $\langle\hat{n}\rangle_P = 4$ with $\overline{n}_P \approx 1.923$ (which gives a slightly lower resource count) for the EPCS in the smaller Bargmann index limit; the values of the optimal phase estimation are equal to $\delta\phi_N^2 \approx 0.060$, $\delta\phi_C^2 \approx 0.030$, $\delta\phi_B^2 \approx 0.038$, and $\delta\phi_P^2 \approx 0.004$. For larger photon number $N = 20$, we find that $\delta\phi_N^2 \approx 0.00250$, $\delta\phi_C^2 \approx 0.00201$, $\delta\phi_B^2 \approx 0.00222$, and $\delta\phi_P^2 \approx 0.00016$. This indicates that, even with a slight resource disadvantage $\delta\phi_P^2 < \delta\phi_C^2 < \delta\phi_B^2 < \delta\phi_N^2$, there is still a phase estimation advantage for larger photon numbers. These results may provide a new perspective on quantum metrology by possibly replacing the previously acclaimed performance limit.

We now discuss the parity measurement that detects whether the number of photons in a given output mode is even or odd. The measurement is applied in mode b and the uncertainty in the estimation of the phase shift, $\Delta\phi$, upon measurement of the parity operator $\hat{\Pi}_b = (-1)^{\hat{n}}$ is given by [38]

$$(\Delta\phi)^2 = \frac{(\Delta\Pi_b)^2}{(|\partial\langle\hat{\Pi}_b\rangle/\partial\phi|)^2}, \tag{16}$$

where $(\Delta\Pi_b)^2 = \langle\hat{\Pi}_b^2\rangle - \langle\hat{\Pi}_b\rangle^2 = 1 - \langle\hat{\Pi}_b\rangle^2$ since $\hat{\Pi}_b^2 = \mathbb{1}$. For the input state (4), the expectation value of the parity operator is

$$\langle\hat{\Pi}_b\rangle_P = \frac{2 + (1 - |\xi_P|^2 e^{i\phi})^{-2k} + (1 - |\xi_P|^2 e^{-i\phi})^{-2k}}{2 + 2(1 - |\xi_P|^2)^{-2k}}, \tag{17}$$

$$\langle\hat{\Pi}_b\rangle_B = \frac{2 + (\gamma)^{-\beta}\mathcal{I}_\beta(2\gamma) + (\delta)^{-\beta}\mathcal{I}_\beta(2\delta)}{2 + 2(\sqrt{\gamma\delta})^{-\beta}\mathcal{I}_\beta(2\sqrt{\gamma\delta})}, \tag{18}$$

where $\beta = 2k - 1$, $\gamma = e^{i\frac{\phi}{2}}|\xi_B|$, and $\delta = e^{-i\frac{\phi}{2}}|\xi_B|$. As shown in Fig. 2(d), we clearly see that the parity measurement on the EPCS and EGBCS does not saturate the optimal phase uncertainty given by the Quantum CRB for these states, but it is still better than the Heisenberg limit given by NOON and ECSs.

III. LOSSY REGIME CASE

Next, we determine a lower bound for the uncertainty of the parameter estimation employing entangled SU(1,1) CSs in the realistic scenario of the photon loss. To this end, we apply two BS transformations characterized by the transmission rate T, considering the scenario of equal losses in both arms of the interferometer, i.e., $T_1 = T_2 = T$, with loss modes c and d located after the phase operation.

When the output state is a mixed state ρ_E^{out}, the QFI is given by

$$F_K^\alpha = \sum_{i,j} \frac{2}{\lambda_i + \lambda_j} \left|\langle\lambda_i|\partial\rho_E^{\text{out}}/\partial\phi|\lambda_j\rangle\right|^2, \tag{19}$$

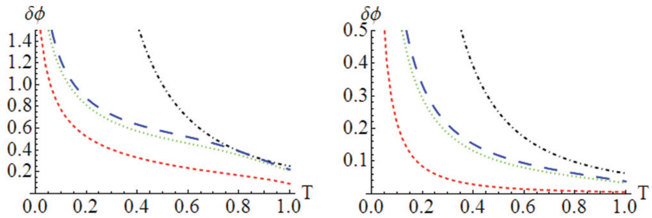
where λ_i and $|\lambda_i\rangle$ are the eigenvalues and eigenvectors of ρ_E^{out}, respectively.

