Non-Markovian quantum jumps from measurements in bipartite Markovian dynamics

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(Received 29 May 2013; published 25 July 2013)

The quantum jump approach allows to characterize the stochastic dynamics associated with an open quantum system submitted to a continuous measurement action. In this paper we show that this formalism can consistently be extended to non-Markovian system dynamics. The results rely on studying a measurement process performed on a bipartite arrangement characterized by a Markovian Lindblad evolution. Both renewal and nonrenewal extensions are found. The general structure of nonlocal master equations that admit an unraveling in terms of the corresponding non-Markovian trajectories is also found. By studying a two-level system dynamics, it is demonstrated that non-Markovian effects such as an environment-to-system flow of information may be present in the ensemble dynamics.

DOI: 10.1103/PhysRevA.88.012124 PACS number(s): 03.65.Yz, 42.50.Lc, 03.65.Ta, 02.50.Ga

I. INTRODUCTION

One of the central achievements of the theory of open quantum systems is the possibility of assigning to a given master equation an ensemble of stochastic realizations. They can be put in one-to-one correspondence with a well-defined continuous-in-time measurement process performed over the system of interest. When the measurement apparatus is able to detect transitions between the system's levels [1–5], the realizations consist in a sequence of disruptive instantaneous changes, associated with the measurement recording events, while in the intermediate time regime the ensemble dynamics is smooth, being defined by a nonunitary dynamics. These basic ingredients, which define the quantum jump approach (QJA) [6–8], are well understood for Markovian dynamics, that is, those where the evolution of the system density matrix is local in time.

In the last ten years, an ever-increasing interest has been paid to the development of a consistent non-Markovian generalization of the standard (Markovian) open quantum system theory [8]. In the generalized scheme the system density-matrix evolution is characterized by (time-convoluted) memory contributions [9–21]. Both a theoretical interest as well as a wide range of physical applications motivate this line of research.

Relevant achievements in the study of non-Markovian master equations were formulated on the basis of stochastic phenomenological approaches [10–13] and related concepts [14–21]. On the other hand, much less progress has been achieved in the formulation of stochastic processes that can be read as the result of a continuous measurement action performed over a system characterized by a nonlocal-in-time (non-Markovian) evolution. In fact, while there exist different stochastic dynamics that on average recover a non-Markovian density matrix evolution, its reading in terms of a continuous measurement process is problematic. Remarkable examples are the non-Markovian quantum state diffusion model [22] and the unraveling of local-in-time master equations characterized by negative transition rates [23]. The realizations associated with these approaches can only be read in the context of hiddenlike-variable models [22,23].

The main goal of this paper is to demonstrate that it is possible to formulate a consistent generalization of the QJA such that on average the ensemble of measurement realizations recovers a nonlocal non-Markovian density matrix evolution. The basic idea of our analysis is to study the QJA in a bipartite Markovian arrangement. Then, we search for the conditions (interaction symmetries) that allows to formulate a closed stochastic dynamic for the system of interest. The coupling with the second or auxiliary system introduces the memory effects. In contrast with previous approaches [22,23], the reading of the stochastic realizations in terms of a continuous-in-time measurement process is guaranteed by construction.

We show that a renewal non-Markovian measurement process can be obtained from the bipartite dynamics. Renewal means that the interval statistics between successive events is always the same, being defined by a probability distribution called the waiting-time distribution [3]. A nonrenewal dynamics is also defined. As in the standard Markovian formalism, the occurrence of each case depends on the properties of the resetting state [5] associated with each measurement event. The structure of the corresponding non-Markovian master equations is also found.

We remark that there exist previous studies where the QJA is formulated for a system that interacts with extra unobserved "classical" degrees of freedom [24–26]. While our approach relies on a similar underlying dynamic (strictly, here no classicality condition is imposed), we demonstrate that over a similar basis it is possible to get a consistent non-Markovian generalization of the QJA. In fact, in contrast with previous contributions [24–26], we focus the analysis on the possibility of establishing a closed stochastic system dynamics, that is, without involving "explicitly" the degrees of freedom of the auxiliary system.

The paper is outlined as follows. In Sec. II, in order to introduce the notation as well as basic results on which our analysis relies, we provide a summary of the standard Markovian QJA. In Sec. III we demonstrate that the basic structure of the standard QJA can be embedded in a bipartite Markovian dynamics, providing in this way the theoretical background for its non-Markovian generalization. Possible

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(bipartite) interactions that lead to a closed system dynamics are found. The non-Markovian density matrix evolution is determined for both renewal and nonrenewal measurement processes. In Sec. IV we study a particular example that explicitly shows the consistency of the present proposal. Furthermore, it demonstrate that non-Markovian features such as an environment-to-system flow of information [27] may be present in the ensemble dynamics. The conclusions are presented in Sec. V. In Appendix A we provide a derivation of the statistics of the measurement events in the standard case. In Appendix B we work out an alternative derivation of the non-Markovian system density matrix evolution based on the measurement statistics.

II. MARKOVIAN QUANTUM JUMPS

The standard QJA allows defining the (stochastic) dynamics of an open quantum system when it is subjected to a measurement process. The basic ingredients of the formalism are the system density matrix evolution, the definition of the apparatus measurement action, the conditional dynamics between detection events, and their statistical characterization. Below, we review these elements.

We write the evolution of the system density matrix ρ_t^s as

$$\frac{d}{dt}\rho_t^s = \left(\hat{\mathcal{L}}_0 + \sum_\alpha \gamma_\alpha \hat{\mathcal{C}}[V_\alpha]\right)\rho_t^s, \tag{1}$$

where $\hat{\mathcal{L}}_0$ is an arbitrary superoperator that may include Hamiltonian as well as dissipative (Lindblad) superoperators [8]. From now on the caret denotes a superoperator. The second contribution in (1) is defined by an addition of Lindblad channels

$$\hat{\mathcal{C}}[V]\rho = V\rho V^\dagger - \tfrac{1}{2}\{V^\dagger V,\rho\}_+, \tag{2}$$

each one characterized by the operator V_α and the transition rate γ_α. With $\{\cdot,\cdot\}_+$ we denote an anticommutation operation.

We assume that the system is monitored by only one measurement apparatus, which is sensible to all Lindblad transitions channels $\hat{\mathcal{C}}[V_\alpha]$. Hence, the master equation (1) is rewritten as

$$\frac{d}{dt}\rho_t^s = (\hat{\mathcal{D}} + \hat{\mathcal{J}})\rho_t^s. \tag{3}$$

The superoperator $\hat{\mathcal{J}}$ reads

$$\hat{\mathcal{J}}\rho = \sum_\alpha \gamma_\alpha V_\alpha \rho V_\alpha^\dagger. \tag{4}$$

It defines the system transformation after a measurement event. In fact, when a recording event happens, consistently with quantum measurement theory [8], the system density matrix suffers the disruptive transformation $\rho \to \hat{\mathcal{M}}\rho$ (jump or state collapse),

$$\hat{\mathcal{M}}\rho = \frac{\hat{\mathcal{J}}\rho}{\mathrm{Tr}_s[\hat{\mathcal{J}}\rho]} = \frac{\sum_\alpha \gamma_\alpha V_\alpha \rho V_\alpha^\dagger}{\left\{\sum_\alpha \gamma_\alpha \mathrm{Tr}_s[V_\alpha^\dagger V_\alpha \rho]\right\}}, \tag{5}$$

where $\mathrm{Tr}_s[\cdots]$ denotes a trace operation. On the other hand, in Eq. (3) the superoperator $\hat{\mathcal{D}}$ is defined as

$$\hat{\mathcal{D}}\rho = \hat{\mathcal{L}}_0\rho - \frac{1}{2}\sum_\alpha \gamma_\alpha \{V_\alpha^\dagger V_\alpha,\rho\}_+. \tag{6}$$

In the QJA, this superoperator defines the system dynamics between detection events. In fact, given that in the interval (τ,t) no detection event happens, the system dynamics is defined by the (conditional) normalized propagator

$$\hat{\mathcal{T}}_c(t-\tau)\rho = \frac{\hat{\mathcal{T}}(t-\tau)\rho}{\mathrm{Tr}_s[\hat{\mathcal{T}}(t-\tau)\rho]}. \tag{7}$$

The superoperator $\hat{\mathcal{D}}$ generates the dynamics of the unnormalized propagator $\hat{\mathcal{T}}(t-\tau)$, which reads

$$\hat{\mathcal{T}}(t-\tau)\rho = \exp[(t-\tau)\hat{\mathcal{D}}]\rho. \tag{8}$$

In this way, the trajectories associated with the measurement process are a piecewise deterministic process [8] which combine a deterministic time evolution [Eq. (7)] with jump processes [Eq. (5)].

The propagator $\hat{\mathcal{T}}(t)$ completely defines the statistics of the measurement process. In fact, it allows to calculate the survival probability between measurement events. Given that at time τ the state of the system is ρ_τ, the probability $P_0(t-\tau|\rho_\tau)$ of no detection event happening in the interval (τ,t) is

$$P_0(t-\tau|\rho_\tau) = \mathrm{Tr}_s[\hat{\mathcal{T}}(t-\tau)\rho_\tau]. \tag{9}$$

The probability distribution $w(t-\tau|\rho_\tau)$ of the interval $(t-\tau)$ follows as $w(t-\tau|\rho_\tau) = -(d/dt)P_0(t-\tau|\rho_\tau)$, delivering

$$w(t-\tau|\rho_\tau) = -\mathrm{Tr}_s[\hat{\mathcal{D}}\hat{\mathcal{T}}(t-\tau)\rho_\tau]. \tag{10}$$

By using that $(d/dt)\mathrm{Tr}_s[\rho_t^s]=0$, Eq. (3) implies that $-\mathrm{Tr}_s[\hat{\mathcal{D}}\cdot] = \mathrm{Tr}_s[\hat{\mathcal{J}}\cdot]$, leading to the equivalent expression $w(t-\tau|\rho_\tau) = \mathrm{Tr}_s[\hat{\mathcal{J}}\hat{\mathcal{T}}(t-\tau)\rho_\tau]$. On the other hand, notice that $P_0(t-\tau|\rho_\tau)$, or equivalently $w(t-\tau|\rho_\tau)$, depends explicitly on the state ρ_τ.

