Measuring the canonical phase with phase-space measurements

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(Received 8 March 2013; revised manuscript received 15 May 2013; published 22 July 2013)

We study measurements of single-mode phase observables with emphasis on the canonical phase. We focus on the class of phase-shift covariant phase-space observables, which then yield the phase observables as the angle margins after integrating over the radial part. We consider the possibility of measuring such observables by using a double homodyne detector and its modification. We show that, in principle, the canonical phase distribution of the signal state can be measured via double homodyne detection by first processing the state using a two-mode unitary channel.

DOI: 10.1103/PhysRevA.88.012121 PACS number(s): 03.65.Ta, 42.50.−p

I. INTRODUCTION

In quantum optics, the concept of phase for a single-mode electromagnetic field has remained a somewhat controversial topic. Alternative descriptions for phase observables have been developed, and hundreds of articles and several monographs have been written on the subject since Dirac's famous paper [1] published in 1927 (see, e.g., [2–7] and references therein). A major reason for this variety of phase theories is that in trying to define the phase of a quantum oscillator, one encounters the restrictions of the conventional approach which identifies observables with self-adjoint operators or, equivalently, their spectral measures. In fact, it is well known that no spectral measure satisfies all physically relevant conditions posed on phase observables (see, e.g., [4,8]). This problem has been overcome with the introduction of the more general concept of observables as (normalized) positive operator valued measures (POVMs).

A natural requirement for the description of a phase measurement is *covariance* with respect to phase shifts. In other words, the application of a phase shifter on the field prior to the measurement should only shift the phase distribution without changing its shape. Although there exists an infinite number of phase-shift covariant POVMs, it is generally accepted that the canonical phase measurement for the single-mode radiation field is represented by the London phase distribution [9]. Hence, the canonical phase measurement is described by the canonical phase observable $\mathsf{E}_{\rm can}$,

$$\mathsf{E}_{\rm can}(X) = \sum_{m,n=0}^{\infty} \frac{1}{2\pi} \int_X e^{i(m-n)\theta} d\theta\, |m\rangle\langle n| = \frac{1}{2\pi}\int_X |\theta\rangle\langle\theta| d\theta,$$

where $X \subseteq [0,2\pi)$, $|m\rangle$ are the number states, and $|\theta\rangle = \sum_{m=0}^{\infty} e^{im\theta}|m\rangle$ is the (formal) Susskind-Glogower phase state [10]. We recall that the canonical phase measurement arises as the limiting distribution of the Pegg-Barnett formalism [3]. Additionally, $\mathsf{E}_{\rm can}$ has been independently derived by Helstrom [11] and Holevo [8] in the more general context of quantum estimation theory.

The canonical phase has a number of properties which makes it an optimal choice among other phase observables. For instance, $\mathsf{E}_{\rm can}$ is pure, i.e., an extreme point of the convex set of all POVMs [12]. Any other covariant phase observable F is connected to the canonical phase via a quantum channel Φ as $\mathsf{F}(X) = \Phi^*(\mathsf{E}_{\rm can}(X))$ [13]. The canonical phase is also (essentially) the only covariant phase observable which generates number shifts [14]. Furthermore, $\mathsf{E}_{\rm can}$ and the photon number N are noncoexistent, probabilistically, and value-complementary observables [15]. Finally, the canonical phase distribution of coherent states $|\alpha\rangle$, $\alpha \in \mathbb{C}$, tends to a Dirac δ distribution in the classical limit $|\alpha| \to \infty$ [14]. A list of further properties of $\mathsf{E}_{\rm can}$ can be found, e.g., in [4] (p. 51).

The problem of finding a suitable realistic measurement model for the canonical phase observable is the last big open problem concerning the quantum description of the phase of a single-mode electromagnetic field. Of course, $\mathsf{E}_{\rm can}$ (or any observable of the field) can be "measured" indirectly or "sampled" by first performing quantum state tomography and then constructing the canonical phase distribution. However, such an approach may hardly be regarded as a direct measurement in the spirit of the quantum theory of measurement [16]. Some suggestions for direct measurements of $\mathsf{E}_{\rm can}$ can be found in the literature (see, e.g., [17]), but they have not led to experiments.

It seems that from the experimental point of view, the most easily accessible phase observables are those arising as the angle margins of certain *translation covariant* phase-space observables. Such observables are often referred to as phase-space phase observables, with the most familiar example being the angle margin of the Husimi Q function of the field [18]. A natural measurement scheme for these phase-space measurements is then double homodyne detection (see, e.g., [19]), also known as eight-port homodyne detection, which has also been demonstrated experimentally [20]. Even though the phase-space phase observables seem like a natural choice for measuring the phase of the electromagnetic field, they suffer from certain drawbacks when compared with the canonical phase. For instance, the phase-space phase observables are not pure [21], and the canonical phase $\mathsf{E}_{\rm can}$ gives smaller

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(minimum) variance in large-amplitude coherent states than any phase-space phase observable [14].

The purpose of this paper is to take a step in the direction of obtaining a realistic measurement scheme for the canonical phase. First, in Sec. II, we give the basic framework for our study. We also present the minimal measurement models for the canonical phase. By noting the practical shortcomings of such models, we are led to introduce the class of *phase-shift covariant* phase-space observables in Sec. III. We study the marginal properties of these observables and present some relevant examples, including cases which give the canonical phase as the margin. In Sec. IV, we then study the double homodyne detection scheme as a means of measuring phase-space observables. We show that by modifying the setup by adding an extra two-mode unitary coupling in front of the apparatus, it is, in principle, possible to measure the canonical phase of the signal field. The conclusions are presented in Sec. V.

