

Roton confinement in trapped dipolar Bose-Einstein condensates

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Roton excitations constitute a key feature of dipolar gases, connecting these gases with superfluid helium. We show that the density dependence of the roton minimum results in a spatial roton confinement, particularly relevant in pancake dipolar condensates with large aspect ratios. We show that roton confinement plays a crucial role in the dynamics after roton instability, and that arresting the instability may create a trapped roton gas revealed by confined density modulations. We discuss the local susceptibility against density perturbations, which we illustrate for the case of vortices. Roton confinement is expected to play a key role in experiments.

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I. INTRODUCTION

Dipolar gases have attracted growing attention in recent years. Quantum degenerate gases of magnetic atoms as chromium [1], dysprosium [2], and erbium [3] have already been realized. Moreover, the preparation of heteronuclear molecules in their ro-vibrational ground state [4] opens the path for the creation of a degenerate gas of polar molecules, a goal currently pursued by various groups worldwide [5]. Rydberg atoms provide yet another possible realization of a highly polar gas [6].

The rich physics of dipolar gases arises from dipole-dipole interactions (DDIs) [7]. Dipolar Bose-Einstein condensates (dBECs) feature a geometry-dependent stability [8] and a peculiar dispersion of the elementary excitations. Nonpolar BECs present the usual Bogoliubov spectrum, with a linear (phonon) dispersion at low momenta and a quadratic dispersion at large momenta [9]. Under proper conditions, dBECs present a dispersion minimum at intermediate momenta [10,11] resembling the roton minimum of superfluid He [12].

Roton excitations are crucial in He, reducing the critical superfluid velocity [13] and leading to density modulations at defects [14–18]. Remarkably, similar effects have also been predicted in dBECs [19–22]. Moreover, the roton minimum is crucial for the stability of a dBEC. When the minimum reaches zero energy, the dBEC becomes unstable against finite-momentum excitations (roton instability) [10,11,23], which differs fundamentally from the usual phonon instability.

Roton properties in He may be controlled by means of pressure [24]. Similarly, the roton minimum in dBECs depends on interactions and therefore on density. In this paper we demonstrate that this density dependence leads to a spatial roton confinement in trapped dBECs. Roton confinement has been hinted in recent numerical calculations [21,25] and resembles that of rotons at vortex lines in He [26]. We show that roton confinement is crucial to understand the roton instability in pancake traps. We also discuss how arresting the instability may create a trapped roton gas. We finally analyze other consequences of the roton confinement, such as local susceptibility against density perturbations.

The paper is organized as follows. In Sec. II we present the system under consideration. In Sec. III we briefly review the main results of Ref. [10]. The key idea of local spectrum is introduced in Sec. IV. In Sec. V we obtain the wave functions of the confined rotonlike excitations. Section VI is devoted

to the key role played by roton confinement in the dynamics following roton instability, whereas Sec. VII analyzes the main features of the localized roton instability in time-of-flight pictures. In Sec. VIII we comment on the realization of a confined roton gas by means of a temporal destabilization of the condensate. In Sec. IX we discuss the local susceptibility associated with the idea of local spectrum, focusing on the particular case of vortex lattices. Finally we summarize our conclusions in Sec. X.

II. MODEL

We consider a dBEC of N bosons of mass m and (electric or magnetic) dipole moment d oriented along z . The dBEC is in a pancake harmonic trap $V_t(\mathbf{r})$ of frequencies ω in the xy plane and $\omega_z = \lambda\omega$ along z , with $\lambda \gg 1$. The dBEC wave function $\phi(\mathbf{r}, t)$ obeys the nonlocal Gross-Pitaevskii equation (GPE) [7],

$$i\hbar \frac{\partial}{\partial t} \phi(\mathbf{r}, t) = \left[-\frac{\hbar^2 \nabla^2}{2m} + V_t(\mathbf{r}) + g|\phi(\mathbf{r}, t)|^2 \right] \phi(\mathbf{r}, t) + \int d^3r' V_{dd}(\mathbf{r} - \mathbf{r}') |\phi(\mathbf{r}', t)|^2 \phi(\mathbf{r}, t), \quad (1)$$

where $g = 4\pi\hbar^2 a N/m$ characterizes the short-range interactions, a is the s -wave scattering length, $V_{dd}(\mathbf{r}) = \frac{Nd^2}{r^3}(1 - 3\cos^2\theta)$ is the DDI potential, θ is the angle between \mathbf{r} and the z axis, and $\int d^3r |\phi(\mathbf{r}, t)|^2 = 1$.

