Negativity of quantumness and its interpretations

Takafumi Nakano,[1] Marco Piani,[1] and Gerardo Adesso[2]

[1]*Institute for Quantum Computing and Department of Physics and Astronomy, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1*
[2]*School of Mathematical Sciences, University of Nottingham, University Park, Nottingham NG7 2RD, United Kingdom*

We analyze the general nonclassicality of correlations of a composite quantum system as measured by the negativity of quantumness. The latter corresponds to the minimum entanglement, as quantified by the negativity that is created between the system and an apparatus that is performing local measurements on a selection of subsystems. The negativity of quantumness thus quantifies the degree of nonclassicality on the measured subsystems. We demonstrate a number of possible different interpretations for this measure and for the concept of quantumness of correlations in general. In particular, for general bipartite states in which the measured subsystem is a qubit, the negativity of quantumness acquires a geometric interpretation as the minimum trace distance from the set of classically correlated states. This can be further reinterpreted as minimum disturbance, with respect to trace norm, due to a local measurement or a nontrivial local unitary operation. We calculate the negativity of quantumness in closed form for Werner and isotropic states and for all two-qubit states for which the reduced state of the system that is locally measured is maximally mixed; this includes all Bell diagonal states. We discuss the operational significance and potential role of the negativity of quantumness in quantum information processing.

I. INTRODUCTION

Quantum systems differ from classical ones in a number of ways. This is particularly true for composite systems, which can exhibit quantum features like nonlocality and quantum entanglement [1]. They are certainly striking manifestations of a deviation from classicality. While quantum entanglement can be regarded as one of the most characteristic traits of quantum mechanics [2], a lot of effort has recently been directed towards characterizing a more general notion of *quantumness* of correlations [3–5], almost always present in mixed states also in the absence of entanglement [6]. The study of quantum correlations, or of quantumness in its most essential manifestation in composite systems, has a deep foundational value as it provides a trail to investigate the boundary between the classical and the quantum world from the perspective of quantum measurements [7]. Quantumness of correlations manifests, for instance, when any local complete projective measurement on a subsystem necessarily alters the state of a composite system [3]. There are many other (sometimes equivalent) signatures to reveal quantumness of correlations in a state, and there are correspondingly a number of possible ways to quantify such quantumness (including the *quantum discord* [3,4]), which range from informational to geometric and thermodynamical settings. Some of the most prominent approaches are summarized below, while for a more extensive treatment we defer the reader to a recent review; see [5] and references therein.

From an applicative point of view, quantum information processing aims at harnessing quantum properties to outperform classical information processing [8]. A natural step towards such a goal is that of developing concepts and tools to more precisely determine which states possess or do not possess a certain quantum property, further aiming to quantify and exploit the latter when present. While quantumness of correlations reduces to entanglement for pure states, thus already embodying the key resource for quantum information

processing in the absence of noise [1,8], a number of researchers are investigating the role of general nonclassical correlations in quantum computation [9–12], quantum communication [13,14], and quantum metrology [15,16].

In the following, we always address the notions of classicality or quantumness as referred to the correlations among subsystems in the state of a composite system; we similarly adopt the wording of (non)classical states to mean equivalently (non)classically correlated states. We are not concerned with other definitions of (non)classicality such as those usually adopted in quantum optics to characterize the nature of light [17], which often lead to a very different classification of states into classical and nonclassical [18].

As anticipated, there are a wide variety of approaches to quantify the nonclassicality (of correlations) of the state ρ of a quantum system [5]. To list a few, it can be measured by

Approach 1 (activation) the minimum amount of entanglement created between the system and its measurement apparatus in a local measurement [19–22];

Approach 2 (geometric) the minimum distance between ρ and its closest classical state [23–26];

Approach 3 (disturbance by measurement) the minimum amount of disturbance caused by local projective measurements [3,4,27];

Approach 4 (disturbance by unitary) the minimum disturbance caused by particular local unitaries [28,29].

As summarized in [5], a number of nonclassicality measures can be defined via Approach 2, Approach 3, and Approach 4 using different distance functions such as the Hilbert-Schmidt distance or the quantum relative entropy. Even though at a first glance Approach 1 seems to be very different from the other approaches, it turns out it is intimately connected with them (see also [30]). This is because there are entanglement measures that are similarly defined as the distance from the closest separable state. One such example is given by the relative entropy of entanglement [31], which is a well-known upper bound to the distillable entanglement [32].

In this paper we develop an extensive study of a promising measure of nonclassical correlations recently defined in [20] along Approach 1 : the *negativity of quantumness* (NoQ). NoQ [20] corresponds to the adoption of negativity [33] as an entanglement measure in Approach 1. In [22] it was proven that Approach 1 leads to a quantitative hierarchy of correlations that formalizes the intuition that "the general quantumness of correlations is more than entanglement." In particular, the entanglement generated in a complete projective local measurement is provably always greater than the entanglement present in the state whose nonclassicality of correlations is under scrutiny. This fact is independent of the specific choice of entanglement measure. Nonetheless, given the usefulness of negativity [33] in the study and quantification of entanglement—a usefulness that comes in particular by its being easy to calculate—it is natural to focus on the specific hierarchy it generates. Furthermore, in this article we prove there are other reasons to focus on NoQ.

Indeed, we find that NoQ can be further interpreted from the perspective of Approach 2, Approach 3, and Approach 4 under suitable conditions. In particular, in the special case of a bipartite state ρ_{AB}, when furthermore the measured subsystem A is a two-level quantum object (qubit), it turns out that the NoQ acquires a geometric interpretation as the minimum distance between ρ_{AB} and the set of classical states, with the distance measured in *trace norm*. This can be proven to be further equivalent to the minimal state change, again in trace norm, after either a local measurement on A or a (nontrivial) local unitary evolution on A. The equivalence between the last two approaches when A is a qubit is here proven for any norm-based distance, complementing the original results of [28,29] which were specific to the Hilbert-Schmidt norm. All these results are derived and described throughout the paper with relevant examples. Our work also provides a proof to a conjecture raised by Khasin *et al.* about a bound for the negativity (of entanglement) [34]. We obtain closed analytical expressions for the NoQ in relevant cases such as Werner and isotropic states of arbitrary dimension [35,36], and a family of two-qubit states where qubit A is maximally mixed, which includes Bell diagonal states. In the latter instance, the problem is recast into an appealing geometrical formulation. The closed formula for two-qubit states with one maximally mixed marginal, together with the hierarchical relation between quantumness of correlations and entanglement of Ref. [22] allows, e.g., a consistent study and comparison of the evolution of entanglement—as measured by negativity—and quantumness of correlations—as measured by the one-sided NoQ—under the action of a family of qubit channels, e.g., a semigroup.

The paper is organized as follows. In Sec. II we fix some notation adopted throughout the paper. In Sec. III we review the main definitions and approaches to quantify quantumness of correlations as sketched above. Section IV is focused on the definition and formulation of the NoQ. The main properties and interpretations of the NoQ are discussed in Sec. V, along with its interplay with the usual negativity of entanglement. In Sec. VI we prove in general the equivalence of the various approaches when the measured subsystem is a qubit. In Sec. VII we calculate the NoQ for relevant families of bipartite states. We conclude the main body of the paper

in Sec. VIII. The appendixes contain a number of technical proofs and extensions.

II. NOTATION

We deal only with finite-dimensional Hilbert spaces and we identify linear operators with matrices. We denote by $\langle A, B \rangle := \mathrm{Tr}(A^\dagger B)$ the *Hilbert-Schmidt inner product* of two matrices A and B.

The *Schatten p-norms* $\| \cdot \|_p$ are a family of norms parametrized by a real number $p \geqslant 1$. If $\sigma_i(A)$'s are the singular values of a matrix A, the p-norm of the latter is defined as

$$\|A\|_p := \left(\sum_i \sigma_i^p(A) \right)^{1/p}. \tag{1}$$

We refer the reader to Appendix A for a concise summary of the main properties of p-norms. In particular, we are going to focus our attention on the 1-norm (also called *trace norm*) $\|A\|_1 = \sum_i \sigma_i(A) = \mathrm{Tr} \sqrt{A^\dagger A}$ and on the 2-norm $\|A\|_2 = \sqrt{\sum_i \sigma_i(A)^2} = \sqrt{\mathrm{Tr}(A^\dagger A)} = \sqrt{\langle A, A \rangle}$ (also called Hilbert-Schmidt norm). For any norm $\| \cdot \|$, one can define an associated distance on matrices by means of $\|A - B\|$. In particular, the distance associated with the 1-norm is also called *trace distance* (up to a factor $1/2$), while the distance associated with the 2-norm is known as *Hilbert-Schmidt distance*. The trace distance between two mixed states (i.e., positive semidefinite operators of trace one) has a direct operational interpretation linked to the probability of success in distinguishing the two states via a measurement [8].

We also make use of the l_1-norm, which is a basis-dependent norm defined as the sum of the absolute values of the entries of a matrix: $\|A\|_{l_1} = \sum_{i,j} |A_{i,j}|$. See Appendix B for more details.