II. PRELIMINARIES

Throughout this paper, we consider a single-mode electromagnetic field as our physical system. The Hilbert space of the system is therefore $\mathcal{H} \simeq L^2(\mathbb{R})$, and it is spanned by the orthonormal basis consisting of the number states $|n\rangle$, $n \in \mathbb{N} = \{0,1,2,\ldots\}$, which in the coordinate representation are identified with the Hermite functions. We denote by a^*, a, and $N = a^*a$ the creation, annihilation, and number operators, respectively, related to this basis. We denote by $\mathcal{L}(\mathcal{H})$ and $\mathcal{T}(\mathcal{H})$ the sets of bounded and trace class operators, respectively, on $\mathcal{H}$. The states of the field are represented by positive trace class operators $\rho \in \mathcal{T}(\mathcal{H})$ with unit trace, i.e., density operators, and the observables are represented by normalized positive operator valued measures (POVMs) $\mathsf{E} : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{H})$ on the Borel σ algebra of the topological space Ω of possible measurement outcomes [22]. We say that an observable is sharp if it is projection valued, that is, a spectral measure. The measurement outcome probabilities are given by the probability measures $X \mapsto \mathrm{tr}[\rho\mathsf{E}(X)]$. For our purposes, the relevant measurement outcome spaces are the torus $\mathbb{T}$, which is identified with $[0,2\pi)$ with addition modulo 2π, the space of non-negative real numbers $\mathbb{R}_+$, and the phase space $\mathbb{C} \simeq \mathbb{R}^2 \simeq \mathbb{R}_+ \times \mathbb{T}$.

It is a fundamental result of the quantum theory of measurement that any observable has a measurement realization in the form of a measurement dilation [23]. Indeed, given a POVM $\mathsf{E} : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{H})$, there exists a Hilbert space $\mathcal{K}$, a unit vector $\xi \in \mathcal{K}$, a unitary operator $U : \mathcal{H}\otimes\mathcal{K} \to \mathcal{H}\otimes\mathcal{K}$, and a sharp observable $\mathsf{Z} : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{K})$ such that

$$\mathrm{tr}[\rho\mathsf{E}(X)] = \mathrm{tr}[U(\rho \otimes |\xi\rangle\langle\xi|)U^* I \otimes \mathsf{Z}(X)] \tag{1}$$

for all $X \in \mathcal{B}(\Omega)$ and all states ρ. From the physical point of view, this tells us that E can be measured by first preparing a probe system into a pure state $|\xi\rangle\langle\xi|$, coupling the object system with the probe via the unitary coupling U, and then performing a measurement of the sharp pointer observable Z on the probe system. The problem with constructing a realistic measurement model for a given observable is therefore twofold: first, one needs to find the mathematical components which reproduce the probabilities (1), and, second, one needs to find a way to realize these components in a laboratory.

We now turn our attention to the problem of covariant phase observables.

Definition 1. An observable $\mathsf{E} : \mathcal{B}([0,2\pi)) \to \mathcal{L}(\mathcal{H})$ is a *covariant phase observable* if

$$e^{i\theta N}\mathsf{E}(\Theta)e^{-i\theta N} = \mathsf{E}(\Theta \dot{+} \theta) \tag{2}$$

for all $\Theta \in \mathcal{B}([0,2\pi))$ and $\theta \in [0,2\pi)$, where $\dot{+}$ denotes addition modulo 2π.

Condition (2) already sets the general structure of the observables. Indeed, if E is a covariant phase observable, then by the structure theorem for phase observables (see, e.g., [24]) there exists a unique positive semidefinite matrix $(c_{mn})_{m,n\in\mathbb{N}}$ with unit diagonal (i.e., $c_{mm} \equiv 1$) such that

$$\mathsf{E}(\Theta) = \sum_{m,n=0}^{\infty} c_{mn}\frac{1}{2\pi}\int_{\Theta} e^{i\theta(m-n)}d\theta\, |m\rangle\langle n|$$

for all $\Theta \in \mathcal{B}([0,2\pi))$. We say that $(c_{mn})_{m,n\in\mathbb{N}}$ is the *phase matrix* associated to E. Clearly, the constant phase matrix $c_{mn} \equiv 1$ corresponds to the canonical phase $\mathsf{E}_{\mathrm{can}}$.

Using the results of [25], it is possible to calculate the minimal measurement models for an arbitrary phase observable. The minimality here means that the auxiliary space is, in some sense, as "small" as possible, meaning that there are no unnecessary degrees of freedom present. In the minimal measurement model for the canonical phase (up to unitary equivalence), the Hilbert space of the probe is $\mathcal{K} = L^2([0,2\pi))$, the pointer observable is the canonical spectral measure $\mathsf{T} : \mathcal{B}([0,2\pi)) \to \mathcal{L}(\mathcal{K})$, $(\mathsf{T}(\Theta)\psi)(\theta) = \chi_\Theta(\theta)\psi(\theta)$, where χ_Θ is the characteristic function of Θ, and the unitary coupling can be constructed by extending the map $[U(|n\rangle \otimes \xi)](\theta) = \frac{1}{\sqrt{2\pi}}\psi_\theta \otimes e^{-i\theta n}$ where $\xi \in \mathcal{K}$ is a fixed unit vector determining the initial pure state of the probe and $\psi_\theta \in \mathcal{H}$ are unit vectors.

III. PHASE-SHIFT COVARIANT PHASE-SPACE OBSERVABLES

A major practical problem with the minimal measurement model is the realization of the pointer observable T. However, things get much simpler if we instead consider T as the angle margin of the canonical spectral measure M on $\mathbb{C}$, i.e., $\mathsf{M} : \mathcal{B}(\mathbb{C}) \to \mathcal{L}(\mathcal{K}')$ where $\mathcal{K}' = L^2(\mathbb{C}) \simeq L^2(\mathbb{R}_+) \otimes L^2([0,2\pi))$ and $[\mathsf{M}(X \times \Theta)\Psi](r,\theta) = \chi_{X\times\Theta}(r,\theta)\Psi(r,\theta)$. This would then correspond to interpreting T as the observable $X \mapsto I \otimes \mathsf{T}(X)$ on the larger Hilbert space. In this case, $L^2(\mathbb{C})$ may be simply realized as the Hilbert space of a two-mode field, whereas M, and therefore T, can be measured by performing homodyne detection on the two modes. The resulting system observable is then a phase-space observable $\mathsf{P} : \mathcal{B}(\mathbb{C}) \to \mathcal{L}(\mathcal{H})$.