III. HOMOGENEOUS CASE

We first consider $\omega = 0$, briefly summarizing the results of Ref. [10] (for more details, see the Appendix). The ground-state wave function is $\phi_0(z) \exp(-i\mu t/\hbar)$, where μ is the chemical potential and $\phi_0(z)$ fulfills a one-dimensional (1D) local GPE with a regularized coupling constant $g + g_d$, with $g_d = 8\pi Nd^2/3$. Assuming a transverse Thomas-Fermi (TF) profile $\phi_0(z) = \sqrt{n_0(1 - z^2/L^2)}$, with n_0 the peak density, one obtains $\mu = (g + g_d)n_0$. Excitations of energy ϵ are evaluated by substituting $\phi(\boldsymbol{\rho}, z, t) = e^{-i\mu t/\hbar} [\phi_0(z) + u(\boldsymbol{\rho}, z)e^{-i\epsilon t/\hbar} - v(\boldsymbol{\rho}, z)^* e^{i\epsilon t/\hbar}]$ [with $\boldsymbol{\rho} \equiv (x, y)$] into Eq. (1), and keeping only linear terms in u and v . The excitations have a defined in-plane momentum \mathbf{q} , and hence $f_{\pm}(\boldsymbol{\rho}, z) \equiv [u(\boldsymbol{\rho}, z) \pm v(\boldsymbol{\rho}, z)] = f_{\pm}(z)e^{i\mathbf{q} \cdot \boldsymbol{\rho}}$. The lowest eigenenergy obtained from the Bogoliubov–de Gennes (BdG) equations for

each \mathbf{q} builds the dispersion $\epsilon(q)$. Interestingly, $\epsilon(q)$ may present a rotonlike minimum at intermediate q . Assuming $qL \gg 1$ and $\mu E(q)|\beta - 2|/(\beta + 1) \lesssim \hbar^2 \omega_z^2$ (with $\beta \equiv g_d/g$ and $E(q) \equiv \hbar^2 q^2/2m$) we obtain an approximate expression of the dispersion, $\epsilon_h^2(q, \mu) = E(q)^2 - G(\beta)E(q)\mu + \hbar^2 \omega_z^2$, with $G(\beta) \equiv \frac{(\beta-2)(5\beta+2)}{3(1+\beta)(2\beta+1)}$, as well as the associated eigenstate, $f_+(z) \simeq \phi_0(z)$. The expression $\epsilon_h(q, \mu)$ agrees well with the numerical $\epsilon(q)$ and shows that the roton depth depends explicitly on μ and hence on density.

IV. LOCAL SPECTRUM

Interesting insights about the roton physics for $\omega > 0$ are provided by the concept of *local spectrum* $\epsilon(q, \rho)$, which we introduce here. We compute from Eq. (1) the ground-state $n_0(\mathbf{r}) = |\phi_0(\mathbf{r})|^2$, which in the TF regime is approximated by $n_0(\mathbf{r}) = \tilde{n}_0(1 - \rho^2/R^2 - z^2/L^2)$ [27]. We obtain for each ρ the z profile $n_0^{\text{1D}}(z) = n_0(\mathbf{r})/\int dz n_0(\mathbf{r})$ and the *local* chemical potential $\mu_l(\rho)$. Solving the corresponding 1D BdG equations [10] we obtain $\epsilon_h(q, \mu_l(\rho))$, which approximates the local spectrum $\epsilon(q, \rho)$ consistently with the LDA. The local chemical potential $\mu_l(\rho)$ decreases quadratically with ρ , and hence at the trap center ($\rho = 0$) the roton energy is lowest: $\epsilon_r/\hbar\omega_z \equiv \epsilon(q = q_r, \rho = 0)/\hbar\omega_z \simeq \sqrt{1 - [G(\beta)\mu_l(0)/2\hbar\omega_z]^2}$. The local spectrum presents a minimum both in momentum, at $q_r l_z \simeq \sqrt{G(\beta)\mu_l(0)/\hbar\omega_z}$ (note that $q_r l_z \sim 1$, with $l_z \equiv \sqrt{\hbar/m\omega_z}$; see [10]), and in space, at $\rho = 0$ (Fig. 1). For larger ρ the minimum becomes shallower and eventually disappears. Hence, remarkably, the inhomogeneous BEC density results in a spatial *roton confinement*. Around the minimum,

$$\epsilon(q, \rho) \simeq \epsilon_r + \frac{\hbar^2(q - q_r)^2}{2m_*} + \frac{1}{2}m_*\omega_*^2\rho^2, \quad (2)$$

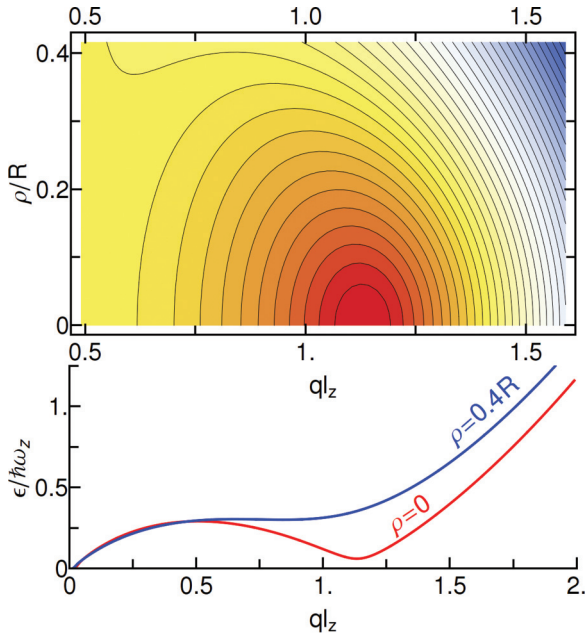


FIG. 1. (Color online) (Top) Local spectrum of a BEC of 2×10^5 Er atoms $\omega_z = 2\pi \times 1$ kHz and $\lambda = 40$. Note a minimum (dark red region in the top panel) in both space and momentum. (Bottom) Change of the roton minimum for two different radial positions.