From the previous statistical objects it is possible to define the "conditional distribution" [3]

$$w_c(t-\tau|\rho_\tau) = \frac{w(t-\tau|\rho_\tau)}{P_0(t-\tau|\rho_\tau)}. \tag{11}$$

It defines the probability density for recording a detection event at time t, given that no counts are recorded in the interval (τ,t), and given that the last one was recorded at time τ. Therefore, $w_c(t-\tau|\rho_\tau)$ gives the probability density for a jump at time t given that we know that no event occurred up to the present time since the last one [3]. Trivially, from Eqs. (9) and (10) it can be written as

$$w_c(t-\tau|\rho_\tau) = \frac{-\mathrm{Tr}_s[\hat{\mathcal{D}}\hat{\mathcal{T}}(t-\tau)\rho_\tau]}{\mathrm{Tr}_s[\hat{\mathcal{T}}(t-\tau)\rho_\tau]}. \tag{12}$$

A. Stochastic dynamics

With the previous elements, it is possible to define the dynamics of a stochastic density matrix $\rho_s^{\mathrm{st}}(t)$ such that its average over realizations, denoted by an overbar, recovers the system state

$$\rho_t^s = \overline{\rho_s^{\mathrm{st}}(t)}. \tag{13}$$

Each realization corresponds to a given recording realization of the measurement apparatus. Its structure can be established by studying the counting statistics of the measurement process (see Appendix A).

Given the initial state ρ_0^s, we can evaluate $P_0(t-0|\rho_0^s)$. The time t_1 of the first detection event follows by solving the equation $P_0(t_1-0|\rho_0^s)=r$, where r is a random number in the interval $(0,1)$. The dynamics of $\rho_s^{\rm st}(t)$ in the interval $(0,t_1)$ is defined by Eq. (7). At $t=t_1$ the disruptive transformation $\rho_s^{\rm st}(t_1)\rightarrow\hat{\mathcal{M}}\rho_s^{\rm st}(t_1)$ is applied. The subsequent dynamics is the same. In fact, after the n_{th} measurement event at time t_n, $\rho_s^{\rm st}(t_n)\rightarrow\hat{\mathcal{M}}\rho_s^{\rm st}(t_n)$, the time t_{n+1} for the next detection event follows from

$$P_0(t_{n+1}-t_n|\hat{\mathcal{M}}\rho_s^{\rm st}(t_n)), \tag{14}$$

equated to r, where again r is a random number in the interval $(0,1)$. The dynamics in the interval (t_n,t_{n+1}) is defined by the conditional propagator (7).

The previous algorithm determines the realizations over finite time intervals [6]. It is also possible to obtain the evolution over infinitesimal intervals. Its structure remains the same [Eqs. (5) and (7)]. Nevertheless, instead of Eq. (14), the jump statistics are determined from $w_c(t-\tau|\rho_\tau)$, Eq. (11). Given that the last event happened at time τ and that no detection was found in the interval (τ,t), the probability ΔP of having a detection event in the *infinitesimal interval* $(t,t+dt)$ is (by definition) [3]

$$\Delta P = w^c(t-\tau|\rho_s^{\rm st}(\tau))\,dt. \tag{15}$$

From Eqs. (6), (7), and (12), we can write

$$\Delta P = -dt\,{\rm Tr}_s\big[\hat{\mathcal{D}}\rho_s^{\rm st}(t)\big] = dt\sum_\alpha \gamma_\alpha {\rm Tr}_s\big[V_\alpha^\dagger V_\alpha \rho_s^{\rm st}(t)\big]. \tag{16}$$

The happening or not of a detection follows by comparing ΔP with a random number in the interval $(0,1)$. This alternative algorithm generates the same realizations as the previous one [6]. Nevertheless, in this last scheme the Markovian property of the underlying master equation is self-evident in the expression for ΔP. In fact, ΔP does not depend on the "history" of $\rho_s^{\rm st}(t)$ in the interval (τ,t). It depends only on $\rho_s^{\rm st}(t)$.

B. Renewal and nonrenewal measurement processes

An extra understanding of the QJA is achieved by specifying the operators $\{V_\alpha\}$ that determine the measurement transformation Eq. (5). When the system state after a measurement event (resetting state) is always the same, the statistics of the time interval between events is defined by a unique probability distribution (the waiting-time distribution). In this case, the measurement process is a renewal one. This situation arises when the measurement apparatus is sensible to all transitions $(|u\rangle \rightsquigarrow |r_\alpha\rangle)$ between a given system state $|u\rangle$ and a set of alternative states $\{|r_\alpha\rangle\}$. Therefore, the operators $\{V_\alpha\}$ have the structure

$$V_\alpha = |r_\alpha\rangle\langle u|, \tag{17}$$

which in turn, from Eq. (5), imply the measurement transformation

$$\hat{\mathcal{M}}\rho = \bar{\rho}_s \equiv \sum_\alpha p_\alpha |r_\alpha\rangle\langle r_\alpha|, \quad p_\alpha = \frac{\gamma_\alpha}{\{\sum_\alpha \gamma_\alpha\}}. \tag{18}$$

Hence, the conditional dynamics [Eq. (7)] always starts in the same resetting state $\bar{\rho}_s$ [5,6]. Furthermore, the survival probability and waiting-time distribution [Eqs. (9) and (10), respectively], after the first event ($\rho_\tau \rightarrow \hat{\mathcal{M}}\rho = \bar{\rho}_s$) are always the same, being defined as

$$P_0(t) = {\rm Tr}_s[\hat{\mathcal{T}}(t)\bar{\rho}_s], \qquad w(t) = -{\rm Tr}_s[\hat{\mathcal{D}}\hat{\mathcal{T}}(t)\bar{\rho}_s]. \tag{19}$$

In consequence, the interval statistics does not depend explicitly on the time τ of the last events and it is always the same. The operators (17) arise, for example, in optical systems such as two-level fluorescent systems, where $\bar{\rho}_s$ is a pure state, and three-level Λ configurations [5,6].

In general, the operators may read

$$V_\alpha = |r_\alpha\rangle\langle u_\alpha|, \tag{20}$$

that is, the measurement apparatus is sensible to different transitions $|u_\alpha\rangle \rightsquigarrow |r_\alpha\rangle$. This case may happen when the natural frequencies of the different transitions are indistinguishable for the measurement apparatus; for example in cascade optical systems [5]. The measurement transformation

$$\hat{\mathcal{M}}\rho = \frac{\sum_\alpha \gamma_\alpha \langle u_\alpha|\rho|u_\alpha\rangle |r_\alpha\rangle\langle r_\alpha|}{\{\sum_\alpha \gamma_\alpha \langle u_\alpha|\rho|u_\alpha\rangle\}} \tag{21}$$

delivers a state that depends on the predetection state. Hence, it is not possible to define a unique statistical object as in the previous case, that is, the survival probability and waiting-time distribution correspond to the general expressions Eqs. (9) and (10), respectively.

III. NON-MARKOVIAN QUANTUM JUMPS FROM BIPARTITE MARKOVIAN DYNAMICS

The previous elements and results that define the QJA, without introducing any new element, can also be established for bipartite dynamics. Here, in addition to the system of interest S we consider an auxiliary or ancilla system A. Their joint dynamics is Markovian. Furthermore, we assume that the measurement apparatus is sensible to the same system transitions as before. Thus, we can define a stochastic density matrix $\rho_{\rm st}^{\rm sa}(t)$ such that its average over realizations recovers the bipartite density matrix $\rho_t^{\rm sa}=\overline{\rho_{\rm st}^{\rm sa}(t)}$. The density matrix of S is recovered by a partial trace operation over the auxiliary system A,

$$\rho_t^s = {\rm Tr}_a\big[\rho_t^{\rm sa}\big] = {\rm Tr}_a\big[\overline{\rho_{\rm st}^{\rm sa}(t)}\big]. \tag{22}$$

Trivially, by introducing the stochastic matrix

$$\rho_{\rm st}^s(t) = {\rm Tr}_a\big[\rho_{\rm st}^{\rm sa}(t)\big], \tag{23}$$

we recover Eq. (13), that is, $\rho_t^s=\overline{\rho_{\rm st}^s(t)}$. At this point, we ask about the existence of different S-A interactions and evolutions under which it is possible to get a closed stochastic dynamics for $\rho_{\rm st}^s(t)$, that is, without involving explicitly the ancilla state. In addition to this constraint, here we search for interaction structures that introduce a minimal modification of the standard approach, that is, it should be possible to define a measurement transformation [Eq. (5)], a conditional interevent dynamic [Eq. (7)], and a survival probability [Eq. (9)].