For any observable $\mathsf{P} : \mathcal{B}(\mathbb{C}) \to \mathcal{L}(\mathcal{H})$, we define the angle and radial margins $\mathsf{P}_{\mathrm{angle}} : \mathcal{B}([0,2\pi)) \to \mathcal{L}(\mathcal{H})$ and $\mathsf{P}_{\mathrm{rad}} : \mathcal{B}(\mathbb{R}_+) \to \mathcal{L}(\mathcal{H})$ via

$$\mathsf{P}_{\mathrm{angle}}(\Theta) = \mathsf{P}(\mathbb{R}_+ \times \Theta), \quad \mathsf{P}_{\mathrm{rad}}(X) = \mathsf{P}(X \times [0,2\pi)).$$

Any phase observable E can be obtained as the angle margin of a phase-space observable in a trivial manner. Namely, for any probability measure $\mu : \mathcal{B}(\mathbb{R}_+) \to [0,1]$, the observable

$\mathsf{P}:\mathcal{B}(\mathbb{C})\to\mathcal{L}(\mathcal{H})$ for which $\mathsf{P}(X\times\Theta)=\mu(X)\mathsf{E}(\Theta)$ is a suitable choice. This observable has the additional property that it transforms covariantly under the action of the phase shifter unitary, i.e.,

$$\begin{aligned}e^{i\theta N}\mathsf{P}(X\times\Theta)e^{-i\theta N}&=\mu(X)e^{i\theta N}\mathsf{E}(\Theta)e^{-i\theta N}\\&=\mu(X)\mathsf{E}(\Theta\dot{+}\theta)=\mathsf{P}(X\times(\Theta\dot{+}\theta)).\end{aligned}$$

We will now focus our attention on phase-space observables having this symmetry property.

Definition 2. An observable $\mathsf{P}:\mathcal{B}(\mathbb{C})\to\mathcal{L}(\mathcal{H})$ is a *phase-shift covariant phase-space observable* if

$$e^{i\theta N}\mathsf{P}(X\times\Theta)e^{-i\theta N}=\mathsf{P}(X\times(\Theta\dot{+}\theta))$$

for all $X\in\mathcal{B}(\mathbb{R}_+)$, $\Theta\in\mathcal{B}([0,2\pi))$, and $\theta\in[0,2\pi)$.

The following proposition then gives the general structure of such observables.

Proposition 1. An observable $\mathsf{P}:\mathcal{B}(\mathbb{C})\to\mathcal{L}(\mathcal{H})$ is a phase-shift covariant phase-space observable if and only if there exists a probability measure $\mu:\mathcal{B}(\mathbb{R}_+)\to[0,1]$ and (a weakly μ-measurable field of) vectors $\eta_m(x)\in\mathcal{H}$ satisfying the condition

$$\int_0^\infty\|\eta_m(x)\|^2d\mu(x)=1$$

such that

$$\begin{aligned}\mathsf{P}(X\times\Theta)=\sum_{m,n=0}^\infty\int_{X\times\Theta}&\langle e^{-im\theta}\eta_m(x)|e^{-in\theta}\eta_n(x)\rangle\\&\times\frac{d\theta}{2\pi}d\mu(x)|m\rangle\langle n|\end{aligned}\qquad(3)$$

for all $X\in\mathcal{B}(\mathbb{R}_+)$ and $\Theta\in\mathcal{B}([0,2\pi))$.

Proof. Consider a fixed set $X\in\mathcal{B}(\mathbb{R}_+)$. Since $e^{i\theta N}\mathsf{P}(X\times\Theta)e^{-i\theta N}=\mathsf{P}(X\times(\Theta\dot{+}\theta))$ for all $\Theta\in\mathcal{B}([0,2\pi))$, following the proof of the structure theorem for covariant phase observables [24], there exists a positive semidefinite matrix $(c_{mn}(X))_{m,n\in\mathbb{N}}$ such that

$$\mathsf{P}(X\times\Theta)=\sum_{m,n=0}^\infty c_{mn}(X)\,\frac{1}{2\pi}\int_\Theta e^{i(m-n)\theta}d\theta|m\rangle\langle n|.$$

Now, for all $m,n\in\mathbb{N}$, the map $X\mapsto c_{mn}(X)$ is a complex measure which is absolutely continuous with respect to the probability measure

$$\begin{aligned}X\mapsto\mu(X)&=\sum_{n=0}^\infty\lambda_n\langle n|\mathsf{P}(X\times[0,2\pi))|n\rangle\\&=\sum_{n=0}^\infty\lambda_nc_{nn}(X),\end{aligned}$$

where $\lambda_n>0$ for all n and $\sum_n\lambda_n=1$. It follows from [26] that there exist vectors $\eta_m(x)\in\mathcal{H}$ such that

$$c_{mn}(X)=\int_X\langle\eta_m(x)|\eta_n(x)\rangle d\mu(x).\qquad(4)$$

Hence, Eq. (3) holds. The converse claim is clearly true. ■

Using Prop. 1, we can now determine the angle and radial margins of any phase-shift covariant phase-space observable. Indeed, by setting $X=\mathbb{R}_+$ in Eq. (3), we obtain

$$\mathsf{P}_{\rm angle}(\Theta)=\sum_{m,n=0}^\infty c_{mn}\,\frac{1}{2\pi}\int_\Theta e^{i\theta(m-n)}\,d\theta|m\rangle\langle n|,$$

where the phase matrix elements are given by

$$c_{mn}=\int_0^\infty\langle\eta_m(x)|\eta_n(x)\rangle d\mu(x).$$

Similarly, the radial margin is seen to be

$$\mathsf{P}_{\rm rad}(X)=\sum_{m=0}^\infty c_{mm}(X)|m\rangle\langle m|,$$

where $c_{mm}(X)$ is as in Eq. (4).

The map $(X,m)\mapsto c_{mm}(X)$ is a Markov kernel on $\mathcal{B}(\mathbb{R}_+)\times\mathbb{N}$, and the observable $\mathsf{P}_{\rm rad}$ can therefore be viewed as a smeared number observable. More generally, we say that an observable $\mathsf{F}:\mathcal{B}(\Omega)\to\mathcal{L}(\mathcal{H})$ is a smeared number observable if there exists a Markov kernel $m:\mathcal{B}(\Omega)\times\mathbb{N}\to[0,1]$ such that

$$\mathsf{F}(X)=\sum_{n=0}^\infty m(X,n)|n\rangle\langle n|$$

for all $X\in\mathcal{B}(\Omega)$. The next proposition shows that among phase-space observables, the ones possessing the property of phase-shift covariance are, in a sense, archetypes of joint observables for phase and smeared photon number.