where the effective roton mass $m_* \equiv m\sqrt{1/(q_r l_z)^4 - 1/4}$ and the effective harmonic frequency $\omega_* \equiv \omega_z \frac{m q_r l_z^2}{m_* R \sqrt{2}}$ define the roton localization length $l_* \equiv \sqrt{\hbar/m_*\omega_*} = 2^{1/4}(R/q_r)^{1/2}$. Note that $l_*/R \sim \sqrt{l_z/R}$, and hence $l_* \ll R$ for $\lambda \gg 1$. Moreover, $q_r l_* \sim \sqrt{R/l_z} \gg 1$ if $\lambda \gg 1$, which justifies the use of LDA above.

V. LOCALIZED ROTON WAVE FUNCTION

We now calculate the localized roton wave functions using the LDA and the formalism of Ref. [10] (for more details, see the Appendix). Assuming $l_* \ll R$, we approximate $f_+(\rho, z) \simeq F(\rho)\phi_0(\rho, z)$, where $F(\rho)$ has a narrow momentum distribution $\tilde{F}(\mathbf{q})$ centered around q_r with a width $\delta q \propto 1/l_* \ll q_r$. The finite width of $\tilde{F}(\mathbf{q})$ must be now considered and as a result $\tilde{F}(\mathbf{q})$ fulfills the eigenvalue equation $\epsilon^2 \tilde{F}(\mathbf{q}) \simeq [\epsilon_r + \hat{H}]^2 \tilde{F}(\mathbf{q})$, with $\hat{H} \equiv \frac{\hbar^2}{2m_*}(q - q_r)^2 - \frac{1}{2}m_*\omega_*^2\nabla_{\mathbf{q}}^2$. Interestingly, \hat{H} resembles the Hamiltonian of a trapped BEC in the presence of spin-orbit coupling [28,29], where the Rashba-like dispersion $\sim (q - q_r)^2$ acts as a ringlike potential in \mathbf{q} space. The eigenfunctions of \hat{H} , $\eta_{n,s}(q)e^{is\varphi}/\sqrt{q}$, fulfill

$$\left[\frac{E_{n,s}}{\hbar\omega_*} - \frac{s^2 - \frac{1}{4}}{2(q l_*)^2} \right] \eta_{n,s} = \left[-\frac{1}{2l_*^2} \frac{d^2}{dq^2} + \frac{l_*^2}{2}(q - q_r)^2 \right] \eta_{n,s}. \quad (3)$$

For $q_r l_* \gg 1$, we expand around $q \simeq q_r$, obtaining the eigenenergies $E_{n,s}/\hbar\omega_* \simeq (s^2 - 1/4)/[2(q_r l_*)^2] + n + 1/2$, characterized by the angular momentum s around the Rashba-like ring, and the radial harmonic excitations, n , with frequency ω_* . The lowest roton states have $n = 0$, being in real space of the form $\psi_s(\rho) \sim e^{is\varphi} e^{-\rho^2/2l_*^2} J_s(q_r \rho)$, with J_s the Bessel function.

VI. LOCAL ROTON INSTABILITY

The localized states ψ_s are crucial for the dynamics following roton instability [30]. We consider a stable BEC prepared with an initial scattering length $a_i > a_c$. The critical a_c for the onset of instability depends nontrivially on λ and g_d [7]; for the parameters of Fig. 2 it is $a_c = 8.48 a_0$, with a_0 the Bohr radius. We are interested in the instability dynamics after a sudden quench to $a_f < a_c$. Unstable modes lead to a modulational instability characterized by a growing density perturbation $\delta n(\mathbf{r}, t) \equiv n(\mathbf{r}, t) - n_0(\mathbf{r}) \propto \sqrt{n_0(\mathbf{r})} \text{Re}(f_-(\mathbf{r}))$. As for the $\omega = 0$ case, $f_- \simeq [E(q_r)/\epsilon] f_+ \propto \sqrt{n_0(\mathbf{r})} F(\rho)$. Hence, the most unstable roton modes result in a localized modulation at the trap center, $\delta n(\mathbf{r})/n_0(\mathbf{r}) \propto \text{Re}(\psi_s(\rho))$.