A. Bipartite Markovian embedding

Taking into account the evolution Eq. (1), we write the bipartite evolution as

$$\frac{d}{dt}\rho_t^{\text{sa}} = \left(\hat{\mathbb{L}}_0 + \sum_{\alpha lm} \gamma_{\alpha lm} \hat{\mathcal{C}}[V_{\alpha lm}]\right)\rho_t^{\text{sa}}. \tag{24}$$

The operators $V_{\alpha lm}$ are defined as

$$V_{\alpha lm} = V_\alpha \otimes |a_l\rangle\langle a_m|. \tag{25}$$

The set of states $\{|a_l\rangle\}$ provides an orthogonal and normalized basis of the ancilla Hilbert space. The system operators $\{V_\alpha\}$ are the same as before. Notice that the diagonal contributions, defined by the operators $V_{\alpha mm} = V_\alpha \otimes |a_m\rangle\langle a_m|$, correspond to system's transitions that only happen when the ancilla system is in the state $|a_m\rangle$. The nondiagonal contributions $V_{\alpha lm} = V_\alpha \otimes |a_l\rangle\langle a_m|$ correspond to system transitions that occur simultaneously with the ancilla transition $|a_m\rangle \rightsquigarrow |a_l\rangle$.

In Eq. (24), the superoperator $\hat{\mathbb{L}}_0$ includes not only the system evolution [$\hat{\mathcal{L}}_0$ in Eq. (1)] but also an arbitrary evolution for the ancilla system as well as the system-ancilla interaction. For this last contribution, we only demand that it must not include any interaction proportional to the transitions defined by the operators $\{V_\alpha\}$. On the other hand, the measurement apparatus remains the same, that is, it only detects the system transitions. Therefore, we split the bipartite master equation (24) as

$$\frac{d}{dt}\rho_t^{\text{sa}} = (\hat{\mathbb{D}} + \hat{\mathbb{J}})\rho_t^{\text{sa}}, \tag{26}$$

where the superoperator $\hat{\mathbb{J}}$ reads

$$\hat{\mathbb{J}}\rho = \sum_{\alpha lm} \gamma_{\alpha lm} V_{\alpha lm} \rho V_{\alpha lm}^\dagger. \tag{27}$$

The measurement transformation [see Eq. (5)] in the bipartite Hilbert space becomes

$$\hat{\mathbb{M}}\rho = \frac{\hat{\mathbb{J}}\rho}{\text{Tr}_{\text{sa}}[\hat{\mathbb{J}}\rho]} = \frac{\sum_{\alpha lm} \gamma_{\alpha lm} V_{\alpha lm}\rho V_{\alpha lm}^\dagger}{\{\sum_{\alpha lm} \gamma_{\alpha lm} \text{Tr}_{\text{sa}}[V_{\alpha lm}^\dagger V_{\alpha lm}\rho]\}}. \tag{28}$$

The goal is to obtain a closed (stochastic) evolution for the system with almost the same elements as in the Markovian case. The free parameters are the rates $\gamma_{\alpha lm}$. In order to have the same measurement transformation as before [Eq. (5)], for arbitrary bipartite states ρ_{sa} one must demand the condition

$$\text{Tr}_a[\hat{\mathbb{M}}\rho_{\text{sa}}] = \hat{\mathcal{M}}[\rho_s], \tag{29}$$

where evidently $\rho_s = \text{Tr}_a[\rho_{\text{sa}}]$. There exist different ways of satisfying this condition. Here, for simplicity, we choose the constraint

$$\hat{\mathbb{M}}\rho_{\text{sa}} = \hat{\mathcal{M}}[\rho_s] \otimes \bar{\rho}_a, \tag{30}$$

where $\bar{\rho}_a$ is a particular ancilla density matrix. Notice that after a measurement event, the system and ancilla become uncorrelated. Trivially, this measurement transformation satisfies the previous condition Eq. (29).

The conditional system dynamics between collision events can be written as in Eq. (7), but now the unconditional propagator reads

$$\hat{\mathcal{T}}(t-\tau) = \text{Tr}_a\{\exp[\hat{\mathbb{D}}(t-\tau)]\bar{\rho}_a\}. \tag{31}$$

It arises from the partial trace over the ancilla system of the bipartite conditional propagator $\hat{\mathbb{T}}(t-\tau) = \exp[\hat{\mathbb{D}}(t-\tau)]$, and the condition (30). The superoperator $\hat{\mathbb{D}}$ is

$$\hat{\mathbb{D}}\rho = \hat{\mathbb{L}}_0\rho - \frac{1}{2}\sum_{\alpha lm} \gamma_{\alpha lm}\{V_{\alpha lm}^\dagger V_{\alpha lm}, \rho\}_+. \tag{32}$$

As we have chosen the stronger separability condition (30), the propagator defined by $\hat{\mathcal{T}}(t)$ [Eq. (31)] not only is completely positive but also its time evolution is given by an homogeneous equation. In fact, in a Laplace domain, $f(z) \equiv \int_0^\infty dt e^{-zt} f(t)$, Eq. (31) becomes $\hat{\mathcal{T}}(z) = \text{Tr}_a[\frac{1}{z-\hat{\mathbb{D}}}\bar{\rho}_a]$. This expression can be rewritten as $\hat{\mathcal{T}}(z) = \{\text{Tr}_a[(z-\hat{\mathbb{D}})^{-1}(z-\hat{\mathbb{D}})\bar{\rho}_a]\}^{-1} \times \{[\hat{\mathcal{T}}(z)]^{-1}\}^{-1}$. Using in the curly brackets that $M^{-1} \times N^{-1} = (N \times M)^{-1}$, where M and N are arbitrary matrices, it follows that $\hat{\mathcal{T}}(z) = ([\hat{\mathcal{T}}(z)]^{-1}\{z\text{Tr}_a[(z-\hat{\mathbb{D}})^{-1}\bar{\rho}_a] - \text{Tr}_a[(z-\hat{\mathbb{D}})^{-1}\hat{\mathbb{D}}\bar{\rho}_a]\})^{-1}$, which in turn leads to the expression $\hat{\mathcal{T}}(z) = [z - \hat{\mathcal{D}}(z)]^{-1}$, where the system superoperator $\hat{\mathcal{D}}(z)$ is

$$\hat{\mathcal{D}}(z) = \left\{\text{Tr}_a\left[\frac{1}{z-\hat{\mathbb{D}}}\bar{\rho}_a\right]\right\}^{-1}\text{Tr}_a\left[\frac{1}{z-\hat{\mathbb{D}}}\hat{\mathbb{D}}\bar{\rho}_a\right]. \tag{33}$$

Hence, in the time domain we get

$$\frac{d}{dt}\hat{\mathcal{T}}(t) = \int_0^t dt' \hat{\mathcal{D}}(t-t')\hat{\mathcal{T}}(t'), \tag{34}$$

where the memory superoperator $\hat{\mathcal{D}}(t)$ is defined by its Laplace transform (33). We notice that in the Markovian case $\hat{\mathcal{T}}(t) = \exp[t\hat{\mathcal{D}}]$ [see Eq. (8)] implying the local-in-time evolution $(d/dt)\hat{\mathcal{T}}(t) = \hat{\mathcal{D}}\hat{\mathcal{T}}(t)$. Thus, in the present approach the conditional evolution between measurement events becomes nonlocal in time. This property also implies that in general, even for pure initial conditions $|\Psi\rangle$, the conditional evolution cannot be decomposed into pure states [6–8], that is,

$$\hat{\mathcal{T}}(t)(|\Psi\rangle\langle\Psi|) \neq |\Psi(t)\rangle\langle\Psi(t)|. \tag{35}$$

Under the assumption Eq. (30), the previous analysis demonstrates that it is possible to obtain a closed evolution for the system dynamics. It remains to determine the statistics of the measurement events. As the bipartite dynamics is Markovian, here we also have a well-defined survival probability [see Eq. (9)]. By using Eq. (30), it is possible to write

$$P_0(t-\tau|\rho_\tau) = \text{Tr}_{\text{sa}}\{\exp[(t-\tau)\hat{\mathbb{D}}]\rho_\tau \otimes \bar{\rho}_a\} \tag{36a}$$

$$= \text{Tr}_s[\hat{\mathcal{T}}(t-\tau)\rho_\tau], \tag{36b}$$

where $\hat{\mathcal{T}}(t)$ is given by Eq. (31). Notice that ρ_τ is a system state. Furthermore, this expression has the same structure as Eq. (9). The definition of the conditional propagator $\hat{\mathcal{T}}(t)$ is the unique difference. The corresponding waiting-time distribution [Eq. (10)] here reads

$$w(t-\tau|\rho_\tau) = -\text{Tr}_{\text{sa}}\{\hat{\mathbb{D}}\exp[(t-\tau)\hat{\mathbb{D}}]\rho_\tau \otimes \bar{\rho}_a\}. \tag{37}$$

From Eq. (34) the equivalent expression follows:

$$w(t-\tau|\rho_\tau) = -\int_0^{t-\tau} dt' \text{Tr}_s[\hat{\mathcal{D}}(t-t')\hat{\mathcal{T}}(t')\rho_\tau], \tag{38}$$

which leads to a natural non-Markovian generalization of Eq. (10). On the other hand, the conditional waiting-time distribution Eq. (11) here becomes

$$w_c(t-\tau|\rho_\tau) = \frac{-\int_0^{t-\tau} dt' \mathrm{Tr}_s[\hat{\mathcal{D}}(t-t')\hat{\mathcal{T}}(t')\rho_\tau]}{\mathrm{Tr}_s[\hat{\mathcal{T}}(t-\tau)\rho_\tau]}. \quad (39)$$

B. Stochastic dynamics

As in the Markovian case, the previous objects [Eqs. (30), (34), and (36)] completely define the system realizations associated with the measurement process. Therefore, the algorithm associated with Eq. (14) remains exactly the same. The unique modification is the definition of the propagator $\hat{\mathcal{T}}(t)$, which in turn modifies the conditional dynamics as well as the measurement event statistics.