Proposition 2. Let $\mathsf{F}:\mathcal{B}(\mathbb{R}_+)\to\mathcal{L}(\mathcal{H})$ be a smeared number observable and let $\mathsf{E}:\mathcal{B}([0,2\pi))\to\mathcal{L}(\mathcal{H})$ be a covariant phase observable. Then, E and F have a joint observable if and only if they have a joint observable which is a phase-shift covariant phase-space observable.

Proof. Assume that E and F have a joint observable $\mathsf{M}:\mathcal{B}(\mathbb{C})\to\mathcal{L}(\mathcal{H})$ and define the biobservable $\mathsf{P}':\mathcal{B}(\mathbb{R}_+)\times\mathcal{B}([0,2\pi))\to\mathcal{L}(\mathcal{H})$ via

$$\mathsf{P}'(X,\Theta)=\frac{1}{2\pi}\int e^{-i\theta N}\mathsf{M}(X\times(\Theta\dot{+}\theta))e^{i\theta N}d\theta,\qquad(5)$$

where the integral is understood in the weak* sense. By [27, Theorem 6.1.5], P' extends to a unique observable $\mathsf{P}:\mathcal{B}(\mathbb{C})\to\mathcal{L}(\mathcal{H})$ which is clearly phase-shift covariant. By setting $\Theta=[0,2\pi)$ (respectively, $X=\mathbb{R}_+$) in Eq. (5) and using the phase-shift invariance of F (respectively, phase-shift covariance of E), we see that $\mathsf{P}_{\rm rad}=\mathsf{F}$ (respectively, $\mathsf{P}_{\rm angle}=\mathsf{E}$). Hence, E and F have a joint observable which is a phase-shift covariant phase-space observable. The converse statement is trivial. ■

We note that the canonical phase observable is obtained as the angle margin of a phase-shift covariant phase-space observable if and only if

$$\langle\eta_m|\eta_n\rangle=\int_0^\infty\langle\eta_m(x)|\eta_n(x)\rangle d\mu(x)=1$$

for all $m,n\in\mathbb{N}$. In particular, we must have $\langle\eta_0|\eta_n\rangle=1$ for all $n\in\mathbb{N}$ and, since η_n is a unit vector, this implies that $\eta_n=\eta_0$ for all $n\in\mathbb{N}$. But in such a case the radial margin is the trivial observable $\mathsf{P}_{\rm rad}(X)=c_{00}(X)I$. This is no surprise, since the canonical phase observable is in this sense exclusive: any observable which is jointly measurable with the canonical phase is necessarily a postprocessing of it (for a proof, see Appendix). This means that given such an observable $\mathsf{F}:\mathcal{B}(\Omega)\to\mathcal{L}(\mathcal{H})$, there exists a (weak) Markov kernel $m:\mathcal{B}(\Omega)\times[0,2\pi)\to[0,1]$ such that

$$\mathsf{F}(X)=\int m(X,\theta)d\mathsf{E}_{\rm can}(\theta).$$

A simple calculation then shows that in the case of the smeared number observable, this forces the observable to be trivial, i.e., $m(X,n)$ cannot depend on n.

A similar situation happens if we insist that the radial margin is the sharp number observable. In this case, we have $c_{mm}(X)=\delta_m(X)$ (the Dirac measure concentrated on m) so that, in particular, $\int_{\{m\}}\|\eta_m(x)\|^2\,d\mu(x)=1$ and we must have $\eta_m(x)=0$ for μ-almost all $x\in\mathbb{R}_+\setminus\{m\}$. It follows that $c_{mn}=\langle\eta_m|\eta_n\rangle=\delta_{mn}$ and the angle margin is therefore the trivial one: $\mathsf{P}_{\rm angle}(\Theta)=\frac{1}{2\pi}\int_\Theta d\theta I$. This also follows from the fact that any observable which is jointly measurable with the number observable must commute with it by [28, Theorem 1.3.1]. For phase observables, this is equivalent to the observable being trivial.

We will now proceed to give some examples of phase-shift covariant phase-space observables.

Example 1. As a first example, we consider phase-space observables which, in addition to phase-shift covariance, are covariant with respect to phase-space translations represented by the displacement operators $D(z)=e^{za^*-\bar{z}a}$, $z\in\mathbb{C}$. Generally, any phase-space POVM G which satisfies the covariance condition $D(z)\mathsf{G}(Z)D(z)^*=\mathsf{G}(Z+z)$, $z\in\mathbb{C}$, $Z\in\mathcal{B}(\mathbb{C})$, is generated by a unique positive trace one operator T, giving the corresponding observable the explicit form [29,30]

$$\mathsf{G}^T(Z)=\frac{1}{\pi}\int_Z D(z)TD(z)^*d^2z. \tag{6}$$

However, not all of these observables are covariant with respect to phase shifts. Indeed, this is the case if and only if the generating operator is a mixture of number states, $T=\sum_{k=0}^\infty\lambda_k|k\rangle\langle k|$; see [24]. In order to connect these observables to the general structure discussed above, let us consider $\mathsf{G}^{|k\rangle}$, the observable generated by a single number state $|k\rangle\langle k|$. First note that we may write Eq. (6) as

$$\mathsf{G}^{|k\rangle}(X\times\Theta)=\sum_{m,n=0}^\infty\frac{1}{2\pi}\int_{X\times\Theta}\langle m|D(re^{i\theta})|k\rangle \times\overline{\langle n|D(re^{i\theta})|k\rangle}\,dr^2d\theta|m\rangle\langle n|, \tag{7}$$

where the matrix elements of the displacement operators are given by

$$\langle m|D(re^{i\theta})|k\rangle=(-1)^{\max\{0,k-m\}}\sqrt{\frac{(\min\{m,k\})!}{(\max\{m,k\})!}}e^{i\theta(m-k)} \times r^{|m-k|}L^{|m-k|}_{\min\{m,k\}}(r^2)e^{-\frac{r^2}{2}} \tag{8}$$