After applying the BS transformations, the output state is written by $|\Psi\rangle \equiv \hat{U}_{\text{BS}_{ac}}\hat{U}_{\text{BS}_{bd}}|\Psi_E^{\text{out}}\rangle|0\rangle_c|0\rangle_d$. Here, the transmission rate parameter in the BSs characterizes the robustness of the phase estimation for the input state against the photon loss. Tracing over modes c and d, the mixed state can be written in four components

$$\rho_E^{\text{out}} = (\mathcal{N}\mathcal{N}_{\xi,k})^2 \sum_i^4 \rho_i, \tag{20}$$

where

$$\rho_1 = \sum_{p,p',m=0}^{\infty} \sqrt{\eta_{m+p,k}\eta_{m+p',k}} \frac{\xi^p \overline{\xi}^{p'} T^{\frac{p}{2}} T^{\frac{p'}{2}}}{\sqrt{p!p'!}} \frac{|\xi|^{2m}(1-T)^m}{m!} \times (|p0\rangle\langle p'0|),$$



(a) $\alpha = 1$ (b) $\alpha = 2$

FIG. 3. (Color online) The graphs show the phase-estimation precision $\delta\phi_{\min}$ for losses in both arms of the interferometer versus transmissivity T for four states ($\langle n\rangle_K = 4$) with different orders of nonlinearity, $\alpha = 1$ and $\alpha = 2$. The dash-dotted black and dotted green lines indicate the NOON state and ECSs, respectively. The long dashed blue line for the EBGCS and the dashed red line for the EPCS show the starting points of $\delta\phi_P^1 = 0.084$ and $\delta\phi_P^2 \approx 0.004$ at $T = 1$.

$$\rho_2 = \sum_{p=0}^{\infty}\sum_{p'=0}^{\infty} e^{ip^\alpha\phi}\sqrt{\eta_{p,k}\eta_{p',k}}\,\frac{\xi^p\bar{\xi}^{p'}T^{\frac{p}{2}}T^{\frac{p'}{2}}}{\sqrt{p!p'!}}(|0p\rangle\langle p'0|),$$

$$\rho_3 = \sum_{p=0}^{\infty}\sum_{p'=0}^{\infty} e^{-i(p')^\alpha\phi}\sqrt{\eta_{p,k}\eta_{p',k}}\,\frac{\xi^p\bar{\xi}^{p'}T^{\frac{p}{2}}T^{\frac{p'}{2}}}{\sqrt{p!p'!}}(|p0\rangle\langle 0p'|),$$

$$\rho_4 = \sum_{p,p',m=0}^{\infty}\frac{\sqrt{\eta_{m+p,k}\eta_{m+p',k}}}{e^{-i(p^\alpha-(p')^\alpha)\phi}}\,\frac{\xi^p\bar{\xi}^{p'}T^{\frac{p}{2}}T^{\frac{p'}{2}}(|\xi|^2(1-T))^m}{m!\sqrt{p!p'!}} \times (|0p\rangle\langle 0p'|), \qquad (21)$$

where the factor $\eta_{n,k}$ depends on the kind of the SU(1,1) CSs; $\eta_{n,k}$ equals $(2k)_n$ and $1/(2k)_n$ for EPCS and EBGCS, respectively. $\mathcal{N}$ is the normalization factor given by the hypergeometric function.

Using the eigenvalues and eigenvectors of the truncated density matrix ρ_E^{out}, we obtain numerically the optimal phase estimation of ρ_E^{out} for symmetric loss cases in a nonlinear phase operation, which is illustrated in Fig. 3. The results show that entangled SU(1,1) CSs clearly improve and still outperform the phase enhancement achieved by NOON and ECSs under conditions of loss, for essentially the whole range of the transmission rate. This effect wins out over the fact that the photon losses in both modes do not destroy the superposition effects and the coherent states maintain their nonclassical properties, which could be of significant utility under loss conditions. Due to the concavity of Fisher information, the engineering of optimal input states for different lossy rates has been investigated (so-called optimal states) [11], so entangled SU(1,1) CSs also offer an advantage over these states even for large photon number.

IV. CONCLUSIONS

We evaluated analytically and numerically the phase sensitivity of the two-mode state in the context of SU(1,1) Lie algebra using QFI and showed that the state can outperform the Heisenberg limit given by NOON states and ECSs possessing the same average particle number under perfect and lossy conditions. We used a general form of the nonlinearity in terms of the power of the number operator, by combining linear and nonlinear interferometers, and showed that entangled SU(1,1) CSs still outperform the phase enhancement achieved by NOON states and ECSs for different regions of loss and nonlinearity order even for a large number of photons. Although a final parity measurement would not saturate our derived phase uncertainty bound, such a realistic measurement approach could still demonstrate an advantage over NOON states and ECSs with current technology. These results show that the two-mode field has a potential for supersensitive phase estimation with phase sensitivity better than the Heisenberg limit and ECSs. Finally, we encourage the experimental scientists to perform this task, which may open new perspectives on quantum metrology for future research avenues.

ACKNOWLEDGMENTS

We acknowledge enlightening discussions with Fabio Benatti, Felipe Fanchini, and Adil Benmoussa. We thank the International Centre for Theoretical Physics (ICTP) for the kind hospitality.

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