On the other hand, the infinitesimal-time-step algorithm defined by Eq. (15) can also be applied. Nevertheless, in contrast to Eq. (16), here it is not possible to write a simple expression for ΔP in terms of either $\rho_{\mathrm{st}}^s(t)$ or its history [see Eq. (39)]. Therefore, in this generalized non-Markovian approach the infinitesimal algorithm, while it can be formally implemented, it does not provide an efficient numerical simulation method, nor does it have a simple physical interpretation.

C. Symmetries of the bipartite dynamics

It remains to demonstrate that in fact there exist different bipartite Lindblad equations that fulfil the condition (30), where the bipartite measurement transformation is given by Eq. (28). From Eq. (25), it can be written as

$$\hat{\mathbb{M}}\rho = \frac{\sum_{\alpha lm} \gamma_{\alpha lm} V_\alpha \langle a_m|\rho|a_m\rangle V_\alpha^\dagger \otimes |a_l\rangle\langle a_l|}{\{\sum_{\alpha lm} \gamma_{\alpha lm} \mathrm{Tr}_s[V_\alpha \langle a_m|\rho|a_m\rangle V_\alpha^\dagger]\}}. \quad (40)$$

The result of calculating $\mathrm{Tr}_a[\hat{\mathbb{M}}\rho]$ can be written in terms of $\hat{\mathcal{M}}$ [Eq. (5)] only if $\gamma_{\alpha lm} = \gamma_{\alpha l} d_m$, where d_m is an arbitrary dimensionless coefficient. Equation (40) becomes

$$\hat{\mathbb{M}}\rho = \frac{\sum_{\alpha m} \gamma_\alpha V_\alpha d_m \langle a_m|\rho|a_m\rangle V_\alpha^\dagger \otimes \bar{\rho}_a^\alpha}{\{\sum_{\alpha m} \gamma_\alpha \mathrm{Tr}_s[V_\alpha d_m \langle a_m|\rho|a_m\rangle V_\alpha^\dagger]\}}, \quad (41)$$

where $\bar{\rho}_a^\alpha \equiv \sum_l (\gamma_{\alpha l}/\gamma_\alpha)|a_l\rangle\langle a_l|$ and $\gamma_\alpha \equiv \sum_l \gamma_{\alpha l}$. With the operator definitions (17) and (20), Eq. (41) can satisfy the weaker condition (29). Nevertheless, the resulting bipartite state is a classical correlated one (with vanishing discord). To satisfy the separability condition (30), which leads to the homogeneous dynamics (34), the states $\bar{\rho}_a^\alpha$ must not depend on the index α. Hence, we demand $\gamma_{\alpha l} = \gamma_\alpha c_l$, where c_l is also an arbitrary dimensionless coefficient. The rates $\gamma_{\alpha lm}$ become

$$\gamma_{\alpha lm} = \gamma_\alpha c_l d_m, \qquad \sum_l c_l = 1, \quad (42)$$

which from Eq. (40) leads to

$$\hat{\mathbb{M}}\rho = \hat{\mathcal{M}}\left[\sum_m d_m \langle a_m|\rho|a_m\rangle\right] \otimes \bar{\rho}_a. \quad (43)$$

The ancilla resetting state $\bar{\rho}_a$ is

$$\bar{\rho}_a = \sum_l c_l |a_l\rangle\langle a_l|. \quad (44)$$

For simplicity, we assumed $\sum_l c_l = 1$. If this condition is not met, it can always be satisfied by a renormalization of the Lindblad channel rates, $\gamma_\alpha \to \gamma_\alpha / \sum_l c_l$.

The expression (42) can be read as a symmetry condition on the bipartite Lindblad evolution Eq. (24). It leads to Eq. (43), which does not recover explicitly Eq. (30). The fulfillment of this constraint can be achieved by choosing different sets of values for the coefficients d_m, which depend on the specific structure of $\hat{\mathcal{M}}$.

1. Renewal case

When the measurement transformation $\hat{\mathcal{M}}$ leads to a renewal process, Eqs. (17) and (18), independently of the coefficients d_m it follows that $\hat{\mathcal{M}}[\sum_m d_m \langle a_m|\rho|a_m\rangle] = \bar{\rho}_s$. Therefore, Eq. (43) leads to

$$\hat{\mathbb{M}}\rho = \bar{\rho}_s \otimes \bar{\rho}_a. \quad (45)$$

Evidently this expression satisfies the condition (30). Furthermore, it tells us that the stochastic dynamics developing in the bipartite S-A Hilbert space is also a renewal measurement process.

Similarly to the Markovian case, after the first detection event the statistics of the time interval between consecutive events is defined by a unique survival probability

$$P_0(t) = \mathrm{Tr}_s[\hat{\mathcal{T}}(t)\bar{\rho}_s], \quad (46)$$

or equivalently a unique waiting-time distribution

$$w(t) = -\int_0^t dt' \mathrm{Tr}_s[\hat{\mathcal{D}}(t-t')\hat{\mathcal{T}}(t')\bar{\rho}_s]. \quad (47)$$

These expressions follow from Eqs. (36) and (38) after introducing the resetting property defined by Eq. (45). They generalize the Markovian expressions (19).

2. Nonrenewal case

When the measurement transformation $\hat{\mathcal{M}}$ does not lead to a renewal process [Eqs. (20) and (21)], the coefficients d_m cannot be arbitrary. In fact, the only way of satisfying the condition (30) is by choosing $d_m = 1$ (after a rate renormalization we can also take d_m equal to an arbitrary real constant). As the states $\{|a_m\rangle\}$ are a complete basis of the ancilla Hilbert space, for any bipartite state ρ_{sa} it follows that $\sum_m \langle a_m|\rho_{\mathrm{sa}}|a_m\rangle = \mathrm{Tr}_a[\rho_{\mathrm{sa}}] = \rho_s$. Thus, Eq. (43) recovers Eq. (30),

$$\hat{\mathbb{M}}\rho = \hat{\mathcal{M}}[\rho_s] \otimes \bar{\rho}_a. \quad (48)$$

Notice that this result is valid for both the nonrenewal and renewal cases. Nevertheless, the condition $d_m = 1$ is "necessary" only in the former case. The symmetry condition on the rates $\gamma_{\alpha lm}$ [Eq. (42)] then reads

$$\gamma_{\alpha lm} = \gamma_\alpha c_l, \qquad \sum_l c_l = 1. \quad (49)$$

In contrast to the renewal case, here the measurement statistics remains defined by the general expression Eqs. (36) and (38).

D. Density matrix evolution

Under the symmetry conditions defined by Eqs. (42) and (49) the stochastic dynamics of $\rho_{\mathrm{st}}^s(t)$ [Eq. (23)] has the same structure as in the Markovian case. For both renewal and nonrenewal measurement processes, the main difference from the Markovian case is the conditional dynamics. It remains to calculate the time evolution of the system density matrix ρ_t^s, Eq. (22). In Appendix B we perform this calculation by averaging the realizations of $\rho_{\rm st}^s(t)$, that is, from $\rho_t^s = \overline{\rho_{\rm st}^s(t)}$. Here, using an alternative procedure, the evolution of the system state is obtained from the bipartite dynamics (24) by using that $\rho_t^s = {\rm Tr}_a[\rho_t^s]$.

For simplicity, we take a separable bipartite initial condition

$$\rho_0^{\rm sa} = \rho_0^s \otimes \bar{\rho}_a, \tag{50}$$

where ρ_0^s is an arbitrary system state and $\bar{\rho}_a$ is the ancilla resetting state defined by Eq. (44). The bipartite Lindblad evolution (24) can formally be integrated as

$$\rho_t^{\rm sa} = \exp[\hat{\mathbb{D}}t]\rho_0^{\rm sa} + \int_0^t dt' \exp[\hat{\mathbb{D}}(t-t')]\hat{\mathbb{J}}\left[\rho_{t'}^{\rm sa}\right]. \tag{51}$$

The superoperators $\hat{\mathbb{J}}$ and $\hat{\mathbb{D}}$ were defined in Eqs. (27) and (32), respectively. By using the rates condition Eq. (42) and the operator definition (25), we get

$$\hat{\mathbb{J}}\left[\rho_t^{\rm sa}\right] = \sum_\alpha \gamma_\alpha V_\alpha {\rm O}\left[\rho_t^{\rm sa}\right] V_\alpha^\dagger \otimes \bar{\rho}_a. \tag{52}$$

To shorten the notation we defined the superoperator

$${\rm O}[\rho_t^{\rm sa}] \equiv \sum_m d_m \langle a_m|\rho_t^{\rm sa}|a_m\rangle. \tag{53}$$

Taking the partial trace over the ancilla degrees of freedom, Eq. (51) leads to

$$\rho_t^s = \hat{\mathcal{T}}(t)\rho_0^s + \int_0^t dt' \hat{\mathcal{T}}(t-t') \sum_\alpha \gamma_\alpha V_\alpha {\rm O}\left[\rho_{t'}^{\rm sa}\right] V_\alpha^\dagger, \tag{54}$$

which in turn, from Eq. (34), allows us to write

$$\frac{d\rho_t^s}{dt} = \int_0^t dt' \hat{\mathcal{D}}(t-t')\rho_{t'}^s + \sum_\alpha \gamma_\alpha V_\alpha {\rm O}\left[\rho_t^{\rm sa}\right] V_\alpha^\dagger.] \tag{55}$$