and L_n^α is the associated Laguerre polynomial. We can then define the probability measure $\mu:\mathcal{B}(\mathbb{R}_+)\to[0,1]$ via $d\mu(r)=e^{-r^2}dr^2$, as well as the vectors

$$\eta_m^k(r)=(-1)^{\max\{0,k-m\}}\sqrt{\frac{(\min\{m,k\})!}{(\max\{m,k\})!}} \times r^{|m-k|}L^{|m-k|}_{\min\{m,k\}}(r^2)\,\varphi_k\in\mathcal{H},$$

where $\varphi_k\in\mathcal{H}$ is an arbitrary fixed unit vector. The normalization condition $\int\|\eta_m^k(r)\|^2\,d\mu(r)=1$ is then satisfied, and it is merely a simple observation that $\mathsf{G}^{|k\rangle}$ takes the form of Eq. (3). Note that the angle margin $\mathsf{G}^{|k\rangle}_{\rm angle}$ is never the canonical phase observable [14,24].

Example 2. A second example can be obtained by modifying the above considerations. Indeed, if we replace the vectors $\eta_m^k(r)$ by $\xi_m^k(r)=L_{\min\{m,k\}}(r^2)\varphi_k$ while keeping the measure μ unchanged, we get an observable $\mathsf{F}^{|k\rangle}:\mathcal{B}(\mathbb{C})\to\mathcal{L}(\mathcal{H})$ with the explicit form

$$\mathsf{F}^{|k\rangle}(X\times\Theta)=\sum_{m,n=0}^\infty\frac{1}{2\pi}\int_{X\times\Theta}e^{i\theta(m-n)}L_{\min\{m,k\}}(r^2) \times L_{\min\{n,k\}}(r^2)\,e^{-r^2}dr^2d\theta|m\rangle\langle n|. \tag{9}$$

Unlike the observable $\mathsf{G}^{|k\rangle}$, this observable is not translation covariant, though it is clearly covariant with respect to phase shifts. It should also be noted that in this case there do not exist bounded (or unitary) operators $\widetilde{D}(re^{i\theta})$ such that $\langle n|\widetilde{D}(re^{i\theta})^*|m\rangle=e^{i\theta(n-m)}L_{\min\{m,n\}}(r^2)e^{-\frac{r^2}{2}}$ for all r and θ.

The significant features of these observables are in their margins. For the angle margin phase observable $\mathsf{F}^{|k\rangle}_{\rm angle}$, we get the phase matrix elements

$$c_{mn}=\int_0^\infty L_{\min\{m,k\}}(r^2)L_{\min\{n,k\}}(r^2)\,e^{-r^2}dr^2 =\delta_{\min\{m,k\},\,\min\{n,k\}}.$$

In particular, the angle margin of $\mathsf{F}^{|0\rangle}$ is the canonical phase. On the contrary, if we increase the value of k, we find that the observable becomes in some sense more trivial. Indeed, since $c_{mn}=\delta_{mn}$ for $m,n\leqslant k$, the phase matrix always contains a $(k+1)\times(k+1)$ identity matrix in the upper left corner. As for the radial margin, we get

$$\mathsf{F}^{|k\rangle}_{\rm rad}(X)=\sum_{m=0}^k c_{mm}(X)|m\rangle\langle m|+c_{kk}(X)\sum_{m=k+1}^\infty|m\rangle\langle m|,$$

which shows that by increasing the value of k, the observable becomes, vaguely speaking, more and more nontrivial. We also immediately recognize the obvious fact that for $k=0$ corresponding to the angle margin being the canonical phase, the radial margin is trivial.

Example 3. As a final example, we consider the case where the probability measure is a Dirac measure $\mu=\delta_{r_0}$ concentrated at some point $r_0>0$. The corresponding phase-space observable $\mathsf{P}^{r_0}:\mathcal{B}(\mathbb{C})\to\mathcal{L}(\mathcal{H})$ can then be written as

$$\mathsf{P}^{r_0}(X\times\Theta)=\sum_{m,n=0}^\infty\langle\eta_m|\eta_n\rangle\,\delta_{r_0}(X)\int_\Theta e^{i\theta(m-n)}\frac{d\theta}{2\pi}|m\rangle\langle n|.$$

We now immediately see that the angle margin $\mathsf{P}^{r_0}_{\rm angle}$ is the phase observable with the phase matrix elements $c_{mn}=\langle\eta_m|\eta_n\rangle$, and the radial margin is the trivial (sharp) observable $\mathsf{P}^{r_0}_{\rm rad}(X)=\delta_{r_0}(X)I$.

IV. DOUBLE HOMODYNE DETECTION SCHEME

We will now turn our attention to the double homodyne detector, a well-established method for measuring translation

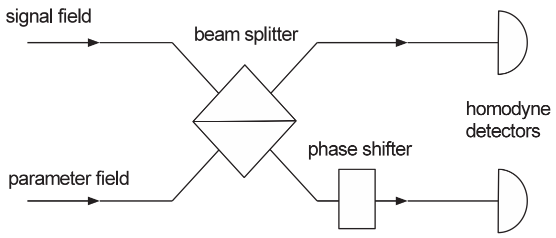

FIG. 1. Schematic of a double homodyne detector. The signal field is mixed with a parameter field by means of a 50:50 beam splitter, after which a phase shift of $-\pi/2$ is performed on one of the modes. Balanced homodyne detection is then performed on both modes.

covariant phase-space observables related to a single-mode electromagnetic field [19,31,32]. This scheme is based on the fact that the usual single homodyne detector provides a measurement of an arbitrary field quadrature. Indeed, when the signal field under investigation is coupled to a strong auxiliary field in a coherent state $|z\rangle$ using a 50:50 beam splitter, and the scaled photon number difference $\frac{1}{|z|}(I\otimes N - N\otimes I)$ of the two output modes is measured, then for a sufficiently large $|z|$, the measured observable is approximately the rotated quadrature observable Q_θ, where $\theta = \arg z$ and $\mathsf{Q}_\theta(X) = e^{i\theta N}\mathsf{Q}(X)e^{-i\theta N}$, with Q being the canonical spectral measure on the real line (for more detail, see, e.g., [33]).