The most unstable mode is $\psi_0(\rho)$, but other modes may contribute to the instability due to the small energy difference between low- s levels $[\sim \hbar\omega_*/(q_r l_*)^2]$. As a nontrivial consequence of that, the density pattern that develops after the quench is influenced by the initial population of the excitations exponentially amplified during the destabilization. We mimic this dependence by considering a small initial seeding $\phi(\mathbf{r}, t = 0) = \phi_0(\mathbf{r})e^{i\chi(\mathbf{r})}$, where ϕ_0 is the ground state calculated for a_i , and $|\chi(\mathbf{r})|/\pi < \xi$ is a random phase sampled from a homogeneous uniform distribution with a variable amplitude ξ . Although this allows us to discuss the

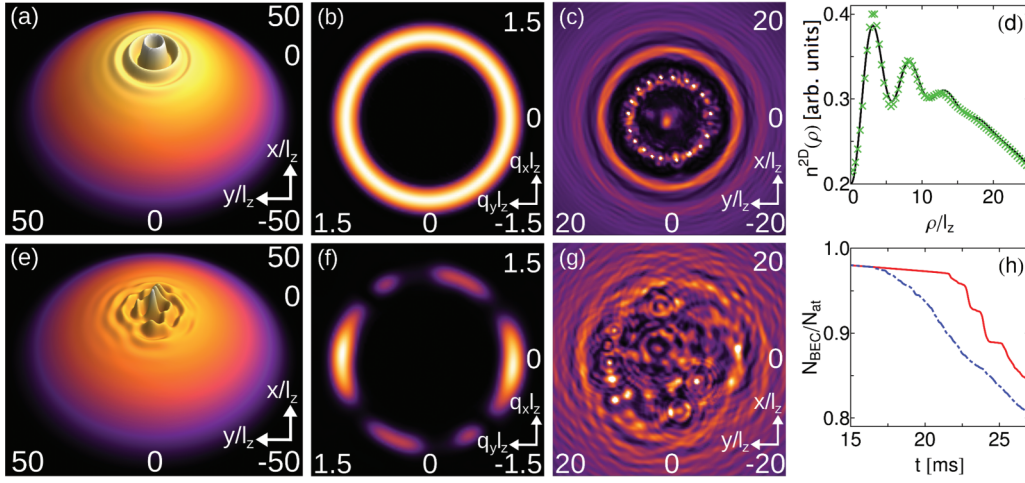


FIG. 2. (Color online) Roton instability for 10^5 Er atoms, with $\omega_z = 2\pi \times 450$ Hz ($l_z \simeq 0.3 \mu\text{m}$), $\lambda = 30$, $a_i = 8.49a_0 > a_c = 8.48a_0$, and $a_f = 0$ (see text). (a) Column density $n^{2D}(\rho) = \int n(\mathbf{r})dz$ showing concentric rings ($s = 0$) formed after $t = 19$ ms for small initial fluctuations ($\xi = 10^{-10}$; see text). (e) Modulational instability after $t = 15.5$ ms consisting of several s states for large initial fluctuations ($\xi = 10^{-6}$, see text). (b) and (f) Momentum distribution of (a) and (e), respectively (we have suppressed the large peak at $q = \sqrt{q_x^2 + q_y^2} = 0$). (c) and (g) Postcollapse dynamics after $t = 23$ ms for (a) and $t = 19.5$ ms for (e), respectively. (d) Radial cut of (a) (green crosses) and theoretical column density of the form $n^{2D}(\rho) \propto (1 - \rho^2/R^2)^{3/2}(1 + AJ_0(q_r, \rho)e^{-\rho^2/2l_z^2})$ (solid black line), with $R \simeq 51 l_z$, $A \simeq -0.4$, $q_r l_z \simeq 1.25$, and $l_* = 2^{1/4}(R/q_r)^{1/2} \simeq 7.6 l_z$. The value of q_r is calculated using the local spectrum picture. (h) Remnant atoms for the cases (a) (solid red line) and (e) (dashed blue line).

possible collapse scenarios, the actual amplitude of the initial fluctuations depends on a_i and on the temperature T , and its analysis lies beyond the scope of this paper [31].

Figures 2 show the results of our simulations of Eq. (1) for an erbium dBEC [32]. For a small initial population of the unstable modes, the modulation instability proceeds at a sufficiently slow pace and $\psi_0(\rho)$ dominates. As a result, a localized pattern of concentric rings develops [Fig. 2(a)], in excellent agreement with $\delta n(\mathbf{r})/n_0(\mathbf{r}) \propto \psi_0(\rho)$; see Fig. 2(d) [33]. The corresponding momentum distribution presents a Rashba-like ring [Fig. 2(b)] [34]. In contrast, for larger initial fluctuations the pattern growth is too fast to select ψ_0 only, and the created density pattern results from a (shot-to-shot dependent) linear combination of different ψ_s , being characterized by a superposition of eccentric collapse centers [Fig. 2(e)]. The corresponding momentum distribution still presents a ringlike structure but with an azimuthal modulation arising from the linear combination of various ψ_s [Fig. 2(f)].