The relative entropy of a density matrix ρ with respect to a density matrix σ is defined as $S(\rho\|\sigma) := \mathrm{Tr}[\rho(\log_2 \rho - \log_2 \sigma)] = -S(\rho) - \mathrm{Tr}(\rho \log_2 \sigma)$. Here $S(\rho) := -\mathrm{Tr}(\rho \log_2 \rho)$ is the von Neumann entropy. The relative entropy is not a distance as, for example, it is not symmetric in its arguments, but for the sake of our investigation we can and do treat it as if it was a distance measure. This is done routinely in quantum information theory [37].

A *channel* is a completely positive trace-preserving (CPTP) linear map on operators [8]; it admits a Kraus representation,

$$\Lambda[B] = \sum_i K_i B K_i^\dagger, \tag{2}$$

with K_i's the corresponding Kraus operators satisfying the trace-preservation condition $\sum_i K_i^\dagger K_i = \mathbb{I}$. The dual $\Lambda^\dagger$ of a channel Λ—and, in general, of a linear map—is defined via the Hilbert-Schmidt inner product through the relation

$$\langle A, \Lambda[B] \rangle = \langle \Lambda^\dagger[A], B \rangle, \quad \forall A, B.$$

It easy to verify that for a channel with Kraus decomposition (2) the dual map has Kraus decomposition

$$\Lambda^\dagger[A] = \sum_i K_i^\dagger A K_i.$$

Hence, Λ being trace-preserving implies that $\Lambda^\dagger$ is unital; i.e., $\Lambda^\dagger[\mathbb{I}] = \mathbb{I}$; $\Lambda^\dagger$ is furthermore a channel—i.e., also trace-preserving—if Λ is unital itself.

We will also make use of the Pauli matrices $\sigma_1 = \sigma_x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_2 = \sigma_y = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, and $\sigma_3 = \sigma_z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, which, alongside with the identity $\sigma_0 = \mathbb{I} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, form a basis for the space of 2×2 matrices.

III. CLASSICALITY AND NONCLASSICALITY OF CORRELATIONS: NOTIONS AND MEASURES

A. Classicality of quantum states

One key quantum feature is the fact that a measurement will, in general, disturb the state of the system being measured. Most importantly, a local measurement will typically lead to a decrease in (total) correlations—quantified, e.g., in terms of quantum mutual information—between the measured system and any other systems that might have been initially correlated with it [3,4,38]. Because of the orthogonality and perfect distinguishability of the states that form an orthonormal basis, a complete projective measurement can be thought as a quantum-to-classical mapping, where the information extracted from the quantum system is recorded in a classical register. In this sense, only correlations between a classical system and the remaining unmeasured systems are left after the local measurement. It has been actually suggested that such surviving correlations should be deemed *classical* [4]. It can be proven that a local measurement that does not destroy any amount of correlation exists for a state if and only if the measured system could be considered classical to start with [3,27,38], so that such a measurement does not disturb the system. To be more precise, the following notions will be adopted.

Definition III.1: Classicality of a quantum state (i). Let $\rho_{\{1,2,\ldots,n\}}$ be an n-partite quantum state. For any $i \in \{1,2,\ldots,n\}$, $\rho_{\{1,2,\ldots,n\}}$ is *classical on the ith system* if there exists a complete local projective measurement on the ith subsystem which leaves $\rho_{\{1,2,\ldots,n\}}$ invariant.

Complete projective measurements are described by a set of orthogonal rank-1 projectors, say $\{|a_i\rangle\langle a_i|\}$, such that they sum up to the identity of the space, i.e., $\sum_i |a_i\rangle\langle a_i| = \mathbb{I}$. Therefore, the state is invariant under such measurement if and only if the original state has block-diagonal form with respect to the basis $\{|a_i\rangle\}$, and this can be used as an alternative definition of classicality of the state.

Definition III.2: Classicality of a quantum state (ii). Let $\rho_{\{1,2,\ldots,n\}}$ be an n-partite quantum state. For any $i \in \{1,2,\ldots,n\}$, $\rho_{\{1,2,\ldots,n\}}$ is *classical on the ith system* if $\rho_{\{1,2,\ldots,n\}}$ can be represented as

$$\rho_{\{1,2,\ldots,n\}} = \sum_j |a_j\rangle\langle a_j|_i \otimes \sigma^j_{\{1,\ldots,i-1,i+1,\ldots,n\}}, \qquad (3)$$

where $\{|a_j\rangle\}$ is an orthonormal basis of the ith system and

$$\sigma^j_{\{1,\ldots,i-1,i+1,\ldots,n\}} = \langle a_j | \rho_{\{1,2,\ldots,n\}} | a_j \rangle.$$

In this paper, mainly bipartite quantum states are considered, $\rho \equiv \rho_{AB}$. A bipartite quantum state which is classical only on one subsystem (say A) is called *classical-quantum* ($\mathcal{CQ}$). Such a state will exhibit zero quantumness (of correlations between A and B) with respect to local measurements

on A. Similarly, a state classical on both subsystems is called *classical-classical* ($\mathcal{CC}$) [38].

Classicality of subsystems is a much stronger notion than separability, i.e., absence of entanglement, as recalled below.

Definition III.3: Separable and entangled states. A state ρ_{AB} is *separable* in the bipartition A versus B, also indicated as the $A:B$ bipartition, if it can be represented as [1]

$$\rho_{AB} = \sum_j p_j \tau^j_A \otimes \sigma^j_B, \qquad (4)$$

with $(p_j)_j$ a probability distribution and τ^j_A and σ^j_B quantum states for A and B, respectively. A state ρ_{AB} is $A:B$ *entangled* if it is not $A:B$ separable.

It is clear from (3) and (4) that every state ρ_{AB} that is classical on A is $A:B$ separable, while the opposite does not hold.

Given a state which is not classical on some subsystem i, i.e., such that the state would get disturbed by any possible complete projective measurement on subsystem i, it is natural to try to quantify the amount of nonclassicality of correlations in that state, in particular from an operational perspective. As mentioned in Sec. I, there are different approaches to quantifying nonclassicality that we briefly review.

B. Nonclassicality by system-apparatus entanglement

We now recall more in detail how a measure—or rather a family of measures—of nonclassical correlations can be based on Approach 1. The definition goes through the consideration of a particular set of states produced during the measurement of a (sub)system, when such process is modeled by a particular unitary interaction—which we call *measurement interaction*—between the system and a measurement apparatus. We call the states that are the result of such an interaction *premeasurement states* [7].

Definition III.4: Measurement interaction. A *measurement interaction* $V_{A \mapsto AA'}$ on system A is a linear isometry from A to a bipartite system AA', where A' has the same dimension as A. This isometry is defined by the following mapping of an orthonormal basis of A, $\{|a_k\rangle\}$:

$$V_{A \mapsto AA'} |a_k\rangle_A = |a_k\rangle_A |k\rangle_{A'}, \quad \forall k,$$

where $\{|k\rangle_{A'}\}$ is the computational basis of system A'. This operation is defined in a basis-dependent way, i.e., the choice of different local orthonormal basis of A results in distinct measurement interactions. Therefore, the operation is sometimes denoted as $V^{\{|a_k\rangle\}}_{A \mapsto AA'}$ when the basis needs to be specified.

Definition III.5: Premeasurement state. Let $\rho_{\{1,2,\ldots,n\}}$ be an n-partite quantum state. For some choice of subsystems $\Sigma \subseteq \{1,2,\ldots,n\}$ and of measurement bases $\{|a_k\rangle\}$ (one for each subsystem $i \in \Sigma$), the corresponding premeasurement state for ρ is

$$\tilde{\rho}_\Xi := \left(\bigotimes_{i \in \Sigma} V_{i \mapsto ii'}\right) \rho \left(\bigotimes_{i \in \Sigma} V_{i \mapsto ii'}\right)^\dagger, \qquad (5)$$

where $\Xi = \{1,2,\ldots,n\} \cup \Sigma'$; i.e., $\tilde{\rho}_\Xi$ is the joint state of the initial quantum systems and their measurement apparatuses.

As already mentioned, measurement interactions depend on the choice of a specific basis for each subsystem in Σ. The action of measurement interactions defined for distinct bases results—in general—in distinct final premeasurement states. This leads to the consideration of the entire class of potential premeasurement states.

The next theorem asserts that the system-apparatus separability of the premeasurement states characterizes the classicality of the correlations of the original system.

Theorem III.1: Activation protocol [19–22]. A n-partite quantum state $\rho_{\{1,2,\ldots,n\}}$ is classical on its subsystems $\Sigma \subseteq \{1,2,\ldots,n\}$ if and only if there exists a corresponding $\{1,2,\ldots,n\} : \Sigma'$ separable premeasurement state $\tilde{\rho}_{\{1,2,\ldots,n\}\cup\Sigma'}$.

One can introduce a family of quantifiers of the nonclassical correlations present in a quantum state exploiting this intimate connection between the nonclassicality of a quantum state and the entanglement properties of the corresponding premeasurement states.