If all $d_m = 1$, it follows that ${\rm O}[\rho_t^{\rm sa}] = \rho_t^s$. Hence, from Eq. (55) we get the closed density matrix evolution

$$\frac{d\rho_t^s}{dt} = \int_0^t dt' \hat{\mathcal{D}}(t-t')\rho_{t'}^s + \sum_\alpha \gamma_\alpha V_\alpha \rho_t^s V_\alpha^\dagger. \tag{56}$$

Notice that this evolution contains both convoluted as well as local-in-time contributions. It is valid for both renewal and nonrenewal measurement processes. On the other hand, in the case of renewal processes the coefficients d_m may be arbitrary and the previous expression does not apply. By using the specific form of the operators V_α [Eq. (17)] it follows that $\sum_\alpha \gamma_\alpha V_\alpha {\rm O}[\rho_t^{\rm sa}] V_\alpha^\dagger = \bar{\rho}_s \gamma \langle u|{\rm O}[\rho_t^{\rm sa}]|u\rangle$, where $\gamma = \sum_\alpha \gamma_\alpha$ and the system resetting state $\bar{\rho}_s$ is defined by Eq. (18). By using in Eq. (55) that $(d/dt){\rm Tr}_s[\rho_t^s] = 0$, it follows that $\gamma\langle u|{\rm O}[\rho_t^{\rm sa}]|u\rangle = -\int_0^t dt' {\rm Tr}_s[\hat{\mathcal{D}}(t-t')\rho_{t'}^s]$, implying the closed density matrix evolution

$$\frac{d\rho_t^s}{dt} = \int_0^t dt' \hat{\mathcal{D}}(t-t')\rho_{t'}^s - \bar{\rho}_s \int_0^t dt' {\rm Tr}_s\left[\hat{\mathcal{D}}(t-t')\rho_{t'}^s\right]. \tag{57}$$

In the present approach, this expression corresponds to the more general master equation consistent with a renewal measurement process. Notice that Eq. (56) is a particular case of this more general expression. By comparing both equations, we realize that it applies when $\gamma\langle u|\rho_t^s|u\rangle = \int_0^t dt' {\rm Tr}_s[\hat{\mathcal{D}}(t-t')\rho_{t'}^s]$.

E. Arbitrary master equations

Equations (56) and (57) represent one of the central results of this section. They correspond to master equations that admit an unraveling in terms of an ensemble of trajectories associated with a continuous measurement action defined by the set of operators $\{V_\alpha\}$. Equation (56) is valid for both renewal and nonrenewal measurement processes [see Eqs. (17) and (20), respectively] while Eq. (57) is only valid for renewal processes [Eq. (17)]. Now we ask about which conditions an arbitrary non-Markovian master equation must to satisfy to admit the non-Markovian unraveling defined previously.

One condition is the possibility of rewriting the master equation with the structure defined by Eq. (56) or (57). On the other hand, the ensemble representation can only be assigned if the memory superoperator $\hat{\mathcal{D}}(t)$ through the relation $(d/dt)\hat{\mathcal{T}}(t) = \int_0^t dt' \hat{\mathcal{D}}(t-t')\hat{\mathcal{T}}(t')$ [Eq. (34)] defines a well-behaved survival probability $P_0(t-\tau|\rho) = {\rm Tr}_s[\hat{\mathcal{T}}(t-\tau)\rho]$ [Eq. (36)] for "arbitrary" system states ρ. A well-behaved survival probability means that it is a decaying function, that is, for arbitrary times $\tau < t_1 < t_2$, it must satisfy $P_0(t_2-\tau|\rho) \leqslant P_0(t_1-\tau|\rho)$, implying

$${\rm Tr}_s[\hat{\mathcal{T}}(t_2)\rho] \leqslant {\rm Tr}_s[\hat{\mathcal{T}}(t_1)\rho], \quad t_1 < t_2. \tag{58}$$

Taking into account that the realizations can be determined from $P_0(t|\rho)$, the fulfillment of the previous inequality guarantees the possibility of assigning a non-Markovian unraveling to a master equation with the structure (56) or (57).

IV. NON-MARKOVIAN RENEWAL TWO-LEVEL TRANSITIONS

Here, we work out an example that explicitly shows the consistency of the previous results. Both the system of interest and the ancilla are two-level systems. Their states are denoted $|\pm\rangle$ and $\{|1\rangle,|2\rangle\}$, respectively. The Markovian dynamic of the bipartite state $\rho_t^{\rm sa}$ [Eq. (24)] here reads

$$\frac{d}{dt}\rho_t^{\rm sa} = -\frac{i}{\hbar}\left[H_0,\rho_t^{\rm sa}\right] + (\gamma\mathcal{C}[\sigma_{11}] + \gamma'\mathcal{C}[\sigma_{21}])\rho_t^{\rm sa}. \tag{59}$$

The bipartite Hamiltonian contribution is defined by the operator

$$H_0 = \hbar\Omega\sigma_x \otimes \sigma_x, \tag{60}$$

where σ_x is the x Pauli matrix in the basis of each Hilbert space. The remaining Lindblad contributions [Eq. (2)] with rates γ and γ' are defined by the operators

$$\sigma_{11} = \sigma \otimes |1\rangle\langle 1|, \quad \sigma_{12} = \sigma \otimes |1\rangle\langle 2|. \tag{61}$$

The lowering system operator is defined as $\sigma = |-\rangle\langle +|$.

Notice that σ_{11} leads to system transitions between the upper and lower states $|+\rangle \rightsquigarrow |-\rangle$ that can only happen when the ancilla is in the state $|1\rangle$. In addition, σ_{12} leads to the same system transitions but in this case they occur simultaneously with the ancilla transition $|2\rangle \rightsquigarrow |1\rangle$. Thus, the dissipative dynamics drives the system to its ground state. On the other hand, the unitary evolution can excite the system to its upper state. In consequence, the interplay between both contributions leads to successive system transitions $|+\rangle \rightsquigarrow |-\rangle$. Each transition can be associated with a recording event in the measurement apparatus.

It is simple to check that Eq. (59) has the structure defined by Eqs. (24) and (25), and also fulfills the symmetry condition Eq. (42). Consistently with the previous analysis, the superoperator $\hat{\mathbb{J}}$ [Eq. (27)] is defined as

$$\hat{\mathbb{J}}\rho = \gamma\sigma_{11}\rho\sigma_{11}^{\dagger} + \gamma'\sigma_{12}\rho\sigma_{12}^{\dagger}, \tag{62}$$

leading to the expression

$$\hat{\mathbb{J}}\rho = (\gamma\,\langle +1|\,\rho\,|+1\rangle + \gamma'\,\langle +2|\,\rho\,|+2\rangle)\,|-\rangle\,\langle -|\otimes|1\rangle\,\langle 1|\,.$$

From here, the measurement transformation [Eq. (28)] associated with each event reads

$$\hat{\mathbb{M}}\rho = |-\rangle\langle -|\otimes|1\rangle\langle 1|. \tag{63}$$

Therefore, the state after a detection is independent of the previous bipartite state ρ, which in turn implies that the measurement process is a renewal one [see Eq. (45)]. The bipartite conditional dynamics between events is defined by the superoperator [Eq. (32)]

$$\hat{\mathbb{D}}\rho = -\frac{i}{\hbar}[H_0,\rho] - \tfrac{1}{2}\{(\gamma\sigma_{11}^{\dagger}\sigma_{11} + \gamma'\sigma_{12}^{\dagger}\sigma_{12}),\rho\}_{+}. \tag{64}$$

In order to obtain simple analytical expressions from now on we analyze the case $\gamma' = \gamma$. Notice that it is also possible to take $\gamma' = 0$ with $\gamma > 0$, or $\gamma = 0$ with $\gamma' > 0$.

The conditional propagator $\hat{\mathcal{T}}(t)$ [Eq. (31)] can be defined when acting on an arbitrary initial condition ρ. By defining the state $\tilde{\rho}_t = \hat{\mathcal{T}}(t)\rho$, the time evolution of $\hat{\mathcal{T}}(t)$ [Eq. (34)] can be written in terms of the matrix elements

$$\tilde{p}_t^{\pm} \equiv \langle \pm|\tilde{\rho}_t|\pm\rangle, \quad \tilde{c}_t^{\pm} \equiv \langle \pm|\tilde{\rho}_t|\mp\rangle. \tag{65}$$

For the populations we get

$$\frac{d\tilde{p}_t^{+}}{dt} = -\int_0^t dt' k_{t-t'}^{+}\tilde{p}_{t'}^{+} + \int_0^t dt' k_{t-t'}^{-}\tilde{p}_{t'}^{-}, \tag{66a}$$

$$\frac{d\tilde{p}_t^{-}}{dt} = -\int_0^t dt' k_{t-t'}^{-}\tilde{p}_{t'}^{-} + (1-\tilde{\delta})\int_0^t dt' k_{t-t'}^{+}\tilde{p}_{t'}^{+}. \tag{66b}$$

Here, the constant $\tilde{\delta}$ must be taken as $\tilde{\delta} \to 1$. Thus, the last term does not contribute. The memory kernels are

$$k_t^{+} = \gamma\delta(t) + \frac{\Omega^2}{2}e^{-t\gamma/2}, \quad k_t^{-} = \frac{\Omega^2}{2}e^{-t\gamma/2}. \tag{67}$$

The coherence evolves as

$$\frac{d\tilde{c}_t^{\pm}}{dt} = -\int_0^t dt' \tilde{k}_{t-t'}\tilde{c}_{t'}^{\pm} + \int_0^t dt' \check{k}_{t-t'}\tilde{c}_{t'}^{\mp}, \tag{68}$$

where the kernels $\tilde{k}_t$ and $\check{k}_t$ are

$$\tilde{k}_t = \frac{\gamma}{2}\delta(t) + \frac{\Omega^2}{4}(1+e^{-t\gamma}), \quad \check{k}_t = \frac{\Omega^2}{4}(1+e^{-t\gamma}). \tag{69}$$

Due to the symmetries of the problem, the populations and coherences evolve independently of each other.