The global (phononlike) collapse studied in chromium and erbium dBECs [3,35] results in large atom losses and in a d -wave pattern in time-of-flight (TOF) pictures. Remarkably, the roton instability discussed above leads to a very different collapse dynamics. Three-body losses become crucial in the collapse dynamics and we included them by adding $-i\hbar \frac{L_3}{2} N^2 |\phi(\mathbf{r}, t)|^4 \phi(\mathbf{r}, t)$ [35] to Eq. (1), with a loss rate $L_3 = 10^{-28} \text{ cm}^{-6} \text{ s}^{-1}$. The concentric rings appearing in Fig. 2(a) eventually undergo a sequential collapse and azimuthal instability, starting from the inner (denser) ones towards the outer ones [Fig. 2(c)] leading to a steplike atom number decrease [Fig. 2(h)]. The superimposed eccentric collapse centers appearing in Fig. 2(e) lead to a complex postcollapse behavior with characteristic mutually interfering jets expelled out of each local collapse center [Fig. 2(g)]. In this case the atom decrease is smooth [Fig. 2(h)].

VII. TIME-OF-FLIGHT PICTURES

Although the most straightforward way of studying roton confinement is, of course, given by *in situ* measurements [36], TOF pictures are expected to show a clear difference as well when compared to the cloverleaf pattern characteristic of the phonon collapse (as observed in chromium [35] and, more recently, in erbium [3]). More specifically, TOF pictures may reveal clear traces of the Rashba-like ring characterizing the localized rotonlike excitations.

Due to the fast expansion after release, TOF pictures are expected to reproduce well the *in situ* momentum distribution at the time of releasing. We have calculated the momentum distribution of the condensate at different stages during the instability dynamics. Our results for the same case studied in Fig. 2(a) are depicted in Fig. 3, where we show both the column density on the q_x - q_y plane (i.e., integrated along q_z) and that on the q_x - q_z plane (i.e., integrated along q_y). We show in particular the results obtained during the first stages of the collapse dynamics. For the parameters of Fig. 2(a), the Rashba-like ring is clearly visible during the first stages of the collapse, being destroyed later on due to the subsequent collapse of the spatial ringlike density modulations. As a result the ring TOF pattern is washed out. In later stages TOF pictures are characterized by the appearance of large momentum excitations on the xy plane (induced by the collapse of the local rings) which becomes visible in the q_x - q_z distribution as pronounced jets along x .

The low-momentum peak always appears saturated in our pictures. It was eliminated from Figs. 2(b) and 2(g), since we were only interested in showing the Rashba-like feature in momentum space. The large low-momentum peak is the result of two effects. The first one is purely geometric. The $q = 0$ peak is always relatively strong, since it is not spread in the angular variable φ (as it is the ring feature discussed above). The second reason is more physical, and it is linked with the