Definition III.6: Nonclassicality by system-apparatus entanglement [19–22]. Let $\rho_{\{1,2,\ldots,n\}}$ be a n-partite quantum state and $\Sigma \subseteq \{1,2,\ldots,n\}$ be a set of its subsystems. Then the measure of nonclassicality of correlation (or quantumness, in short) present in $\rho_{\{1,2,\ldots,n\}}$ as revealed on subsystems Σ with respect to an entanglement measure E is defined as

$$Q_E^\Sigma(\rho_{\{1,2,\ldots,n\}}) := \min_{\bigotimes_{i\in\Sigma} \mathcal{B}(i)} E_{\{1,2,\ldots,n\}:\Sigma'}(\tilde{\rho}_{\{1,2,\ldots,n\}\cup\Sigma'}), \quad (6)$$

where $\mathcal{B}(i)$ denotes a local orthonormal basis of subsystem $i \in \Sigma$, the bipartite entanglement is measured across the bipartite cut $\{1,2,\ldots,n\} : \Sigma'$, and the minimum is taken over different premeasurement states for different measurement interaction defined by $\bigotimes_{i\in\Sigma} \mathcal{B}(i)$.

C. Other measures of nonclassicality

In this section, we will briefly summarize other approaches to define measures of nonclassicality of quantum states, namely Approach 2 and Approach 3. The approach based on Approach 4 and its relevance to our other results are discussed in Sec. VI.

1. Measures of nonclassicality based on disturbance

Since the nonclassicality of a quantum state is defined by the unavoidable disturbance caused by any local projective measurement, perhaps the most immediate way to study the degree to which a quantum state nonclassical is to quantify the difference between the state before and after a measurement (Approach 3).

Definition III.7: Nonclassicality by measurement disturbance [27,39]. Let $d(\cdot,\cdot)$ be a distance function. The nonclassicality of a n-partite quantum state $\rho_{\{1,2,\ldots,n\}}$ revealed on its subsystems $\Sigma \subseteq \{1,2,\ldots,n\}$, measured by the minimum disturbance caused by local projective measurements on each subsystem in Σ, with respect to the distance function $d(\cdot,\cdot)$, is defined as

$$\mathcal{D}_{d(\cdot,\cdot)}^\Sigma(\rho_{\{1,2,\ldots,n\}})$$
$$:= \min_{\bigotimes_{i\in\Sigma} \mathcal{B}(i)} d\left(\rho_{\{1,2,\ldots,n\}}, \bigotimes_{i\in\Sigma} \Pi_{\mathcal{B}(i)}[\rho_{\{1,2,\ldots,n\}}]\right), \quad (7)$$

where $\Pi_{\mathcal{B}(i)}$ denotes a complete projective measurement on subsystem i on a complete orthonormal basis $\mathcal{B}(i)$ and the minimum is taken over different choices of local bases for these projective measurements. If the distance function $d(\cdot,\cdot)$ is derived from a norm $\|\cdot\|$, we use the shorthand notation $\mathcal{D}_{\|\cdot\|}^\Sigma(\rho_{\{1,2,\ldots,n\}})$.

Example III.1. We list a few measures of quantumness of correlations for bipartite states ρ_{AB} based on this notion.

(i) *Zero-way deficit* [39]:

$$\Delta^\emptyset(\rho_{AB}) := \mathcal{D}_{S(\cdot||\cdot)}^{AB}(\rho_{AB})$$
$$= \min_{\mathcal{B}(A)\otimes\mathcal{B}(B)} S(\rho_{AB}\|(\Pi_{\mathcal{B}(A)}\otimes\Pi_{\mathcal{B}(B)})[\rho_{AB}]). \quad (8)$$

The distance function here is the quantum relative entropy (see Sec. II).

(ii) *One-way deficit* [39]:

$$\Delta^\rightarrow(\rho_{AB}) := \mathcal{D}_{S(\cdot||\cdot)}^A(\rho_{AB})$$
$$= \min_{\mathcal{B}(A)} S(\rho_{AB}\|\Pi_{\mathcal{B}(A)}[\rho_{AB}]).$$

Again the distance function is the quantum relative entropy. Notice that the one-way deficit vanishes on $\mathcal{CQ}$ states while the zero-way deficit is a symmetric measure vanishing only on $\mathcal{CC}$ states.

(iii) *Geometric discord* [24,25]:

$$D_G^A(\rho_{AB}) := \mathcal{D}_{\|\cdot\|_2^2}^A(\rho_{AB}) = \min_{\mathcal{B}(A)} \|\rho_{AB} - \Pi_{\mathcal{B}(A)}[\rho_{AB}]\|_2^2. \quad (9)$$

The distance function here is the square of the Hilbert-Schmidt distance (see Sec. II). For a discussion of some conceptual issues with the use of the geometric discord as quantumness measure, see [26].

2. Distance-based measures of the nonclassicality of correlations

The quantification of nonclassicality based on Approach 2 follows a very common approach in quantum information theory. Here the quantumness of correlations is defined as the minimum distance from the (suitably chosen) set of classical states.

Definition III.8: Nonclassicality by distance from classical states. Let $d(\cdot,\cdot)$ be any distance function. The nonclassicality of a n-partite quantum state $\rho_{\{1,2,\ldots,n\}}$ on its subsystems $\Sigma \subseteq \{1,2,\ldots,n\}$, measured by the distance from the set $\mathcal{C}(\Sigma)$ of states which are classical on each subsystem in Σ, with respect to the distance function $d(\cdot,\cdot)$, is defined as

$$\mathbb{D}_{d(\cdot,\cdot)}^\Sigma(\rho_{\{1,2,\ldots,n\}}) := \min_{\eta\in\mathcal{C}(\Sigma)} d(\rho_{\{1,2,\ldots,n\}},\eta). \quad (10)$$

If the distance function $d(\cdot,\cdot)$ is derived from a norm $\|\cdot\|$, we use the shorthand notation $\mathbb{D}_{\|\cdot\|}^\Sigma(\rho_{\{1,2,\ldots,n\}})$.

Example III.2. Here we provide two examples of nonclassicality measures for a bipartite state ρ_{AB} based on this notion that differ in the choice of distance measures and relevant set of classical states (recall that $\mathcal{CC}$ stands for the set of states which are classical on systems A and B and $\mathcal{CQ}$ is the set of states which are classical on A):

(i) *Relative entropy of discord* [23]:

$$D_R(\rho_{AB}) := \mathbb{D}_{S(\cdot||\cdot)}^{AB}(\rho_{AB}) = \min_{\eta\in\mathcal{CC}} S(\rho_{AB}||\eta).$$

Here the distance function is the quantum relative entropy. It can be proven that the relative entropy of discord is equivalent to the zero-way deficit (8) [23].

(ii) *Geometric discord* [24]:

$$D_G(\rho_{AB}) := \mathbb{D}^A_{\|\cdot\|_2^2}(\rho_{AB}) = \min_{\eta \in \mathcal{CQ}} ||\rho_{AB} - \eta||_2^2. \qquad (11)$$

Here the distance function is (the square of) the Hilbert-Schmidt distance. It is easily verified that the two definitions (9) and (11) for the geometric discord are equivalent [25].

IV. NEGATIVITY OF QUANTUMNESS

The activation protocol (Definition III.6) allows us to define a measure of nonclassical correlations Q_E for each entanglement measure E we may want to consider. Through this mapping, some entanglement measures generate known nonclassicality measures and others generates new ones. Negativity [33] is a widely used entanglement measure with a very appealing property: It is easily computable.[1] For the rest of the paper, we study the measure of nonclassicality of quantum states based on the activation protocol (Definition III.6) and on the choice of negativity as entanglement measure, $E \equiv \mathcal{N}$. It is in order to recall that the negativity can vanish for some entangled states, known as Positive-under-Partial-Transposition (PPT) entangled states, whose entanglement is not distillable by local operations and classical communication [40]. However, this does not compromise its use to define a measure of nonclassicality of correlations through the activation protocol. This is because entangled premeasurement states of the form Eq. (5) are always nonpositive under partial transposition (NPT) and distillable, as argued in [20] and shown explicitly in Lemma IV.1 later in this paper. This guarantees that the quantity analyzed in this paper and defined formally below, which we dub NoQ, is a faithful measure of nonclassicality of correlations in composite systems, vanishing if and only if the correlations of the considered states are classical.

The negativity (of entanglement) is defined as follows.

Definition IV.1: Negativity (of entanglement) [33]. Let ρ_{AB} be a bipartite quantum state. The *negativity (of entanglement)* of ρ_{AB} is defined as

$$\mathcal{N}_{A:B}(\rho_{AB}) := \frac{\|\rho_{AB}^\Gamma\|_1 - 1}{2},$$

where the subscript of $\mathcal{N}$ denotes the bipartition with respect to which the entanglement is being measured, the superscript Γ on ρ_{AB} denotes its partial transpose, and $\| \cdot \|_1$ is the Schatten 1-norm. This definition assumes, as we do in the rest of the paper, that we are dealing with normalized states.

It can be immediately verified that the negativity of entanglement is independent both of the choice of the party on which the partial transposition is considered and of the choice of local basis in which the local transposition is taken. The NoQ can then be defined as follows.