Equations (66) and (68) can be solved in a Laplace domain. The survival probability [Eq. (36)] reads $P_0(t|\rho) = \mathrm{Tr}_s[\hat{\mathcal{T}}(t)\rho] = \mathrm{Tr}_s[\tilde{\rho}_t] = \tilde{p}_t^{+} + \tilde{p}_t^{-}$. We get

$$P_0(t|\rho) = \mathrm{Tr}_s[\rho]e^{-\gamma t/2}\left[\left(\frac{\gamma}{2\nu}\right)^2\cosh(\nu t) - \left(\frac{\Omega}{\nu}\right)^2\right] - \mathrm{Tr}_s[\sigma_z\rho]e^{-\gamma t/2}\left[\frac{\gamma}{2\nu}\sinh(\nu t)\right], \tag{70}$$

where the "frequency" ν reads

$$\nu = \sqrt{(\gamma/2)^2 - \Omega^2}. \tag{71}$$

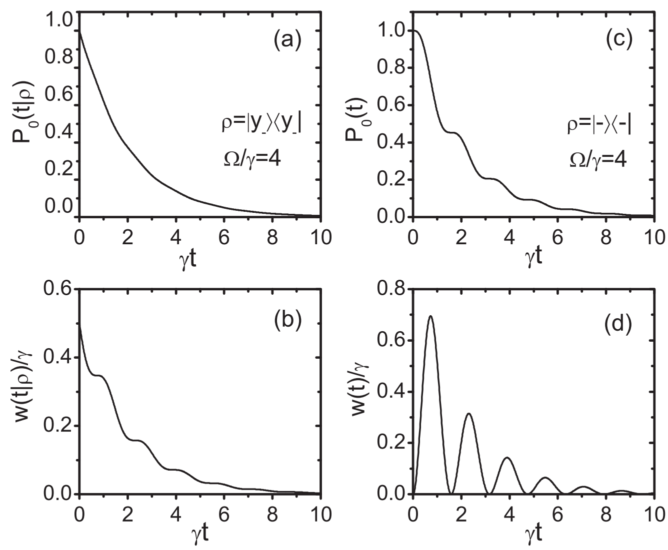

FIG. 1. Survival probability $P_0(t|\rho)$ [Eq. (70)] and its associated waiting-time distribution $w(t|\rho) = -(d/dt)P_0(t|\rho)$ for different initial conditions. In (a) and (b), $\rho = |y_-\rangle\langle y_-|$, where $|y_-\rangle = (1/\sqrt{2})(|+\rangle - i|-\rangle)$. In (c) and (d), $\rho = |-\rangle\langle -|$, which correspond to the resetting state defined by Eq. (63). In all cases, the parameters satisfy $\Omega/\gamma = 4$.

In Eq. (70) the dependence on the system state ρ is given by $\mathrm{Tr}_s[\rho]$ and $\mathrm{Tr}_s[\sigma_z\rho]$, where σ_z is the z Pauli matrix. Using the normalization of ρ it follows that $\mathrm{Tr}_s[\rho] = 1$, while $\mathrm{Tr}_s[\sigma_z\rho] = \langle +|\rho|+\rangle - \langle -|\rho|-\rangle$. Therefore, $P_0(t|\rho)$ depends only on the populations of ρ.

In Fig. 1 we plotted $P_0(t|\rho)$ and its associated waiting-time distribution $w(t|\rho) = -(d/dt)P_0(t|\rho)$ [Eq. (38)] for different initial states ρ. In Figs. 1(a) and 1(b) we took $\rho = |y_-\rangle\langle y_-|$, where $|y_-\rangle$ is an eigenvector of σ_y with eigenvalue -1, $|y_-\rangle = (1/\sqrt{2})(|+\rangle - i|-\rangle)$. In Figs. 1(c) and 1(d) the initial state is $\rho = |-\rangle\langle -|$, that is, the resetting state after a detection event [see Eq. (63)]. Hence, these objects, after the first measurement event, completely define the measurement statistics [Eqs. (46) and (47)]. Consistently with Eq. (58), for both initial conditions the survival probabilities as a function of time are decaying functions. On the other hand, while $\lim_{t\to 0} w(t||y_-\rangle\langle y_-|) \neq 0$, given that $\lim_{t\to 0} w(t) = 0$, an antibunching phenomenon [7,8] characterize the renewal measurement process.

The survival probability allows to generate the random time intervals between detection events. On the other hand, the matrix elements of $\tilde{\rho}_t = \hat{\mathcal{T}}(t)\rho$ [Eq. (65)] also allow one to obtain the corresponding normalized conditional evolution $\hat{\mathcal{T}}(t)\rho/\mathrm{Tr}_s[\hat{\mathcal{T}}(t)\rho] = \tilde{\rho}_t/\mathrm{Tr}_s[\tilde{\rho}_t]$. In each jump, the measurement transformation $\rho \to \mathcal{M}\rho = |-\rangle\langle -|$ applies [see

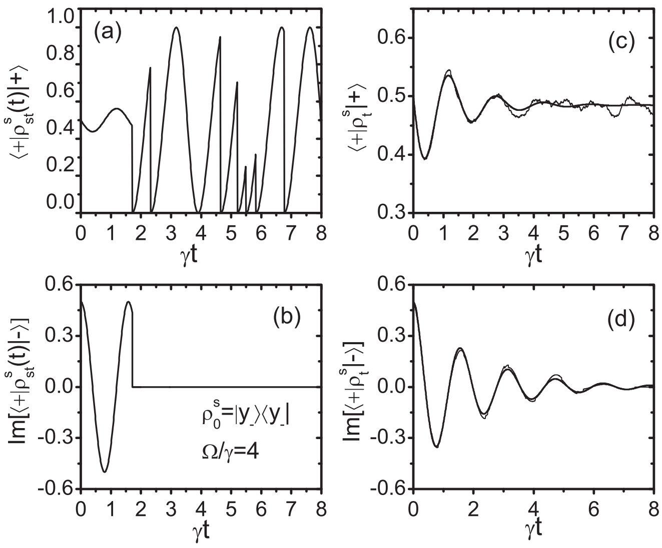

FIG. 2. Realizations of the stochastic density matrix $\rho_{\rm st}^s(t)$ and its ensemble average. In (a) and (b) are plotted the population $\langle+|\rho_{\rm st}^s(t)|+\rangle$ and the imaginary part of the coherence $\langle+|\rho_{\rm st}^s(t)|-\rangle$, respectively. In (c) and (d) are plotted an average over 2×10^3 realizations (noisy curves). The full lines correspond to the analytical solutions Eqs. (73) and (76). In all cases the initial system state is $\rho_0^s=|y_-\rangle\langle y_-|$, while the characteristic parameters satisfy $\Omega/\gamma=4$.

Eq. (63)]. These elements completely define the ensemble of trajectories associated with the stochastic density matrix $\rho_{\rm st}^s(t)$ (see Sec. III B).

In Fig. 2, a realization of $\rho_{\rm st}^s(t)$ is shown through the matrix elements $\langle+|\rho_{\rm st}^s(t)|+\rangle$ [upper population, Fig. 2(a)] and $\langle+|\rho_{\rm st}^s(t)|-\rangle$ [coherence, Fig. 2(b)]. The initial state is $\rho_{\rm st}^s(0)=|y_-\rangle\langle y_-|$. In the behavior of $\langle+|\rho_{\rm st}^s(t)|+\rangle$ it is possible to observe the successive jumps, where the system state collapses to the resetting state $|-\rangle\langle-|$, or equivalently, $\langle+|\rho_{\rm st}^s(t)|+\rangle\to0$. The conditional interevent behavior is periodic.

On the other hand, for the chosen initial condition the coherence $\langle+|\rho_{\rm st}^s(t)|-\rangle$ does not have a real component. Hence, from Fig. 2(b) we conclude that after the first event it dies out. This property follows from the resetting state defined by Eq. (63) and the fact that the conditional evolution [Eqs. (66) and (68)] does not couple the populations and coherences of the system. Notice that for the chosen parameter values an oscillatory behavior characterizes the conditional coherence dynamics.