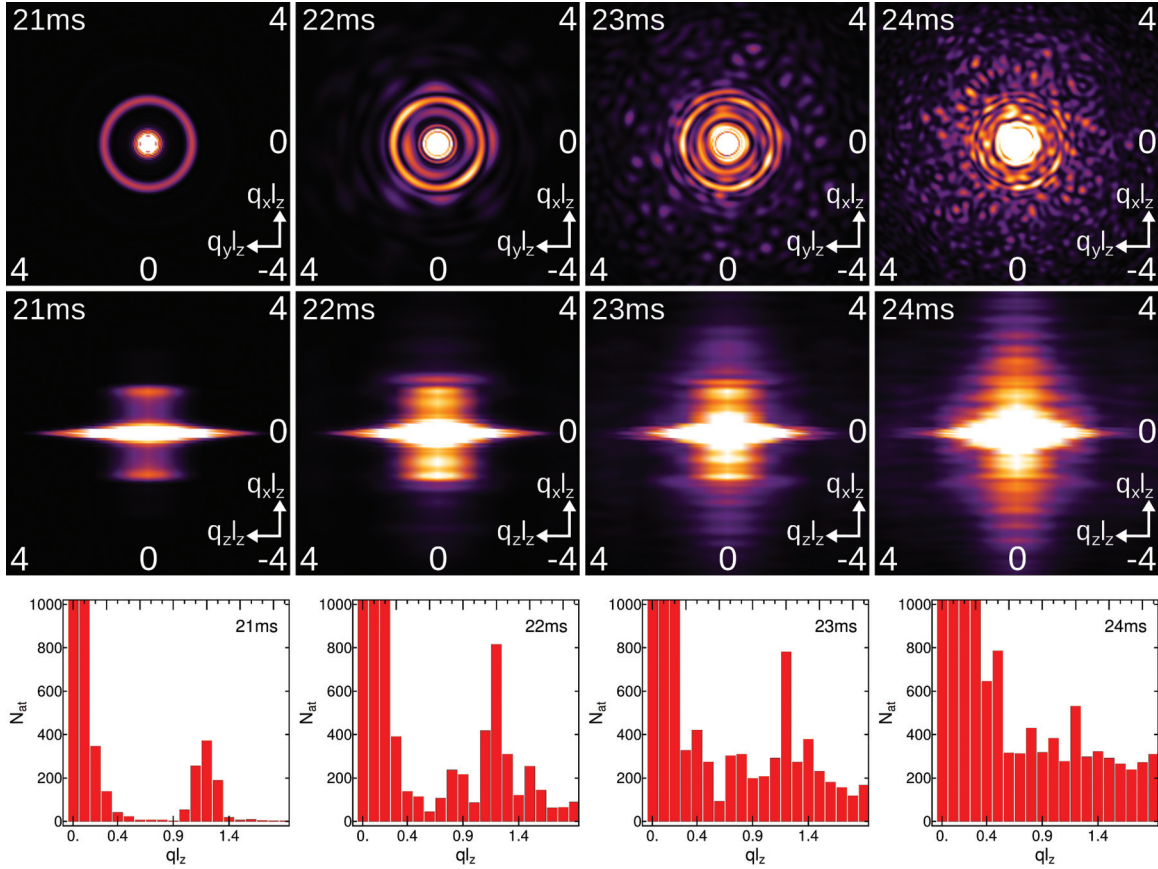


FIG. 3. (Color online) Momentum distribution for different times for the same case depicted in Fig. 2(a), as it would be revealed by a TOF experiment. In the top panel we show the q_x, q_y momentum distribution (integrated over q_z) and in the middle panel the q_x, q_z momentum distribution (integrated over q_y). The bottom panel shows the integrated population at different intervals of the radial momentum $q = \sqrt{q_x^2 + q_y^2}$.

local nature of the collapse. Since only a small fraction of the atoms is actually participating in the modulational instability and in the subsequent collapse, a large fraction of atoms remains at low momentum. Note that the low-momentum peak also broadens due to the generation of low-energy phonons during the collapse. As a result, the low-momentum peak dominates the TOF pictures. This is in itself a clear difference with respect to the phonon collapse, where the initial low-momentum peak transforms into a cloverleaf pattern due to the global nature of the collapse.

However, although the relative number of particles produced in the Rashba-like ring may be small, the actual absolute number of particles may be certainly sizable. In Fig. 3 (bottom panel) we depict at various times the number of particles at different radial momenta. This plot shows that a significant number of particles is found in the ring before the ring feature is eventually washed out (up to 1500 particles at 21.5 ms).

VIII. CONFINED ROTON GAS

Interestingly, the roton instability may be used to create a confined roton gas. The density pattern obtained after quenching down to $a_f < a_c$ may be interpreted as the growth of the roton population. At the initial stage of the instability, the roton population is small compared to N and we may neglect condensate depletion or roton-roton interactions. Once

the roton gas is populated, we stabilize it by quenching up to $a > a_c$. Since m_* and ω_* do not vary significantly around the instability threshold, the created rotons (density patterns) remain confined at the trap center after the quench, as shown in Fig. 4.

IX. LOCAL SUSCEPTIBILITY

A deep roton minimum induces close to a perturbation a large susceptibility against the formation of density modulations with the roton wavelength. This well-known effect in He [14–18] is also relevant for dBECs [20]. The local spectrum

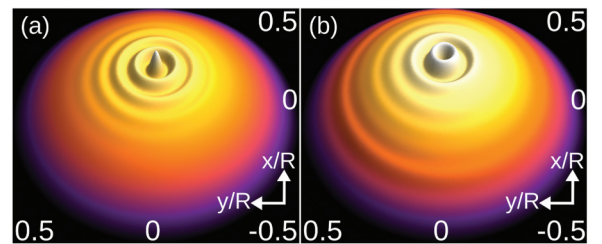


FIG. 4. (Color online) Same parameters as in Fig. 2(a) but arresting the collapse after 16 ms with a quench of a up to $8.55 a_0 > a_c$. Snapshots of the central region right after the quench (a) and 80 ms later (b). Note a clear density modulation confined at the trap center.

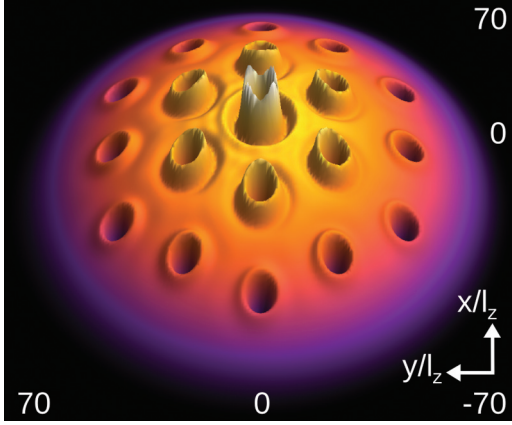


FIG. 5. (Color online) Vortex lattice for a BEC of $N = 10^5$ Er atoms at the threshold of the roton instability. Parameters as in Fig. 1 with a rotational frequency 0.3ω .

picture implies a spatially dependent susceptibility, which we illustrate for the case of vortices [37]. Vortex cores present a craterlike shape in the case of a deep roton minimum [19,20] that disappears when the minimum is shallow or absent. Therefore, vortices at different positions in a trapped BEC present a different core profile. This is shown for a vortex lattice in Fig. 5, where we depict the ground state of an erbium BEC rotating around the z axis with an angular frequency $\Omega = 0.3\omega$. Note that core modulations at the trap center disappear for vortices at the boundary.