Definition IV.2: Negativity of quantumness [20]. The NoQ of a n-partite quantum state $\rho_{\{1,2,\dots,n\}}$ on subsystems $\Sigma \subseteq \{1,2,\dots,n\}$ is defined as

$$Q_\mathcal{N}^\Sigma(\rho_{\{1,2,\dots,n\}}) := \min \mathcal{N}_{\{1,2,\dots,n\}:\Sigma'}(\tilde{\rho}_\Xi), \qquad (12)$$

where the minimum is taken over all premeasurement states $\tilde{\rho}_\Xi$, $\Xi = \{1,2,\dots,n\} \cup \Sigma'$, of the quantum systems $\{1,2,\dots,n\}$ and the measurement apparatuses Σ' associated with the individually measured systems $\Sigma \subseteq \{1,2,\dots,n\}$.

A. Total negativity of quantumness

The total (or two-sided, in the case where there are only two subsystems) quantumness (of correlations) of a quantum state is quantified by the amount of apparatus-system entanglement in a premeasurement state when every subsystem is measured individually, i.e., $\Sigma = \{1,2,\dots,n\}$. Adopting the NoQ as a measure, the corresponding explicit expression for the total NoQ of arbitrary multipartite states was given in [20]. Here we derive this expression again as it will be useful for the analysis made later in this paper.

First, we observe that the premeasurement states for the case of studying total quantumness have a very particular form.

Definition IV.3: Maximally correlated state. A bipartite state τ_{AB} of systems A and B is said to have the *maximally correlated form* if it can be expressed as $\tau_{AB} = \sum_{i,j=1}^n \tau_{ij} |a_i\rangle\langle a_j|_A \otimes |b_i\rangle\langle b_j|_B$ for some $\tau_{ij} \in \mathbb{C}$ with respect to some orthonormal basis of each system $\{|a_i\rangle\}$ and $\{|b_i\rangle\}$ and $n = \min\{\dim A, \dim B\}$. Note that Hermiticity implies $\tau_{ij} = \tau_{ji}^*$. A state that can be represented in a maximally correlated form is called a *maximally correlated state (MCS)*.

Maximally correlated states have some remarkable properties. For example, if a MCS has negative partial transpose the entanglement contained in the state is distillable and, moreover, distillable entanglement and relative entropy of entanglement coincide for this set of states [32]. Moreover, for any quantumness measure, the quantumness of a MCS is the same as its corresponding entanglement as proven in Appendix C.[2] At the same time, there is a simple analytic expression for both the eigenvalues and the eigenvectors of its partial transpose.

Lemma IV.1. The partial transpose[3] $\tau_{AB}^\Gamma = \sum_{i,j=1}^n \tau_{ij} |a_i\rangle\langle a_j|_A \otimes |b_j\rangle\langle b_i|_B$ of a MCS $\tau = \sum_{i,j=1}^n \tau_{ij} |a_i\rangle\langle a_j|_A \otimes |b_i\rangle\langle b_j|_B$ has the following eigenvalues and corresponding eigenvectors:

$$\tau_{ii} \quad \text{for} \quad |a_i\rangle \otimes |b_i\rangle,$$
$$\pm|\tau_{ij}| \quad \text{for} \quad \frac{1}{\sqrt{2}}\left(|a_i\rangle \otimes |b_j\rangle \pm \frac{\tau_{ji}}{|\tau_{ji}|}|a_j\rangle \otimes |b_i\rangle\right),$$
$$\text{with} \quad i > j.$$

Proof. Notice that ρ_{AB}^Γ with respect to the basis $\{|a_i\rangle \otimes |b_j\rangle\}$ is a generalized permutation matrix; i.e., there is only

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[1]Unlike other operationally more meaningful entanglement measures such as entanglement cost, distillable entanglement and entanglement of formation, which typically involve a nontrivial optimization over a large set of parameters [1].

[2]Recently it has been proven that some entanglement measures and some nonclassical correlation measures coincide for a more general class of states [30].

[3]Here it is taken with respect to subsystem B, in the maximally correlated basis $\{|b_i\rangle\}$; while the eigenvalues of the partially transposed state do not depend on this details, the eigenvectors do.

one nonzero entry on each row and column. Therefore, by the action of an appropriate permutation matrix P, it can be transformed into the following form (the off-diagonal blocks have all vanishing entries; we do not indicate all the zero entries):

$$
P\rho_{AB}^{\Gamma}P^{-1} =
\begin{pmatrix}
\tau_{11} & & & & & & & & \\
& \ddots & & & & & & & \\
& & \tau_{nn} & & & & & & \\
& & & 0 & \tau_{12} & & & & \\
& & & \tau_{21} & 0 & & & & \\
& & & & & \ddots & & & \\
& & & & & & 0 & \tau_{n-1,n} & \\
& & & & & & \tau_{n,n-1} & 0 &
\end{pmatrix}.
$$

Each diagonal block corresponds to an invariant subspace (under the action of $P\rho_{AB}^{\Gamma}P^{-1}$). The rows and columns of the upper left block corresponds to the vectors $|a_i\rangle \otimes |b_i\rangle$'s, while the entries of the other two-by-two on-diagonal blocks each correspond to $|a_i\rangle \otimes |b_j\rangle$ and $|a_j\rangle \otimes |b_i\rangle$ for $i \neq j$. Hence, $P\rho_{AB}^{\Gamma}P^{-1}$ has the eigenvalues and eigenvectors above. ∎

From this we get immediately the following.

Corollary IV.1: Negativity of a MCS. The negativity of a MCS $\tau_{AB} = \sum_{i,j=1}^{n} \tau_{ij}|a_i\rangle\langle a_j|_A \otimes |b_i\rangle\langle b_j|_B$ is

$$
\mathcal{N}(\rho_{AB}) = \frac{\sum_{i,j} |\tau_{ij}| - 1}{2}. \tag{13}
$$

The quantity $\sum_{i,j} |\tau_{ij}|$ in the above corollary is the sum of the absolute value of entries of the matrix $[\tau_{ij}]$ and it can be seen as the l_1-norm of the matrix in the maximally correlated basis (see Sec. II and Appendix B). The negativity of a MCS can then be expressed as

$$
\mathcal{N}(\rho_{AB}) = \frac{\|\tau_{AB}\|_{l_1}^{\{|a_i\rangle \otimes |b_j\rangle\}} - 1}{2},
$$

where the superscript indicates that the l_1-norm is taken for the matrix representation of τ_{AB} with respect to the basis $\{|a_i\rangle \otimes |b_j\rangle\}$.

We are now ready to prove the following.

Theorem IV.1: Total negativity of quantumness [20]. Let $\rho_{\{1,2,...,n\}}$ be an n-partite quantum state. Then the total NoQ, i.e., the minimum amount of entanglement with respect to negativity between system-apparatuses when each subsystem is measured independently, is

$$
Q_{\mathcal{N}}^{\{1,2,...,n\}}(\rho_{\{1,2,...,n\}})
$$
$$
= \min_{\bigotimes_{i\in\{1,2,...,n\}} \mathcal{B}(i)} \frac{\|\rho_{\{1,2,...,n\}}\|_{l_1}^{\bigotimes_{i\in\{1,2,...,n\}} \mathcal{B}(i)} - 1}{2},
$$

where the minimum is taken over different choices of factorized basis $\bigotimes_{i\in\{1,2,...,n\}} \mathcal{B}(i)$ for the (local) measurement interaction.

Proof. This result is a direct consequence of every premeasurement state being a MCS (between system and apparatus) and it is true regardless of the number of systems. For the sake of concreteness we only prove it for the bipartite case, the extension to the multiparty case being straightforward.

Let ρ_{AB} be a bipartite state and let $\tilde{\rho}_{ABA'B'}$ be its premeasurement state produced by measurement interactions $V_{A\mapsto AA'}^{\{|a_i\rangle_A\}}$ and $U_{B\mapsto BB'}^{\{|b_i\rangle_B\}}$, where $\mathcal{B}(A) = \{|a_i\rangle\}$ and $\mathcal{B}(B) = \{|b_i\rangle\}$ are some orthonormal bases of subsystem A and B,

$$
\tilde{\rho}_{ABA'B'} = \sum_{ijkl} \rho_{ijkl}|a_i\rangle\langle a_j|_A \otimes |b_k\rangle\langle b_l|_B \otimes |i\rangle\langle j|_{A'} \otimes |k\rangle\langle l|_{B'},
$$

where $\rho_{ijkl} = \langle a_i|\langle b_k|\rho_{AB}|a_j\rangle|b_l\rangle$. Observe that this state has indeed the maximally correlated form in the $(AB):(A'B')$ cut. Therefore, Corollary IV.1 implies

$$
Q_{\mathcal{N}}^{AB}(\rho_{AB}) = \min_{\mathcal{B}(A)\otimes\mathcal{B}(B)} \frac{\sum_{ijkl} |\rho_{ijkl}| - 1}{2} \tag{14}
$$

$$
= \min_{\mathcal{B}(A)\otimes\mathcal{B}(B)} \frac{\|\rho_{AB}\|_{l_1}^{\mathcal{B}(A)\otimes\mathcal{B}(B)} - 1}{2}. \tag{15}
$$

∎

This relation can be rewritten as

$$
Q_{\mathcal{N}}^{AB}(\rho_{AB})
$$
$$
= \min_{\mathcal{B}(A)\otimes\mathcal{B}(B)} \frac{1}{2}\|\rho_{AB} - (\Pi_{\mathcal{B}(A)} \otimes \Pi_{\mathcal{B}(B)})[\rho_{AB}]\|_{l_1}^{\mathcal{B}(A)\otimes\mathcal{B}(B)}, \tag{16}
$$

where $\Pi_{\mathcal{B}(A)}$ and $\Pi_{\mathcal{B}(B)}$ represent complete projective measurements on systems A and B, respectively, on the orthonormal basis $\mathcal{B}(A)$ and $\mathcal{B}(B)$. The superscript in the norm expression indicates that the l_1-norm is taken in the same local basis as for the projective measurement. Therefore, the total NoQ can be given the following interpretation.