In double homodyne detection, the signal field is first coupled to a parameter field in some state σ via a 50:50 beam splitter, after which a phase shift of $-\frac{\pi}{2}$ is performed on one of the output modes (see Fig. 1). Balanced homodyne detection is then performed on each output mode, so that by choosing the phase of the auxiliary coherent field to be zero in both measurements, this corresponds to measuring the canonical spectral measure on $\mathbb{R}^2$ for the two-mode field. With this setup, the measured observable is the phase-space observable $\mathsf{G}^{\sigma'}$, where the generating operator is connected to the state of the parameter field via the conjugation map $(C\varphi)(x) = \overline{\varphi(x)}$ as $\sigma' = C\sigma C$ [31]. In particular, if we want the observable to be phase-shift covariant and thus give rise to a phase observable as the angle margin, we must use a parameter field which is diagonal in the number state representation, $\sigma = \sum_{k=0}^{\infty}\lambda_k|k\rangle\langle k|$. The simplest case is obviously obtained by using the vacuum $\sigma = |0\rangle\langle 0|$.

The problem with using this measurement setup is, of course, caused by the fact that the canonical phase cannot be obtained as the margin of any translation covariant phase-space observable [14,24]. Thus, a modification of the setup is needed. We will next show that a suitable modification is obtained by adding a unitary coupling between the signal and parameter fields prior to the beam splitter. In other words, while in the usual double homodyne detection the signal and parameter fields are uncorrelated before entering the beam splitter, with this modification they will enter the beam splitter in an entangled state. With this method, it is, in principle, possible to measure the observables $\mathsf{F}^{|k\rangle}$ encountered in Example 2.

To this end, first notice that in the usual double homodyne detector, the overall unitary coupling consisting of the beam splitter and the phase shifter is given by $U : \mathcal{H}_{\text{in}} \to \mathcal{H}_{\text{out}}$,

$$(U|m\rangle\otimes|n\rangle)(r,\theta) = \frac{1}{\sqrt{\pi}}\,\langle n|D(re^{i\theta})^*|m\rangle, \qquad (10)$$

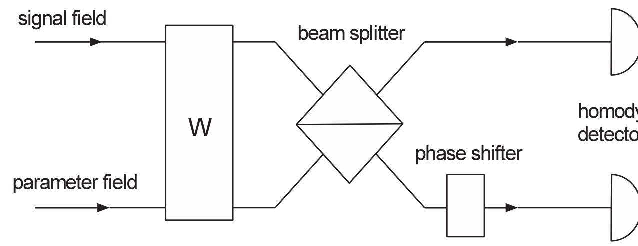

FIG. 2. A modified double homodyne detector where an additional unitary coupling W is performed prior to the beam splitter. By choosing the parameter field to be in the vacuum state, one can obtain the canonical phase observable as the angle margin of the measured phase-space observable.

where $\mathcal{H}_{\text{in}} \simeq \mathcal{H}\otimes\mathcal{H}$ consists of the input signal and parametric field modes and $\mathcal{H}_{\text{out}} \simeq \mathcal{H}\otimes\mathcal{H}$ is the output space. The observable measured with the two homodyne detectors is then (or rather may be chosen to be) the canonical spectral measure $\mathsf{M} : \mathcal{B}(\mathbb{C}) \to \mathcal{L}(\mathcal{H}_{\text{out}})$. If ρ and σ are the states of the signal and parameter fields, respectively, then a direct computation shows that

$$\text{tr}[U(\rho\otimes\sigma)U^*\mathsf{M}(Z)] = \text{tr}[\rho\mathsf{G}^{\sigma'}(Z)]$$

for all $Z \in \mathcal{B}(\mathbb{C})$. Hence, by considering only the angle margin $\mathsf{M}_{\text{angle}}$ as the pointer observable, one can measure the angle margin $\mathsf{G}^{\sigma'}_{\text{angle}}$. In particular, by preparing the parameter field in a state which is diagonal in the number basis, one can measure any phase-space phase observable.

Now it is easily seen from the discussion in Example 2 that the total coupling needed for the measurement of $\mathsf{F}^{|k\rangle}$ is

$$(V|m\rangle\otimes|n\rangle)(r,\theta) = \frac{1}{\sqrt{\pi}}e^{i\theta(n-m)}L_{\min\{m,n\}}(r^2)e^{-\frac{r^2}{2}}.$$

We want to exploit the fact that we already have at our disposal the unitary coupling (10) of the usual double homodyne detector. Therefore, we will look for a unitary operator $W : \mathcal{H}_{\text{in}} \to \mathcal{H}_{\text{in}}$ such that $V = UW$, which would then amount to adding an extra component to the measurement setup prior to the beam splitter (see Fig. 2). We could, of course, consider equally well a unitary operator $W' : \mathcal{H}_{\text{out}} \to \mathcal{H}_{\text{out}}$ such that $V = W'U$, but this leads to a similar treatment.

The action of the operator W is now given by

$$W(|m\rangle\otimes|n\rangle) = \sum_{k,l=0}^{\infty}\langle\!\langle U|k\rangle\otimes|l\rangle|V|m\rangle\otimes|n\rangle\rangle\!\rangle\,|k\rangle\otimes|l\rangle.$$

Let us denote $\alpha_{kl,mn} = \langle\!\langle U|k\rangle\otimes|l\rangle|V|m\rangle\otimes|n\rangle\rangle\!\rangle$ so that

$$\begin{aligned}\alpha_{kl,mn} &= (-1)^{\max\{0,l-k\}}\sqrt{\frac{\min\{k,l\}!}{\max\{k,l\}!}}\\ &\quad\times\frac{1}{\pi}\int_0^\infty\int_0^{2\pi} e^{i\theta(k-l+n-m)}r^{|k-l|}L_{\min\{k,l\}}^{|k-l|}(r^2)\\ &\quad\times L_{\min\{m,n\}}(r^2)e^{-r^2}r\,dr\,d\theta\\ &= \delta_{l,k+n-m}(-1)^{\max\{0,n-m\}}\sqrt{\frac{\min\{k,k+n-m\}!}{\max\{k,k+n-m\}!}}\\ &\quad\times\int_0^\infty x^{\frac{1}{2}|n-m|}L_{\min\{k,k+n-m\}}^{|n-m|}(x)L_{\min\{m,n\}}(x)e^{-x}dx.\end{aligned}$$