X. CONCLUSIONS

In summary, we have shown that an inhomogeneous trapping in pancake dBECs with large aspect ratios leads to a spatial roton confinement which is crucial to understanding the roton instability. The roton dispersion has not yet been observed experimentally, being currently a major goal pursued by several groups. Roton confinement is expected to play a key role in these experiments, since harmonic traps are typically employed and large aspect ratios are needed to study the roton dispersion. In addition to the local susceptibility discussed above [37], roton confinement should be carefully considered when measuring the critical superfluid velocity, performing Bragg scattering [38], or analyzing finite temperature physics, which may be very interesting since the thermal roton cloud is expected to localize at the center of the trap.

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APPENDIX: DERIVATION OF THE LOCALIZED ROTON WAVE FUNCTIONS

1. Homogeneous xy case ($\omega = 0$)

We briefly summarize in this section the main formalism developed in Ref. [10]. For $\omega = 0$, the ground state is of the

form $\phi_0(z)e^{-i\mu t/\hbar}$, where μ is the chemical potential and

$$\left\{ -\frac{\hbar^2}{2m}\nabla^2 + \frac{m}{2}\omega_z^2 z^2 + (g + g_d)|\phi_0(z)|^2 - \mu \right\} \phi_0(z) = 0. \quad (\text{A1})$$

We assume $(g + g_d) > 0$. For $\mu \gg \hbar\omega_z$ the condensate presents a TF density profile, $|\phi_0(z)|^2 = n_0(1 - z^2/L^2)$, with $n_0 = \mu/(g + g_d)$ being the maximum density, and $L = \sqrt{2\mu/m\omega_z^2}$ the TF radius.

Excitations of energy ϵ are evaluated by substituting $\phi(\mathbf{r}, z, t) = e^{-i\mu t/\hbar}[\phi_0(z) + u(\mathbf{r}, z)e^{-i\epsilon t/\hbar} - v(\mathbf{r}, z)^*e^{i\epsilon t/\hbar}]$ [with $\mathbf{r} \equiv (x, y)$] into the 3D nonlocal GPE [Eq. (1) of our Letter], and keeping only linear terms in the Bogoliubov amplitudes u and v . For $\omega = 0$, the excitations have a defined in-plane momentum \mathbf{q} , and we may write $f_{\pm}(\mathbf{r}, z) \equiv [u(\mathbf{r}, z) \pm v(\mathbf{r}, z)] = f_{\pm}(z)e^{i\mathbf{q}\cdot\mathbf{r}}$. The BdG equations read

$$\epsilon f_- = \frac{\hbar^2}{2m} \left[-\frac{d^2}{dz^2} + q^2 + \frac{\nabla^2 \phi_0}{\phi_0} \right] f_+ \equiv H_{\text{kin}} f_+, \quad (\text{A2})$$

$$\epsilon f_+ = H_{\text{kin}} f_- + H_{\text{int}}[f_-], \quad (\text{A3})$$

with

$$H_{\text{int}}[f_-] = 2(g_d + g)f_-(z)|\phi_0(z)|^2 - (3/2)g_d q \phi_0(z) \times \int_{-\infty}^{\infty} dz' f_-(z')\phi_0(z')\exp(-q|z - z'|). \quad (\text{A4})$$

For each q we calculate the eigenenergies. The lowest one, $\epsilon(q)$, builds the dispersion law.

The most interesting behavior occurs for $qL \gg 1$. We may now introduce $f_{\pm}(z) = W(z)\phi_0(z)$. Expressing f_- through W from Eq. (A2), we substitute it into Eq. (A3) and integrate over dz' in $H_{\text{int}}[f_-]$ as the main contribution to the integral comes from a narrow range of distances $|z' - z| \sim 1/q$. This yields (here $\chi = z/L$)

$$\begin{aligned} & \hbar^2\omega^2 \left[\frac{1}{2}(1 - \chi^2)\frac{d^2}{d\chi^2} - \left(1 + \frac{3}{2(1 + \beta)}\right)\chi\frac{d}{d\chi} \right] W(\chi) \\ & = - \left[\epsilon^2 - E(q)^2 - \frac{2\beta - 1}{1 + \beta}\mu E(q)(1 - \chi^2) - \frac{3\hbar^2\omega^2}{2(1 + \beta)} \right] \\ & \times W(\chi) = 0, \end{aligned} \quad (\text{A5})$$

where $E(q) = \hbar^2 q^2/2m$. Here we omitted terms of the order of $E(q)\hbar^2\omega^2/\mu$ and $\hbar^4\omega^4/\mu^2$, since they are small compared to either $\hbar^2\omega^2$ or $E(q)^2$. For each mode of the confined motion (along the z direction), the solution of Eq. (A5) can be written as an expansion series in Gegenbauer polynomials $C_n^\lambda(\chi)$, where $\lambda = (4 + \beta)/2(1 + \beta)$, and $n \geq 0$ is an integer. For $\mu E(q)|2\beta - 1|/(1 + \beta) \lesssim \hbar^2\omega^2$ the lowest solution is of the form $W(\chi) \simeq 1 + \sum_{n>0} a_n C_n^\lambda(\chi)$, with small amplitudes a_n of Gegenbauer polynomials of higher order, while the lowest eigenenergy (and hence the dispersion) is given by

$$\epsilon_h^2(q, \mu) = E(q)^2 - G(\beta)E(q)\mu + \hbar^2\omega_z^2, \quad (\text{A6})$$

$$\text{with } G(\beta) \equiv \frac{(\beta - 2)(5\beta + 2)}{3(1 + \beta)(2\beta + 1)}.$$