Corollary IV.2: Decoherence interpretation of total NoQ. The total NoQ of n-partite quantum state $\rho_{\{1,2,...,n\}}$ is the minimum disturbance caused by a complete projective measurement on every system $\{1,2,...,n\}$ as quantified by the l_1-norm in the basis of the measurement.

Notice that the total NoQ corresponds to the absolute sum of the off-diagonal entries (*coherences*) of the density matrix, minimized over all local product bases [20].

B. Partial negativity of quantumness

Here we study the quantumness of correlations due to the nonclassical nature of single subsystems. Notice that this notion of partial (or one-sided, in the case there are only two subsystems) quantumness is well defined since the activation protocol Theorem III.1 applies.

Theorem IV.2. Partial negativity of quantumness. Let $\rho_{\{1,2,...,n\}}$ be an n-partite quantum state and $\Sigma \subseteq \{1,2,...,n\}$ a subset of subsystems, and denote the elements of Σ as $\{k_1,k_2,...\}$. For each subsystem $k \in \Sigma$, let $\mathcal{B}(k) = \{|a_{i_k}^{(k)}\rangle\}_{i_k}$ be one orthonormal basis of k. Then the *partial NoQ on subsystems* $\Sigma \subset \{1,2,...,n\}$ is

$$
Q_{\mathcal{N}}^{\Sigma}(\rho_{\{1,2,...,n\}}) = \min_{\bigotimes_{k\in\Sigma} \mathcal{B}(k)} \frac{1}{2}\left(\sum_{i_{k_1},i_{k_2},...} \|\rho_{i_{k_1},i_{k_2},...}\|_1 - 1\right),
$$

where $\rho_{i_1,i_2,...} = (\langle a_{i_1}^{(k_1)}|\langle a_{i_2}^{(k_2)}|\cdots)\rho_{\{1,2,...,n\}}(|a_{i_1}^{(k_1)}\rangle|a_{i_2}^{(k_2)}\rangle\cdots)$ and the minimum is taken over different choices of measurement interaction defined by $\bigotimes_{k\in\Sigma} \mathcal{B}(k)$.

Proof. The same proof applies to a system with an arbitrary number of subsystems, but for the sake of clarity and concreteness only the bipartite case is explicitly treated here. Let ρ_{AB}

be a bipartite quantum state; we want to calculate the QoN on subsystem A, $Q_\mathcal{N}^A(\rho_{AB})$. Let $\{a_i\}_{i\in\{1,2,\dots,m\}}$ be an orthonormal basis for subsystem A, which we assume to have finite dimension $\dim A = m$. Suppose that the premeasurement state

$$\tilde{\rho}_{ABA'} = \sum_{i,j=1}^m |a_i\rangle\langle a_j|_A \otimes \rho_{ij} \otimes |i\rangle\langle j|_{A'},$$

where $\rho_{ij} = \rho_{ij}^B = \langle a_i|\rho_{AB}|a_j\rangle$, is created by a measurement interaction $V_{A\mapsto AA'}^{\{|a_i\rangle_A\}}$. After the action by an appropriate permutation matrix, the partially transposed $\tilde{\rho}_{ABA'}$ on A' can be written in a block-diagonal form as we did for the maximally correlated states in the proof of Lemma IV.1:

$$P\tilde{\rho}_{ABA'}^\Gamma P^{-1}$$

$$= \begin{pmatrix} \rho_{11} & & & & & & & \\ & \ddots & & & & & & \\ & & \rho_{mm} & & & & & \\ \hdashline & & & 0 & \rho_{12} & & & \\ & & & \rho_{21} & 0 & & & \\ \hdashline & & & & & \ddots & & \\ \hdashline & & & & & & 0 & \rho_{m-1,m} \\ & & & & & & \rho_{m,m-1} & 0 \end{pmatrix}.$$

The only difference with the MCS case is that here the entries of the block matrices on the diagonal are matrices themselves rather than scalars. It is well known that the eigenvalues of a Hermitian matrix $\begin{pmatrix} 0 & H \\ H^\dagger & 0 \end{pmatrix}$ for an arbitrary matrix H are given by the singular values of H, taken with both positive and negative signs. Therefore, while the Schatten 1-norm of the upper left block is clearly $\sum_{i=1}^m \|\rho_{ii}\|_1$, that of the second block-diagonal matrix is $\|\rho_{1,2}\|_1 + \|\rho_{2,1}\|_1$, etc. Now, the Schatten 1-norm of block-diagonal matrices is simply the sum of Schatten 1-norms of each sub-block (see Appendix A). Therefore,

$$Q_\mathcal{N}^A(\rho_{AB}) = \min_{\mathcal{B}(A)} \frac{1}{2}\left(\sum_{i,j} \|\rho_{ij}\|_1 - 1 \right). \tag{17}$$

∎

V. PROPERTIES AND INTERPRETATIONS OF TOTAL AND PARTIAL NEGATIVITY OF QUANTUMNESS

A. Properties of NoQ

We list here some general properties of NoQ.

(i) *Positivity*: NoQ is non-negative for any quantum state.

(ii) *Faithfulness*: NoQ is faithful; i.e., it is zero if and only if the state is classical on the subsystems that are measured (all the subsystems in the case of the total NoQ).

(iii) *Negativity of quantumness exceeds negativity of entanglement [22]*: For any bipartite quantum state, the total NoQ exceeds the partial NoQ and both are always larger than the entanglement of the state as quantified by the negativity.

The first two properties are very natural requirements for a measure of nonclassical correlations. The first property follows directly from the definition; the second follows from

the activation protocol of Theorem III.1 and the fact that for a MCS, negativity is nonzero if and only if the state is entangled [32]. The third property follows from the hierarchy theorem of [22] and is exploited in the next section.

B. Negativity of entanglement versus negativity of quantumness

In [34], Khasin *et al.* studied the negativity of an arbitrary MCS and made a similar observation as ours; that is to say, the negativity of a MCS can be geometrically interpreted as the l_1-norm distance from the separable state given by considering only the diagonal component of that MCS. Using our notation, their result is the following.

Theorem V.1 [34]. Let $\tau_{AB} = \sum_{i,j=1}^n \tau_{ij}|a_i\rangle\langle a_j|_A \otimes |b_i\rangle\langle b_j|_B$ be a MCS for some $\tau_{ij} \in \mathbb{C}$ with respect to some orthonormal basis of each system $\mathcal{B}(A) = \{|a_i\rangle\}$ and $\mathcal{B}(B) = \{|b_j\rangle\}$. Then,

$$\mathcal{N}_{A:B}(\tau_{AB}) = \frac{1}{2}\|\tau_{AB} - \Pi_{\mathcal{B}(A)} \otimes \Pi_{\mathcal{B}(B)}[\tau_{AB}]\|_{l_1}^{\mathcal{B}(A)\otimes\mathcal{B}(B)},$$

$$\tag{18}$$

where Π_A, Π_B are complete projective measurements on the basis $\mathcal{B}(A), \mathcal{B}(B)$.

Observe that we have already obtained the same result, thanks to Eq. (16) and the equality of entanglement and quantumness for a MCS (see Appendix C or Ref. [30]).

Besides such a result, Khasin *et al.* conjectured that the negativity of any bipartite state is upper-bounded by the above quantity minimized over the choice of local bases. In this section, we prove their conjecture and moreover prove that for a MCS τ_{AB} the state $(\Pi_A \otimes \Pi_B)[\tau_{AB}]$ in Eq. (18) is indeed the closest separable state of τ_{AB} with respect to the l_1-norm in the $\mathcal{B}(A) \otimes \mathcal{B}(B)$ maximally correlated basis.

With respect to the conjecture of Khasin *et al.*, we find the following.

Theorem V.2. For any bipartite state ρ_{AB} it holds

$$\mathcal{N}(\rho_{AB}) \leqslant \min_{\mathcal{B}(A)\otimes\mathcal{B}(B)} \frac{1}{2}\|\rho_{AB} - \Pi_{\mathcal{B}(A)} \otimes \Pi_{\mathcal{B}(B)}[\rho_{AB}]\|_{l_1}^{\mathcal{B}(A)\otimes\mathcal{B}(B)},$$

where the minimization is taken over the choice of local product bases $\mathcal{B}(A) \otimes \mathcal{B}(B)$ and Π_A is a complete projection onto $\mathcal{B}(A)$ (similarly for B).

Proof. We know from [22] that $\mathcal{N}(\rho_{AB}) \leqslant Q_\mathcal{N}(\rho_{AB})$ holds for all ρ_{AB}. The claim is then obtained combining this with Eq. (16). ∎

We are also able to prove the following.