In Figs. 2(c) and 2(d) we plot the population and coherence behaviors obtained by averaging 2×10^3 realizations (noisy curves). In addition we also show the curves corresponding to the exact solution of the density matrix evolution. For the chosen parameter values [$\gamma=\gamma'$ in Eq. (59)] it acquires the structure defined by Eq. (56). By introducing the matrix elements

$$p_t^{\pm}\equiv\langle\pm|\rho_t^s|\pm\rangle,\quad c_t^{\pm}\equiv\langle\pm|\rho_t^s|\mp\rangle, \tag{72}$$

the evolution of the population can be written as in Eq. (66) under the replacement $\tilde{p}_t^{\pm}\to p_t^{\pm}$ and taking $\tilde{\delta}=0$. Therefore, the populations are governed by a memorylike classical rate equation. The solution of these time-convoluted evolutions reads

$$p_t^+=\frac{\Omega^2}{\gamma^2+2\Omega^2}\left\{1+e^{-3\gamma t/4}\left[{\rm q}_c\cosh(\mu t)-{\rm q}_s\frac{\gamma}{\mu}\sinh(\mu t)\right]\right\}, \tag{73}$$

where

$$\mu=\sqrt{(\gamma/4)^2-\Omega^2}. \tag{74}$$

The dimensionless coefficients ${\rm q}_c$ and ${\rm q}_s$ introduce the dependence on the initial conditions,

$${\rm q}_c=p_0^+(\gamma^2/\Omega^2)+(p_0^+-p_0^-), \tag{75a}$$

$${\rm q}_s=[p_0^+(\gamma^2/\Omega^2)+(5p_0^++3p_0^-)]/4. \tag{75b}$$

The lower population follows as $p_t^-=1-p_t^+$. The evolution of the coherences can be written as in Eq. (68) after replacing $\tilde{c}_t^{\pm}\to c_t^{\pm}$. Their explicit solution is

$$c_t^+=e^{-\gamma t/2}\tfrac{1}{2}[a-b\cosh(\nu t)], \tag{76}$$

where the coefficients a and b read

$$a=[c_0^+\gamma^2/2-(c_0^++c_0^-)\Omega^2]/\nu^2, \tag{77a}$$

$$b=(c_0^+-c_0^-)\Omega^2/\nu^2. \tag{77b}$$

In Figs. 2(c) and 2(d), the analytical expressions for both the populations and coherences, Eqs. (73) and (76), respectively, recover the ensemble average behavior. This result explicitly shows the consistency of the proposed approach. On the other hand, Eqs. (73) and (76) lead to a diagonal stationary density matrix

$$\rho_\infty^s=\lim_{t\to\infty}\rho_t^s={\rm diag}\left\{\frac{\Omega^2}{\gamma^2+2\Omega^2},\frac{\gamma^2+\Omega^2}{\gamma^2+2\Omega^2}\right\}. \tag{78}$$

The evolution of the matrix elements (72) can also be rewritten in terms of the system density matrix ρ_t^s. From Eqs. (66) ($\tilde{\delta}\to0$, $\tilde{p}_t^{\pm}\to p_t^{\pm}$, $\tilde{c}_t^{\pm}\to c_t^{\pm}$) and (68) we find

$$\frac{d\rho_t^s}{dt}=\gamma\hat{\mathcal{C}}[\sigma]\rho_t^s+\sum_{i=x,y,z}\int_0^t dt' k_{t-t'}^i\hat{\mathcal{C}}[\sigma_i]\rho_{t'}^s, \tag{79}$$

where the Lindblad channels are defined by Eq. (2), σ_i, $i=x,y,z$, are the Pauli matrices, and the memory functions are

$$k_t^x=\frac{\Omega^2}{8}(e^{-\gamma t/2}+1)^2,\qquad -k_t^y=k_t^z=\frac{\Omega^2}{8}(e^{-\gamma t/2}-1)^2.$$

As expected, the density matrix evolution (79) has the structure defined by Eq. (56), where the local-in-time contribution is directly associated with the system transitions recorded by the measurement apparatus.

A. Genuine non-Markovian effects

Quantum non-Markovian time-convoluted master equations can always be rewritten in terms of local-in-time evolutions with time-dependent rates [21]. If the rates are positive at all times, the measurement dynamics is still consistent with a standard QJA [23]. On the other hand, if the rates assume negative values, the dynamics develops "genuine" non-Markovian effects such as an environment-to-system flow of information. This phenomenon can be detected through different measures [27], which in the Markovian case present a monotonic time decay behavior [8]. Now, we demonstrate

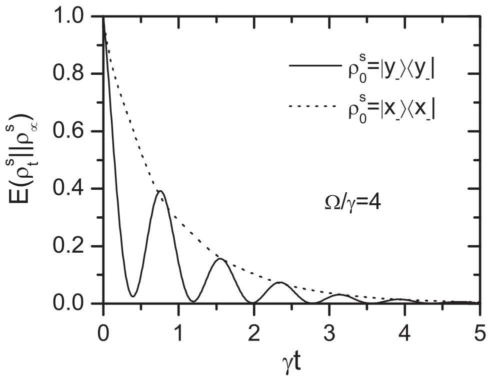

FIG. 3. Relative entropy with respect to the stationary state Eq. (80). The density matrix follows from Eq. (79). For the full line the initial condition is $\rho_0^s = |y_-\rangle\langle y_-|$, while for the dotted line it is $\rho_0^s = |x_-\rangle\langle x_-|$, where $|x_-\rangle = (1/\sqrt{2})(|+\rangle - |-\rangle)$. In both cases $\Omega/\gamma = 4$.

that this phenomenon can also arise in master equations such as Eqs. (56) and (57). As a measure we choose the relative entropy [8] with respect to the stationary state ($\rho_s^\infty = \lim_{t\to\infty} \rho_t^s$)

$$E\left(\rho_t^s||\rho_\infty^s\right) = \mathrm{Tr}_s\left[\rho_t^s\left(\ln_2 \rho_t^s - \ln_2 \rho_s^\infty\right)\right]. \quad (80)$$

In Fig. 3 the density matrix obeys the evolution (79), whose solution is defined by Eqs. (73) and (76). The solid line corresponds to the initial condition and parameter values of Figs. 1 and 2. Evidently, the oscillatory behavior of $E(\rho_t^s||\rho_\infty^s)$ demonstrates that (79) cannot be rewritten in terms of a local-in-time evolution with (time-dependent) positive rates. The same property arises by choosing the initial conditions $\rho_0^s = |\pm\rangle\langle\pm|$, in which case the system dynamics can be mapped with a classical two-level system. In general, the development or not of the revivals strongly depends on the initial conditions. For example, for $\rho_0^s = |x_-\rangle\langle x_-|$, where $|x_-\rangle$ is an eigenvector of σ_x with eigenvalue $-1[|x_-\rangle = (1/\sqrt{2})(|+\rangle - |-\rangle)]$, $E(\rho_t^s||\rho_\infty^s)$ decays in a monotonic way (dotted line). This case can be understood in terms of the symmetries of the underlying bipartite dynamics Eq. (59).

V. SUMMARY AND CONCLUSIONS

In this paper we established a non-Markovian generalization of the standard QJA. The underlying idea consists in embedding the system dynamics in a bipartite Markovian evolution [Eq. (24)]. Assuming that the measurement action is performed only on the system of interest, we demonstrated that there exist symmetry conditions on the Lindblad (bipartite) channels [Eqs. (42) and (49)] that lead to a closed system stochastic dynamics consistent with a quantum measurement theory.

For both renewal and nonrenewal measurement processes, the ensemble of realizations is similar to that of the standard case. At random times, the system state suffers a disruptive transformation, which is associated with each recording event. In the intermediate time intervals, the (conditional) system dynamic is smooth and nonunitary. The main difference from the standard approach is this last ingredient. Here, it is not defined by an exponential propagator [Eq. (34)]. In fact, it arises from a partial trace over the semigroup evolution associated with the Markovian bipartite dynamics [Eq. (31)]. Hence, in general, the stochastic dynamics does not admit an unraveling in terms of pure states [Eq. (35)].

As in the standard case, the jump statistics can be defined by a survival probability [Eq. (36)], which in general depends on the system state. In addition to the stochastic dynamics, we also characterized the system density matrix evolution. The structure of the corresponding non-Markovian quantum master equations is defined by Eqs. (56) and (57). Arbitrary master equations with this structure can be unraveled with the ensemble of trajectories if it is possible to assign a survival probability to the conditional dynamics [Eq. (58)].

The consistency of the formalism was checked by studying the dynamics of a two-level system whose non-Markovian dynamics leads to successive transitions between the upper and lower levels. The simplicity of the model allowed us to obtain short analytical expressions for the measurement statistics [Eq. (70)] as well as for the density matrix elements and the corresponding density matrix evolution [Eq. (79)]. The relevance of the example comes not only from its simplicity. In fact, it also allowed us to demonstrate that the present generalization is consistent with a backflow of information from the environment to the system. This property follows from the nonmonotonic decay of the relative entropy with respect to the stationary state (Fig. 3).

While the present formalism leads to a consistent non-Markovian generalization of the quantum jump approach, it is clear that it can be extended in different directions. For example, one may consider arbitrary initial bipartite states [Eq. (50)] or the introduction of nonseparable bipartite resetting states [Eq. (30)]. A less technical aspect should be to consider the case in which many different measurement apparatuses are monitoring the system or to determine which kind of consistent non-Markovian generalization is not covered by a Markovian embedding.

ACKNOWLEDGMENT

This work was supported by CONICET, Argentina, under Grant No. PIP 11420090100211.