The above integral can be further calculated, but it seems that a simple closed expression does not exist. However, note that this already tells us that $W(|m\rangle \otimes |n\rangle)$ is an eigenvector of the photon number difference operator:

$$(I \otimes N - N \otimes I)\, W(|m\rangle \otimes |n\rangle) = (n-m) W(|m\rangle \otimes |n\rangle).$$

In the special case $n = 0$, which corresponds to the canonical phase observable, the above integrals are easily calculated. We then obtain the expressions $W(|0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle$ and, for $m > 0$,

$$\begin{aligned} &W(|m\rangle \otimes |0\rangle) \\ &\quad = \sum_{k=0}^{\infty} \frac{\Gamma(k+\frac{m}{2}+1)}{\sqrt{k!(k+m)!}} \frac{m}{2k+m} |k+m\rangle \otimes |k\rangle \\ &\quad = \frac{m}{2} \sum_{k=0}^{\infty} \frac{\Gamma(k+\frac{m}{2})}{\sqrt{k!(k+m)!}} |m,k\rangle\rangle, \end{aligned} \tag{11}$$

where $|m,k\rangle\rangle$ are Ban's relative number states [34]

$$|m,k\rangle\rangle = \begin{cases} |k+m\rangle \otimes |k\rangle, & m \geqslant 0, \\ |k\rangle \otimes |k-m\rangle, & m < 0. \end{cases}$$

Note that $W(|m\rangle \otimes |0\rangle)$ is actually a two-mode nonlinear coherent state [35]. Such states are generally of the form $|\alpha, f, m\rangle = \sum_{k=0}^{\infty} C_k |k+m\rangle \otimes |k\rangle$, where the constants $C_k \in \mathbb{C}$ are such that $\sum_k |C_k|^2 = 1$, and the states satisfy the equations

$$f(N \otimes I, I \otimes N)(a \otimes a)|\alpha, f, m\rangle = \alpha |\alpha, f, m\rangle, \quad \alpha \in \mathbb{C},$$

where $f(N \otimes I, I \otimes N)$ is some fixed function of the single-mode number operators. As shown in [35],

$$C_k = \alpha^k \sqrt{\frac{m!}{k!(k+m)!}} \left[\prod_{s=1}^{k} \frac{1}{f(s-1+m, s-1)} \right] C_0,$$

so that $W(|m\rangle \otimes |0\rangle) = |\alpha, f, m\rangle$ where $\alpha = 1$, $C_0 = \Gamma(\frac{m}{2}+1)/\sqrt{m!}$, and $f(n_1, n_2) = 2/(n_1+n_2), n_1, n_2 \in \mathbb{N}$, $n_1 \neq 0$.

We close this section by noting that the action of the total coupling V is easy to calculate for some physically relevant states. For example, if the first mode is in the coherent state $|\alpha\rangle$ and the second mode is in the vacuum state $|0\rangle$, then

$$\begin{aligned} (V|\alpha\rangle \otimes |0\rangle)(r,\theta) &= \frac{e^{-|\alpha|^2/2}}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} e^{-i\theta m} e^{-\frac{r^2}{2}} \\ &= \frac{e^{-r^2/2}}{\sqrt{\pi}} \langle \theta | \alpha \rangle, \end{aligned}$$

that is, we obtain the London distribution $\theta \mapsto \langle\theta|\alpha\rangle$ of the coherent state. Consider then a pair-coherent state

$$\psi_\alpha = C(\alpha) \sum_{m=0}^{\infty} \frac{\alpha^m}{m!} |m\rangle \otimes |m\rangle, \quad \alpha \in \mathbb{C},$$

where $C(\alpha) = J_0(2i|\alpha|)^{-1/2}$ is a normalization constant and J_0 is the zeroth Bessel function of the first kind. Now,

$$\begin{aligned} (V\psi_\alpha)(r,\theta) &= \frac{C(\alpha)}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{\alpha^m}{m!} L_m(r^2) e^{-r^2/2} \\ &= \frac{C(\alpha)}{\sqrt{\pi}} J_0(2r\sqrt{\alpha}) e^{\alpha} e^{-r^2/2}, \end{aligned}$$

whereas, for a two-mode phase coherent state,

$$\widetilde{\psi}_\alpha^q = (1-|\alpha|^2)^{1/2} \sum_{m=0}^{\infty} \alpha^m |m\rangle \otimes |m+q\rangle,$$

$q \in \mathbb{N}$, $\alpha \in \mathbb{C}$, $|\alpha| < 1$, one gets the Gaussian function

$$\begin{aligned} \left(V\widetilde{\psi}_\alpha^q\right)(r,\theta) &= \frac{(1-|\alpha|^2)^{1/2}}{\sqrt{\pi}} e^{iq\theta} \sum_{m=0}^{\infty} \alpha^m L_m(r^2) e^{-r^2/2} \\ &= \frac{(1-|\alpha|^2)^{1/2}}{\sqrt{\pi}(1-\alpha)} e^{iq\theta - [\alpha/(1-\alpha)+1/2]r^2} \end{aligned}$$

(see Eqs. 8.975(3) and 8.975(1) of [36]).

V. CONCLUSIONS

We have studied measurements of covariant phase observables, with the aim of obtaining a realistic measurement model for the canonical phase. Due to the practical problems related to realizing minimal measurement models of the canonical phase, we have instead considered measurements of phase-shift covariant phase-space observables. In particular, we have shown that the canonical phase may be obtained as the angle margin of such an observable. We have then considered the quantum optical double homodyne detection scheme, and its modification, as a means of measuring these phase-space observables. We have constructed a unitary coupling which, when placed in front of the setup, would allow one to measure the canonical phase observable.