Theorem V.3. Consider a MCS $\tau_{AB} = \sum_{ij} \tau_{ij}|a_i\rangle\langle a_j| \otimes |b_i\rangle\langle b_j|$ where $\mathcal{B}(A) = \{|a_i\rangle\}$ and $\mathcal{B}(B) = \{|b_j\rangle\}$ are any orthonormal bases of subsystem A and B. Let us define a state $\sigma_{AB} = \sum_i \tau_{ii}|a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i|$; i.e., σ only has the diagonal components of τ_{AB} in the chosen local bases. Then one of the closest separable states (or, more precisely, one of the closest PPT states) to τ_{AB} with respect to the $\|\cdot\|_{l_1}$ norm in the $\mathcal{B}(A) \otimes \mathcal{B}(B)$ basis is σ_{AB}:

$$\|\tau_{AB} - \sigma_{AB}\|_{l_1}^{\mathcal{B}(A)\otimes\mathcal{B}(B)} = \min_{\xi\in\mathcal{PPT}} \|\tau_{AB} - \xi\|_{l_1}^{\mathcal{B}(A)\otimes\mathcal{B}(B)}$$

$$= \min_{\xi\in\mathcal{SEP}} \|\tau_{AB} - \xi\|_{l_1}^{\mathcal{B}(A)\otimes\mathcal{B}(B)},$$

where $\mathcal{PPT}$ is the set of all AB states with positive partial transpose, $\mathcal{SEP}(\subset \mathcal{PPT})$ is the set of all separable state in AB, and the l_1-norm is taken with respect to the basis

$\mathcal{B}(A) \otimes \mathcal{B}(B)$ in which τ_{AB} has the maximally correlated form.

Proof. See Appendix D. ∎

C. Interpretations of the negativity of quantumness

Any measure of quantumness defined through the activation protocol naturally possesses an operational meaning: the least amount of system-apparatus entanglement which will be created by any measurement interaction. Namely, NoQ quantifies such minimum entanglement in terms of negativity. However, it turns out that NoQ has some more possible interpretations: a geometric interpretation as the minimum distance from classical states (Approach 2) and an operational interpretation in terms of disturbance induced by a measurement (Approach 3). Here we restrict our attention to the study of bipartite states ρ_{AB}. In the bipartite case, we often refer to he total (partial) NoQ as to the two-sided (one-sided) NoQ.

With respect to the operational interpretation in terms of measurement disturbance, when the decohered quantum system A is a qubit the one-sided NoQ can be interpreted as the distinguishability of a quantum state from its partially decohered state.

Theorem V.4. Let ρ_{AB} be a bipartite quantum state with A a qubit. Then the partial NoQ on subsystem A of quantum state ρ_{AB} is equivalent to the minimum trace distance between ρ_{AB} and its decohered state on subsystem A:

$$Q_{\mathcal{N}}^A(\rho_{AB}) = \min_{\mathcal{B}(A)} \tfrac{1}{2} ||\rho_{AB} - \Pi_{\mathcal{B}(A)}[\rho_{AB}]||_1, \qquad (19)$$

where $\Pi_{\mathcal{B}(A)}$ is the complete projective measurement in the basis $\mathcal{B}(A)$ and the minimum is taken for different choices of basis $\mathcal{B}(A)$.

Proof. Suppose the subsystem A is a two-level system, so that the sum in Eq. (17) is limited to $i,j \in \{0,1\}$. Then,

$$\sum_{i,j=0}^{1} ||\rho_{ij}||_1 = ||\rho_{00}||_1 + ||\rho_{11}||_1 + ||\rho_{01}||_1 + ||\rho_{10}||_1$$

$$= 1 + \left\| \begin{pmatrix} 0 & \rho_{01} \\ \rho_{10} & 0 \end{pmatrix} \right\|_1$$

$$= 1 + ||\rho_{AB} - \Pi_{\mathcal{B}(A)}[\rho_{AB}]||_1.$$

Here, the second equality is due on one side to the fact that, ρ_{AB} being a normalized state, $||\rho_{00}||_1 + ||\rho_{11}||_1 = 1$; on the other side, for a block antidiagonal matrix one has $\|(\begin{smallmatrix} 0 & X \\ Y & 0 \end{smallmatrix})\|_1 = \|X\|_1 + \|Y\|_1$. ∎

Furthermore, we find the following equivalence.

Theorem V.5. Let ρ_{AB} be a bipartite quantum state where the quantum system A is two-dimensional. Then

$$\min_{\Pi_{\mathcal{B}(A)}} \|\rho_{AB} - \Pi_{\mathcal{B}(A)}[\rho_{AB}]\|_1 = \min_{\sigma \in \mathcal{CQ}} \|\rho_{AB} - \sigma\|_1. \quad (20)$$

Proof. The inequality

$$\min_{\Pi_{\mathcal{B}(A)}} \|\rho_{AB} - \Pi_{\mathcal{B}(A)}[\rho_{AB}]\|_1 \geqslant \min_{\sigma \in \mathcal{CQ}} \|\rho_{AB} - \sigma\|_1$$

holds trivially. Therefore, we prove the other direction.

Let $\{|i\rangle\}_{\{i=0,1\}}$ be the orthonormal basis of A in which the minimization of right-hand side of Eq. (20) is achieved (i.e., in which σ is explicitly classical on A, for an optimal classical-quantum σ). Now, $\rho - \sigma$ in the chosen basis has the following block-matrix form:

$$\begin{pmatrix} A & \rho_{01} \\ \rho_{10} & B \end{pmatrix},$$

with $\rho_{AB} = (\begin{smallmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{smallmatrix}), \sigma_{AB} = (\begin{smallmatrix} \sigma_{00} & 0 \\ 0 & \sigma_{11} \end{smallmatrix}), A = \rho_{00} - \sigma_{00}$, and $B = \rho_{11} - \sigma_{11}$. Let the singular value decomposition of ρ_{01} be $\rho_{01} = UDV$ and note that

$$\left\| \begin{pmatrix} A & \rho_{01} \\ \rho_{10} & B \end{pmatrix} \right\|_1 = \left\| \begin{pmatrix} U & 0 \\ 0 & V^\dagger \end{pmatrix} \begin{pmatrix} U^\dagger A U & D \\ D & V B V^\dagger \end{pmatrix} \begin{pmatrix} U^\dagger & 0 \\ 0 & V \end{pmatrix} \right\|_1$$

$$= \left\| \begin{pmatrix} U^\dagger A U & D \\ D & V B V^\dagger \end{pmatrix} \right\|_1$$

$$\geqslant \left\| \begin{pmatrix} \Pi[U^\dagger A U] & D \\ D & \Pi[V B V^\dagger] \end{pmatrix} \right\|_1$$

$$= \left\| \begin{pmatrix} D_1 & D \\ D & D_2 \end{pmatrix} \right\|_1,$$

where Π is a complete projective measurement on B that leaves D invariant, $D_1 = \Pi[U^\dagger A U], D_2 = \Pi[V B V^\dagger]$, and we have used that the trace distance is invariant under unitaries and monotone under general quantum operations.

Let $D_1 = \text{diag}(\{a_1, \cdots, a_n\}), D_2 = \text{diag}(\{b_1, \cdots, b_n\})$, and $D = \text{diag}(\{c_1, \cdots, c_n\})$. Notice that D depends only on the state ρ, not on σ nor on the choice of basis. Then

$$\|\rho_{AB} - \sigma\|_1 \geqslant \left\| \begin{pmatrix} D_1 & D \\ D & D_2 \end{pmatrix} \right\|_1$$

$$= \sum_{i=1}^{n} \left\| \begin{pmatrix} a_i & c_i \\ c_i & b_i \end{pmatrix} \right\|_1.$$

Now

$$\left\| \begin{pmatrix} a & c \\ c & b \end{pmatrix} \right\|_1 = \max_U \left| \text{Tr} \left(U \begin{pmatrix} a & c \\ c & b \end{pmatrix} \right) \right|$$

$$\geqslant \left| \text{Tr} \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & c \\ c & b \end{pmatrix} \right) \right| = 2c;$$

hence,

$$\min_{\sigma \in \mathcal{CQ}} \|\rho_{AB} - \sigma\|_1 \geqslant \sum_i 2c_i,$$

and such lower bound can be achieved when $a_i = b_i = 0$ for all i, i.e., when $A = B = 0$. The latter case corresponds to $\sigma_{00} = \rho_{00}$ and $\sigma_{11} = \rho_{11}$, i.e., $\sigma = \Pi_{\mathcal{B}(A)}[\rho_{AB}]$. ∎

Hence, when the single measured system is a qubit, Approach 2 and Approach 3 are equivalent also in the case in which the distance adopted is the *trace distance* rather than the Hilbert-Schmidt distance used in the case of the "standard" geometric discord [24,25]. Combining Eqs. (19) and (20) with the hierarchy of [22], we have that the trace-distance-based quantumness measure always exceeds the negativity (of entanglement) in bipartite states ρ_{AB} where A is a qubit. We have numerical evidence that the equivalence between the three approaches—activation (with negativity), geometric (with trace distance), and disturbance (with trace distance)—breaks down when the measured system is not a qubit.