2. Trapped xy case

We consider in the following $\omega > 0$, with $\lambda = \omega_z/\omega \gg 1$. Due to the xy confinement, we cannot employ anymore a plane wave basis for the Bogoliubov amplitudes, as we

did above. However, we assume that (i) the Bogoliubov amplitudes are, in momentum space, strongly peaked around a momentum $q \gg 1/R$ and (ii) the Bogoliubov amplitudes are localized at the center of the trap, in a region with a radius much smaller than R . These assumptions must be checked self-consistently, but we anticipate that they are fulfilled for a sufficiently large aspect ratio $\lambda \gg 1$. If this is the case, we may proceed as for the homogeneous case, but taking into account that (a) the chemical potential is now ρ dependent, and (b) the Bogoliubov amplitudes are not plane waves anymore. Assuming the above conditions are met, we may write $f_+(\rho, z) \simeq F(\rho)W(x)\phi_0(\rho, z)$ (now $x^2 = \rho^2/R^2 + z^2/L^2$), where the function $F(\rho)$ has a narrow momentum distribution $F(\mathbf{q})$ peaked at q_r with a momentum width $\delta q \propto 1/l_* \ll q_r$ and $\phi_0(\rho, z)$ is the ground state. Taking into account (a) and (b), we rewrite Eq. (A5) in the form

$$\begin{aligned} \hbar^2 \omega^2 \left[\frac{1}{2}(1-x^2) \frac{d^2}{dx^2} - \left(1 + \frac{3}{2(1+\beta)} \right) x \frac{d}{dx} \right] W(x)F(\rho) \\ = - \left[\epsilon^2 - E(\hat{\mathbf{q}})^2 - \frac{2\beta-1}{1+\beta} \mu E(\hat{\mathbf{q}})(1-x^2) - \frac{3\hbar^2 \omega^2}{2(1+\beta)} \right] \\ \times W(x)F(\rho) = 0, \end{aligned} \quad (\text{A7})$$

where $\hat{\mathbf{q}} \equiv -i\nabla$. The lowest energy eigenstates still fulfill $W(x) \simeq 1$, but now we must keep explicitly the spatial

dependence of $F(\rho)$,

$$\epsilon^2 F(\rho) = [E(\hat{\mathbf{q}})^2 - G(\beta)E(\hat{\mathbf{q}})\mu(\rho) + \hbar^2 \omega_z^2] F(\rho). \quad (\text{A8})$$

Expanding around the roton minimum in the local spectrum, and moving to momentum space ($\rho = i\nabla_q$) we obtain

$$\epsilon^2 \tilde{F}(\mathbf{q}) = \left[\epsilon_r + \frac{\hbar^2(q-q_r)^2}{2m_*} - \frac{1}{2}m_*\omega_*^2 \nabla_q^2 \right]^2 \tilde{F}(\mathbf{q}), \quad (\text{A9})$$

where $\tilde{F}(\mathbf{q})$ is the Fourier transform of $F(\rho)$. Hence, $\tilde{F}(\mathbf{q})$ are the eigenstates of the Hamiltonian $\hat{H} \equiv \frac{\hbar^2(q-q_r)^2}{2m_*} - \frac{1}{2}m_*\omega_*^2 \nabla_q^2$, which may be calculated as discussed in Sec. V.

3. Density modulations

Note that density modulations are given by the f_- amplitudes in the form $\delta n(\mathbf{r}, \mathbf{t}) \equiv n(\mathbf{r}, \mathbf{t}) - n_0(\mathbf{r}) \propto \sqrt{n_0(\mathbf{r})} \text{Re}(f_-(\mathbf{r}))$, where Re denotes the real part. For the $\omega = 0$ case, and since $W(\chi) \simeq 1$, from Eq. (A2) one obtains that $f_- \simeq (E(q)/\epsilon)f_+$. For the $\omega > 0$ case, since $W(x) \simeq 1$ still, and we assume that $\tilde{F}(\mathbf{q})$ is narrowly peaked around q_r , we obtain the same result for the trapped case. Hence, apart from a constant, $f_-(\mathbf{r}) \propto f_+(\mathbf{r}) \propto \psi_0(\mathbf{r})F(\rho)$. As a result, $\delta n(\mathbf{r}, \mathbf{t}) \propto n_0(\mathbf{r}) \text{Re}(F(\rho))$.

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