ACKNOWLEDGMENTS

The authors thank Pekka Lahti for comments and suggestions. Financial support from the Academy of Finland (Grant No. 138135) and the Italian Ministry of Education, University and Research (FIRB Project No. RBFR10COAQ) is gratefully acknowledged.

APPENDIX: JOINT MEASUREMENTS INCLUDING THE CANONICAL PHASE

We say that two observables $\mathsf{E}_1 : \mathcal{B}(\Omega_1) \to \mathcal{L}(\mathcal{H})$ and $\mathsf{E}_2 : \mathcal{B}(\Omega_2) \to \mathcal{L}(\mathcal{H})$ are *jointly measurable* if there exists an observable $\mathsf{E} : \mathcal{B}(\Omega_1 \times \Omega_2) \to \mathcal{L}(\mathcal{H})$ such that

$$\mathsf{E}_1(X) = \mathsf{E}(X \times \Omega_2), \quad \mathsf{E}_2(Y) = \mathsf{E}(\Omega_1 \times Y),$$

for all $X \in \mathcal{B}(\Omega_1)$ and $Y \in \mathcal{B}(\Omega_2)$. It is a standard result that for sharp observables, joint measurability is equivalent to commutativity. The same is true also in the case that one (but not both) of the observables is merely a POVM [28, Theorem 1.3.1], in which case the joint observable E is of the product form $\mathsf{E}(X \times Y) \equiv \mathsf{E}_1(X)\mathsf{E}_2(Y) \equiv \mathsf{E}_2(Y)\mathsf{E}_1(X)$. In the case of the canonical phase, it is an elementary exercise to check that any bounded operator which commutes with all $\mathsf{E}_{\mathrm{can}}(X)$ is necessarily a scalar multiple of the identity. Thus, we have the following result.

Proposition 3. The canonical phase observable is not jointly measurable with any nontrivial sharp observable.

In the more general case of a joint measurement with a POVM, we still have the following negative result.

Proposition 4. Let $\mathsf{F} : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{H})$ be an observable. Then F is jointly measurable with the canonical phase observable $\mathsf{E}_{\rm can}$ if and only if there exists a (weak) Markov kernel $m : \mathcal{B}(\Omega) \times [0,2\pi) \to [0,1]$ such that

$$\mathsf{F}(X) = \int m(X,\theta)\, d\mathsf{E}_{\rm can}(\theta) \tag{A1}$$

for all $X \in \mathcal{B}(\Omega)$. In such a case, the joint observable $\mathsf{M} : \mathcal{B}(\Omega \times [0,2\pi)) \to \mathcal{L}(\mathcal{H})$ is unique and is given by

$$\mathsf{M}(X \times \Theta) = \sum_{m,n=0}^{\infty} \frac{1}{2\pi} \int_{\Theta} m(X,\theta)\, e^{i\theta(m-n)} d\theta |m\rangle\langle n| \tag{A2}$$

for all $X \in \mathcal{B}(\Omega)$ and $\Theta \in \mathcal{B}([0,2\pi))$.

Proof. Assume first that F and $\mathsf{E}_{\rm can}$ are jointly measurable, with the joint observable M, i.e., $\mathsf{M}(X \times [0,2\pi)) = \mathsf{F}(X)$ and $\mathsf{M}(\Omega \times \Theta) = \mathsf{E}_{\rm can}(\Theta)$. Consider the minimal Naimark dilation $(\mathcal{K},\mathsf{T},V)$ of $\mathsf{E}_{\rm can}$ where $\mathcal{K} = L^2([0,2\pi))$, T is the canonical spectral measure on $\mathcal{K}$, and $V : \mathcal{H} \to \mathcal{K}$ is the isometry $V|n\rangle = e_n$ where $e_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{-in\theta}$. Since $\mathsf{M}(X \times \Theta) \leqslant \mathsf{M}(\Omega \times \Theta) = \mathsf{E}_{\rm can}(\Theta)$ for all $X \in \mathcal{B}(\Omega)$ and $\Theta \in \mathcal{B}([0,2\pi))$, there exists an observable $\mathsf{R} : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{K})$ which commutes with T and

$$\mathsf{M}(X \times \Theta) = V^*\mathsf{R}(X)\mathsf{T}(\Theta)V \tag{A3}$$

for all $X \in \mathcal{B}(\Omega)$ and $\Theta \in \mathcal{B}([0,2\pi))$ [37, Lemma 4.1]. But the canonical spectral measure T is maximal in the sense that any operator which commutes with T must be a function of it [38, Theorem 1, p. 187]. It follows that there exists a (weak) Markov kernel $m : \mathcal{B}(\Omega) \times [0,2\pi) \to [0,1]$, such that $\mathsf{R}(X) = \int m(X,\theta)\, d\mathsf{T}(\theta)$, and it follows from Eq. (A3) that

$$\mathsf{F}(X) = V^*\mathsf{R}(X)V = \int m(X,\theta)\, d\mathsf{E}_{\rm can}(\theta).$$

Suppose now that there exists a (weak) Markov kernel such that Eq. (A1) holds. Since for each $X \in \mathcal{B}(\Omega)$ the map $\theta \mapsto m(X,\theta)$ is measurable and $m(X,\theta) \leqslant 1$ for almost all θ, we may define a bounded operator $\mathsf{R}(X) \in \mathcal{L}(\mathcal{K})$ by $(\mathsf{R}(X)\varphi)(\theta) = m(X,\theta)\varphi(\theta)$. The map $X \mapsto \mathsf{R}(X)$ is then a POVM which commutes with T. Therefore, the map $(X,\Theta) \mapsto V^*\mathsf{R}(X)\mathsf{T}(\Theta)V$ extends to an observable $\mathsf{M} : \mathcal{B}(\Omega \times [0,2\pi)) \to \mathcal{L}(\mathcal{H})$ whose margins are F and $\mathsf{E}_{\rm can}$. In other words, F and $\mathsf{E}_{\rm can}$ are jointly measurable.

In both of the above instances, the joint observable M satisfies Eq. (A2), and the uniqueness follows from [37, Theorem 4.1(a)] since $\mathsf{E}_{\rm can}$ is an extreme point of the convex set of all observables on $\mathcal{B}([0,2\pi))$ [12]. ■

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