For the two-sided case, we have already seen in Corollary IV.2 that, in general, the total NoQ can be interpreted as the minimum disturbance caused by local complete projective measurements on A and B, as quantified by the l_1-norm

in the basis in which the projection is taken. Moreover, by choosing properly the distance, it can be also interpreted from the perspective of Approach 2.

Theorem V.6. Let ρ_{AB} be a bipartite state. Then its two-sided NoQ is equivalent to the distance in l_1-norm from its closest $\mathcal{CC}$ state where the norm is with respect to the eigenbasis of the classical state,

$$Q_{\mathcal{N}}^{AB}(\rho_{AB}) = \min_{\sigma \in \mathcal{CC}} \tfrac{1}{2}\|\rho_{AB} - \sigma\|_{l_1}^{\mathcal{B}(\sigma)}, \tag{21}$$

where $\mathcal{B}(\sigma)$ denotes the eigenbasis[4] of σ.

Proof. Consider having fixed the local bases $\mathcal{B}(A)$ and $\mathcal{B}(B)$ for the $\mathcal{CC}$ state σ, so that $\mathcal{B}(\sigma) = \mathcal{B}(A) \otimes \mathcal{B}(B)$. Optimize now over the eigenvalues of σ, for fixed $\mathcal{B}(\sigma)$. It is clear that within such a class, the $\mathcal{CC}$ state σ that is optimal for the sake of $\|\rho_{AB} - \sigma\|_{l_1}^{\mathcal{B}(\sigma)}$ is the one that has the same diagonal (in the fixed basis) entries as ρ. For such σ it holds $\|\rho_{AB} - \sigma\|_{l_1}^{\mathcal{B}(\sigma)} = \|\rho_{AB} - (\Pi_{\mathcal{B}(A)} \otimes \Pi_{\mathcal{B}(B)})[\rho_{AB}]\|_{l_1}^{\mathcal{B}(A)\otimes\mathcal{B}(B)}$. The remaining minimization over the choice of local bases is the same as in Eq. (16). ∎

D. The mechanism of the activation protocol for negativity

In this section, we see that the (total) activation protocol using the negativity as an entanglement measure can be described in terms of an isometric mapping due to the measurement interaction: A bipartite quantum state and its closest $\mathcal{CC}$ state are mapped to a premeasurement state and the separable state closest to the latter, respectively.

The following lemma makes it clear what we mean by the fact that the measurement interaction is isometric with respect to the l_1-norm.

Lemma V.1. The l_1-norm is invariant under a measurement interaction provided that the l_1-norm is taken in the basis in which the measurement interaction is defined.

Proof. The proof is immediate and, for the sake of clarity and concreteness, we consider only the case of the measurement of a single system. Consider an operator X on system A and the measurement interaction $V_{A\mapsto AA'}^{\{|a_i\rangle_A\}}$ that acts on an orthonormal basis $\{|a_i\rangle\}$ of A as

$$|a_i\rangle_A \mapsto |a_i\rangle_A |i\rangle_{A'}.$$

Then the matrix representations of X in the basis $\{|a_i\rangle\}$ and that of $V_{A\mapsto AA'}^{\{|a_i\rangle_A\}} X (V_{A\mapsto AA'}^{\{|a_i\rangle_A\}})^{\dagger}$ in the basis $\{|a_i\rangle_A \otimes |i\rangle_{A'}\}$ have nonzero terms that are in one-to-one correspondence, so that

$$\|X\|_{l_1}^{\{|a_i\rangle_A\}} = \left\| V_{A\mapsto AA'}^{\{|a_i\rangle_A\}} X (V_{A\mapsto AA'}^{\{|a_i\rangle_A\}})^{\dagger} \right\|_{l_1}^{\{|a_i\rangle_A\otimes|i\rangle_{A'}\}}$$

holds. ∎

This isometric property of the measurement interaction and the following two observations—actually, restatements of results we obtained in the previous sections—allow us to draw a clear picture of the activation mechanism.

The first observation is that, according to Theorem V.6, the total NoQ of a bipartite quantum state can be interpreted as the l_1-norm distance from its closest classical state:

$$Q_{\mathcal{N}}^{AB}(\rho_{AB})$$
$$= \min_{\sigma \in \mathcal{CC}} \tfrac{1}{2}\|\rho_{AB} - \sigma\|_{l_1}^{\mathcal{B}(\sigma)}$$
$$= \min_{\mathcal{B}(A)\mathcal{B}(B)} \tfrac{1}{2}\|\rho_{AB} - (\Pi_{\mathcal{B}(A)} \otimes \Pi_{\mathcal{B}(B)})[\rho_{AB}]\|_{l_1}^{\mathcal{B}(A)\otimes\mathcal{B}(B)}.$$

The second observation is that, according to Theorem V.3, the closest separable state to a MCS τ_{AB} is the separable state corresponding to the diagonal—in the maximally correlated basis—entries of τ_{AB}:

$$\min_{\xi \in \mathcal{PPT}} \|\tau_{AB} - \xi\|_{l_1}^{\mathcal{B}(A)\otimes\mathcal{B}(B)} = \|\tau_{AB} - \sigma\|_{l_1}^{\mathcal{B}(A)\otimes\mathcal{B}(B)}.$$

Now, the measurement interaction acting with respect to the basis defined by $\Pi_{\mathcal{B}(A)}$ and $\Pi_{\mathcal{B}(B)}$ acts on the state and on its closest classical state as follows:

$$\rho_{AB} \mapsto \tilde{\rho}_{ABA'B'}$$
$$(\Pi_A \otimes \Pi_B)[\rho_{AB}] \mapsto (\Pi_A \otimes \Pi_B)[\tilde{\rho}_{ABA'B'}].$$

Due to Theorem V.3, the state $(\Pi_A \otimes \Pi_B)[\tilde{\rho}_{ABA'B'}]$ is indeed the closest separable state to the premeasurement state $\tilde{\rho}_{ABA'B'}$. In Appendix E we argue that that same isometric mapping "state $\mapsto$ premeasurement state" and "closest classical state $\mapsto$ closest (to the premeasurement state) separable state" hold also for the case of relative entropy used as a distance function.

The following diagram shows the isometric mapping for the NoQ:

$$
\begin{aligned}
\min_{\Pi_A\otimes\Pi_B} \|\tilde{\rho}_{ABA'B'} - (\Pi_A \otimes \Pi_B)[\tilde{\rho}_{ABA'B'}]\|_{l_1}^{\Pi_A\otimes\Pi_B} &=^{(ii)} \min_{\eta\in\mathcal{SEP}} \|\tilde{\rho}_{ABA'B'} - \eta\|_{l_1}^{\mathcal{MCS}} \\
&=_{(i)} \qquad\qquad = \mathcal{N}(\tilde{\rho}_{ABA'B'}) \\
\min_{\Pi_A\otimes\Pi_B} \|\rho_{AB} - (\Pi_A \otimes \Pi_B)[\rho_{AB}]\|_{l_1}^{\Pi_A\otimes\Pi_B} &=_{(iii)} \min_{\sigma\in\mathcal{CC}} \|\rho_{AB} - \sigma\|_{l_1}^{\mathcal{CC}} \\
&\qquad\qquad = Q_{\mathcal{N}}(\rho_{AB}).
\end{aligned}
\tag{22}
$$

In Eq. (22), equality (i) holds because the measurement interaction for the basis $\Pi_A \otimes \Pi_B$ is isometric; relation (ii)

---

[4] An eigenbasis is uniquely defined if there are no degeneracies in the spectrum; if there are, it is implicit that the basis $\mathcal{B}(\sigma)$ is chosen optimally to minimize the distance.

corresponds to the fact that the closest separable state to a MCS is given by its diagonal part; relation (iii) holds because the closest $\mathcal{CC}$ state is again its diagonal part. With $\|\cdot\|_{l_1}^{\mathcal{MCS}}$ we indicate that the l_1-norm is taken in the MCS basis of $\tilde{\rho}_{ABA'B'}$; similarly, with $\|\cdot\|_{l_1}^{\mathcal{CC}}$ we indicate that the l_1-norm is taken in the basis in which σ is excplicitly $\mathcal{CC}$.

VI. THE EQUIVALENCE OF NONCLASSICALITY MEASURES FOR BIPARTITE SYSTEMS WITH A TWO-LEVEL SUBSYSTEM

Recently, Gharibian has shown in [28] that the classicality of a bipartite quantum state ρ_{AB} on subsystem A can be tested by verifying the invariance of the state under some special local unitary operations. Similar results have been obtained independently by Giampaolo *et al.* [29]. Both works, which define Approach 4 in Sec. I, are based on a generalization to mixed states [41] of an approach to the quantification of pure-state entanglement via local unitaries [42]. In [28,29] it was proven that a quantum state is classical on one subsystem A if and only if there exists some operation from a particular set of local unitaries acting on the subsystem A, called *root-of-unity* operations, that leaves the state invariant. Therefore, the minimum disturbance caused by a local unitary from this nontrivial set was suggested as a measure of nonclassicality of correlations. We show that if the subsystem A under investigation is a two-level system (qubit), then the corresponding measure of nonclassicality defined by this approach (Approach 4), and those defined by Approach 1 and Approach 3, are all related.[5]

Definition VI.1: Root-of-unity operation [28,29,41]. Consider a n-dimensional quantum system A. Then the set of all unitary operators on A with spectrum $\{\omega_n^j\}_{j\in\{0,1,\cdots,n-1\}}$ for $\omega_n = e^{2\pi i/n}$ is called the set of *root-of-unity (RU)* operations.[6] We indicate such a set by $RU(A)$.