APPENDIX A: QUANTUM JUMP STATISTICS—MARKOVIAN CASE

Here we derive the statistical description of the ensemble of realizations associated with the Markovian QJA. The solution of Eq. (3) can formally be written as

$$\rho_t^s = \exp[\hat{\mathcal{D}}t]\rho_0^s + \int_0^t dt' \exp[\hat{\mathcal{D}}(t-t')]\hat{\mathcal{J}}[\rho_{t'}^s], \quad (A1)$$

where ρ_0^s is the initial system state. This expression can be iterated, leading to the series expansion

$$\rho_t^s = \sum_{n=0}^{\infty} \rho_t^{(n)}, \quad (A2)$$

where each contribution satisfies the recursive relation

$$\rho_t^{(n)} = \int_0^t dt' \hat{\mathcal{T}}(t-t')\hat{\mathcal{J}}\rho_{t'}^{(n-1)}, \quad (A3)$$

with $\rho_t^{(0)} = \hat{\mathcal{T}}(t)\rho_0^s$. Therefore, it follows that ($n \geqslant 1$)

$$\rho_t^{(n)} = \int_0^t dt_n \cdots \int_0^{t_2} dt_1\, \hat{\mathcal{T}}(t-t_n)\hat{\mathcal{J}} \cdots \hat{\mathcal{T}}(t_2-t_1)\hat{\mathcal{J}}\hat{\mathcal{T}}(t_1)\rho_0^s. \tag{A4}$$

The superoperators $\hat{\mathcal{J}}$ and $\hat{\mathcal{T}}(t)$ are defined by Eqs. (4) and (8), respectively. Each contribution $\rho_t^{(n)}$ can be associated with trajectories with n detection events. Its statistics can be obtained by writing the previous expression in terms of the measurement transformation $\hat{\mathcal{M}}$ [Eq. (5)] and the normalized propagator $\hat{\mathcal{T}}_c(t)$ [Eq. (7)]. We get

$$\rho_t^{(n)} = \int_0^t dt_n \cdots \int_0^{t_2} dt_1 P_n[t,\{t_i\}_1^n] \times \hat{\mathcal{T}}_c(t-t_n)\hat{\mathcal{M}} \cdots \hat{\mathcal{T}}_c(t_2-t_1)\hat{\mathcal{M}}\hat{\mathcal{T}}_c(t_1)\rho_0^s \tag{A5}$$

($n \geqslant 1$) and $\rho_t^{(0)} = P_0(t|\rho_0^s)\hat{\mathcal{T}}_c(t)\rho_0^s$. The function

$$P_n\big[t,\{t_i\}_1^n\big] = \mathrm{Tr}_s\big[\hat{\mathcal{T}}(t-t_n)\hat{\mathcal{J}} \cdots \hat{\mathcal{J}}\hat{\mathcal{T}}(t_2-t_1)\hat{\mathcal{J}}\hat{\mathcal{T}}(t_1)\rho_0^s\big] \tag{A6}$$

is the joint probability density for observing measurement events at times $\{t_i\}_1^n$. It completely characterizes the statistics of the measurement process. By introducing the auxiliary states $\rho_{t_{i+1}} = \hat{\mathcal{T}}_c(t_{i+1},t_i)\hat{\mathcal{M}}\rho_{t_i}$, with $\rho_{t_1} = \hat{\mathcal{T}}_c(t_1,0)\rho_0$, the previous object can be rewritten as

$$P_n\big[t,\{t_i\}_1^n\big] = P_0(t-t_n|\hat{\mathcal{M}}\rho_{t_n}) \times \prod_{j=2}^{n} w(t_j-t_{j-1}|\hat{\mathcal{M}}\rho_{t_{j-1}})w\big(t_1|\rho_0^s\big), \tag{A7}$$

where the survival probability $P_0(t|\rho)$ and the waiting-time distribution $w(t|\rho)$ are defined by Eqs. (9) and (10), respectively.

The structures of both Eqs. (A5) and (A7) are consistent with the stochastic dynamics defined in Sec. II A. The second line of Eq. (A5) consists in successive applications of the measurement transformations $\hat{\mathcal{M}}$ and intermediate evolution with the propagator $\hat{\mathcal{T}}_c(t)$. On the other hand, the weight of each realization, defined by Eq. (A7), has the same structure as a renewal process, that is, there exists a probability distribution (waiting-time distribution) that defines the statistics of the time interval between consecutive detection events. Nevertheless, here the distribution depends on the resetting state, that is, the state after a measurement event.

APPENDIX B: NON-MARKOVIAN MASTER EQUATIONS FROM THE JUMP STATISTICS

We derived the non-Markovian extension of the QJA by studying the standard approach in a Markovian bipartite dynamics. Under the conditions obtained in Sec. III the system's stochastic dynamics becomes closed, that is, it can be written without taking into account explicitly the ancilla dynamics. Here we derive the corresponding non-Markovian master equation [see Eqs. (56) and (57)] by averaging the ensemble of trajectories.

The full counting statistics can be derived from the Markovian evolution Eq. (26) and its formal solution (51). All calculation steps described in Appendix A can be extended, after a trivial change of notation ($\hat{\mathcal{J}} \to \hat{\mathbb{J}}$, $\hat{\mathcal{T}} \to \hat{\mathbb{T}}$), to the bipartite evolution defined in terms of ρ_t^{sa}. By performing a partial trace over the ancilla degrees of freedom on the corresponding expressions, by using the bipartite measurement transformation (30) and the initial bipartite state (50), it is possible to demonstrate that Eqs. (A5) and (A7) are also valid for the non-Markovian system dynamics. Nevertheless, in the non-Markovian case, the propagator $\hat{\mathcal{T}}(t)$ is defined by Eq. (31) [or equivalently Eq. (34)] while the survival probability $P_0(t|\rho)$ and waiting-time distribution $w(t|\rho)$ from Eqs. (36) and (38), respectively.

1. Renewal case

When the measurement process is a renewal one, we can write the joint probability density [Eq. (A7)] as

$$P_n\big[t,\{t_i\}_1^n\big] = P_0(t-t_n)\prod_{j=2}^{n} w(t_j-t_{j-1})w\big(t_1|\rho_0^s\big), \tag{B1}$$

where, in contrast to a Markovian renewal process, here the survival probability $P_0(t)$ and waiting-time distribution $w(t)$ are defined by Eqs. (46) and (47), respectively. From Eq. (A5) and by using the renewal property Eq. (18), the previous expression for $P_n[t,\{t_i\}_1^n]$ allows us to write

$$\rho_t^{(n)} = \int_0^t dt' \hat{\mathcal{T}}(t-t')\bar{\rho}_s\, f^{(n)}(t') \tag{B2}$$

[$\rho_t^{(0)} = \hat{\mathcal{T}}(t)\rho_0^s$], where the function $f^{(n)}(t)$ is defined as

$$f^{(n)}(t) = \int_0^t dt_n \cdots \int_0^{t_2} dt_1 \prod_{j=2}^{n} w(t_j-t_{j-1})w\big(t_1|\rho_0^s\big). \tag{B3}$$

From Eq. (B2) and the expression for the waiting-time distribution $w(t)$ [Eq. (47)], we get the recursive relation

$$\rho_t^{(n)} = -\int_0^t dt' \hat{\mathcal{T}}(t-t')\bar{\rho}_s \int_0^{t'} dt'' \mathrm{Tr}_s\big[\hat{\mathcal{D}}(t'-t'')\rho_{t''}^{(n-1)}\big]. \tag{B4}$$

By adding all these states [see Eq. (A2)] and by using the non-Markovian time evolution of the propagator $\hat{\mathcal{T}}(t)$ [Eq. (34)], after some calculation steps, the system density matrix evolution Eq. (57) is recovered.

2. Nonrenewal case

By using the rate condition Eq. (49) corresponding to the nonrenewal case, it is possible to demonstrate that the superoperator $\hat{\mathbb{J}}$ [Eq. (27)] satisfies the relation

$$\hat{\mathbb{J}}\rho = \hat{\mathcal{J}}\{\mathrm{Tr}_a[\rho]\} \otimes \bar{\rho}_a, \tag{B5}$$

where the system superoperator $\hat{\mathcal{J}}$ is defined by Eq. (4) and the ancilla resetting state $\bar{\rho}_a$ follows from Eq. (44). By writing Eq. (A6) in terms of bipartite objects ($\hat{\mathcal{J}} \to \hat{\mathbb{J}}$, $\hat{\mathcal{T}} \to \hat{\mathbb{T}}$), after introducing Eq. (B5), the joint probability distribution can be written as

$$P_n\big[t,\{t_i\}_1^n\big] = \mathrm{Tr}_s\big[\hat{\mathcal{T}}(t-t_n)\hat{\mathcal{J}} \cdots \hat{\mathcal{J}}\hat{\mathcal{T}}(t_2-t_1)\hat{\mathcal{J}}\hat{\mathcal{T}}(t_1)\rho_0^s\big], \tag{B6}$$

where $\hat{\mathcal{T}}(t)$ and $\hat{\mathcal{J}}$ follow from Eqs. (31) and (4), respectively. Notice that in this case, the only difference with the Markovian case [Eq. (A6)] is the definition of $\hat{\mathcal{T}}(t)$.

In order to obtain the density matrix evolution we need a recursive relation for the states $\rho_t^{(n)}$. Here, this kind of relation can be easily obtained from the recursive relation (A3) when applied to the bipartite dynamics. With the aid of Eq. (B5) we get

$$\rho_t^{(n)} = \int_0^t dt' \hat{\mathcal{T}}(t-t')\hat{\mathcal{J}}\rho_{t'}^{(n-1)}. \tag{B7}$$

Consistently, the same relation arises from Eqs. (A5) and (B6). By adding all states $\rho_t^{(n)}$, and by using the non-Markovian time evolution of the propagator $\hat{\mathcal{T}}(t)$ [Eq. (34)], we recover Eq. (56).

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