Approach 4 is then expressed by the following theorem.

Theorem VI.1: Nonclassicality by local unitary disturbance [28,29]. A bipartite quantum state ρ_{AB} is classical on subsystem A if and only if there exists a local unitary operation $V_A \in RU(A)$ which leaves the quantum state invariant, i.e.,

$$(V_A \otimes \mathbb{I}_B)\rho_{AB}(V_A \otimes \mathbb{I}_B)^\dagger = \rho_{AB}.$$

Now consider the case in which A is a qubit. By definition, a RU $V_A \equiv V$ of the qubit A has eigenvalues ± 1 and thus its spectral decomposition can be expressed as

$$V = |\phi\rangle\langle\phi| - |\phi^\perp\rangle\langle\phi^\perp|,$$

where $\{|\phi\rangle, |\phi^\perp\rangle\}$ is an orthonormal basis of a qubit. Then it is easy to see that the mapping

$$\rho \mapsto \tfrac{1}{2}(\rho + V\rho V^\dagger)$$

corresponds the totally dephasing operation in the basis $\{|\phi\rangle, |\phi^\perp\rangle\}$. This immediately implies

$$\rho - \Pi_A^{|\phi\rangle}[\rho] = \tfrac{1}{2}(\rho - V\rho V^\dagger),$$

where $\Pi_A^{|\phi\rangle}$ denotes the totally dephasing operation on system A in the basis $\{|\phi\rangle, |\phi^\perp\rangle\}$. Therefore, when A is a qubit, the quantification of nonclassicality of correlations by the two different approaches Approach 3 and Approach 4 is

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[5]The analytic equivalence of geometric discord and nonclassicality measure by local unitary invariance for a $2 \times N$-dimensional system was shown in [28,29] for the special case of Hilbert-Schmidt distance, but not from our general operational viewpoint.

[6]Note that this set corresponds to $\{UZU^\dagger \,|\, U \text{ unitary}, Z = \sum_j \omega_n^j |j\rangle\langle j|\}$ for $\{|j\rangle\}$ fixed to be, e.g., the computational basis.

equivalent up to a constant. In the original papers [28,29,41], the minimum disturbance caused by a RU operation was measured by the Hilbert-Schmidt norm, but, in principle, any norm-based distance can be considered, with the equivalence staying true.

VII. ANALYTIC EXAMPLES

In this section, we look at some special classes of states and obtain an analytic expression for the NoQ of these states. Indeed, as we have seen, the expression of NoQ includes an optimization over local bases and finding a closed analytic expression for general states appears to be challenging. The classes of states we study here, two-qubit states with a maximally mixed marginal, Werner states and isotropic states, all have properties—in particular, symmetries—that allow us to simplify the optimization. In [20] it was already proven that for two qudits,

$$Q_{\mathcal{N}}^A\left((1-p)\frac{\mathbb{1}}{d^2} + p|\psi\rangle\langle\psi|\right) = p\mathcal{N}(|\psi\rangle\langle\psi|),$$

where p is a probability, $\mathbb{1}/d^2$ is the maximally mixed state of the two qudits, and $|\psi\rangle$ is an arbitrary pure state. This previous result encompasses also two-qubit states not considered here, as well as isotropic states in arbitrary dimensions. In the latter case, the proof provided below is simpler and based on symmetry considerations.

A. Two-qubit states with one maximally mixed marginal

1. Bell diagonal states

Bell states are maximally entangled two-qubit states of the following form:

$$|\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle \pm |1\rangle|1\rangle),$$

$$|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle \pm |1\rangle|0\rangle).$$

Bell diagonal states are two-qubit states such that all of their eigenvectors are Bell states, i.e., of the form

$$\tau = p_0|\phi^+\rangle\langle\phi^+| + p_1|\psi^+\rangle\langle\psi^+| + p_2|\psi^-\rangle\langle\psi^-| + p_3|\phi^-\rangle\langle\phi^-|,$$

with $\{p_i\}$ a probability vector. The properties of Bell diagonal states that are used for the proof of the theorems in this section are summarized in Appendix F. Most importantly, Bell diagonal states can be completely characterized by three alternative—i.e., besides the above probability vector—real parameters, namely by the three elements R_{11}, R_{22}, and R_{33} of the *correlation matrix* $\mathbf{R} = [R_{\mu\nu}]$, with

$$R_{\mu\nu} = \text{Tr}[(\sigma_\mu \otimes \sigma_\nu)\rho], \quad \mu, \nu = 0, 1, \ldots, 3.$$

We are able to compute explicitly the one-sided NoQ for Bell diagonal states. Our result is summarized in the following theorem.

Theorem VII.1: One-sided NoQ for Bell diagonal states. Let $R_{00}, R_{11}, R_{22}, R_{33}$ be the correlation matrix elements of a Bell diagonal state ρ_{AB}. Rename and reorder $|R_{11}|, |R_{22}|, |R_{33}|$

according to their size as $\lambda_1,\lambda_2,\lambda_3$ with $1 \geqslant \lambda_3 \geqslant \lambda_2 \geqslant \lambda_1 \geqslant 0$. Then

$$Q_{\mathcal{N}}^A(\rho_{AB}) = \frac{\lambda_2}{2}. \tag{23}$$

Proof. Let $|\phi\rangle = (|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)/\sqrt{2}$ and $\phi_{AB} = |\phi\rangle\langle\phi|$, where throughout the proof $|i\rangle$ denotes the computational basis. Then, as explained in the Appendix F, a Bell diagonal state can be expressed as $(\mathbb{I}_A \otimes \Lambda_B)[\phi_{AB}]$, where $\Lambda(X) = \sum_{\mu=0}^{3} p_\mu \sigma_\mu X \sigma_\mu$ is a Pauli channel with $\{p_\mu\}_{\mu \in \{0,1,2,3\}}$ a probability vector. Now let $\{|a_i\rangle : i = 0,1\}$ be a basis for the qubit space. Then the expression of one-sided NoQ in Eq. (17) implies

$$\begin{aligned} Q_{\mathcal{N}}^A(\rho_{AB}) &= \min_{\{|a_0\rangle,|a_1\rangle\}} \big(\| \langle a_0| (\mathbb{I}_A \otimes \Lambda_B[\phi_{AB}]) |a_1\rangle \|_1 \\ &\quad + \| \langle a_1| (\mathbb{I}_A \otimes \Lambda_B[\phi_{AB}]) |a_0\rangle \|_1 \big)/2 \\ &\stackrel{(i)}{=} \min_{\{|a_0\rangle,|a_1\rangle\}} \max_U |\langle U, \langle a_0| (\mathbb{I}_A \otimes \Lambda_B[\phi_{AB}]) |a_1\rangle \rangle| \\ &\stackrel{(ii)}{=} \min_{\{|a_0\rangle,|a_1\rangle\}} \max_U |\langle U, \Lambda_B[\langle a_0|\phi_{AB}|a_1\rangle] \rangle| \\ &\stackrel{(iii)}{=} \min_{\{|a_0\rangle,|a_1\rangle\}} \max_U |\langle \Lambda_B[U], \langle a_0|\phi_{AB}|a_1\rangle \rangle| \\ &\stackrel{(iv)}{=} \min_{\{|a_0\rangle,|a_1\rangle\}} \max_U |\langle \Lambda_B[U], (|a_0^*\rangle\langle a_1^*|_B/2) \rangle| \\ &\stackrel{(v)}{=} \min_V \max_U |\langle 1|_B V^\dagger \Lambda_B[U] V |0\rangle_B |/2, \tag{24} \end{aligned}$$

where (i) holds because of the 1-norm invariance under Hermitian conjugation, so that the two terms on the right-hand side of the previous line are equal; (ii) holds because tracing on A commutes with operations on B; (iii) holds because a Pauli channel is self-dual; (iv) holds because for the maximally entangled state $|\phi\rangle$ one has $\langle\gamma|_A|\phi\rangle_{AB} = |\gamma^*\rangle_B/\sqrt{2}$, where $|\gamma^*\rangle = \gamma_0^*|0\rangle + \gamma_1^*|1\rangle$ if $|\gamma\rangle = \gamma_0|0\rangle + \gamma_1|1\rangle$; finally, (v) holds because $\{|a_i^*\rangle : i = 0,1\}$ is still an orthonormal basis and we can write $|a_i^*\rangle = V|i\rangle$ for V a unitary. Since $\langle 1|_B V^\dagger \Lambda_B[U] V |0\rangle_B|/2$ is an expression only on subsystem B, the subscripts are omitted in the following.

Now, the spectral decomposition of the unitary U can be expressed as

$$\begin{aligned} U &= e^{i\theta_1}|\psi\rangle\langle\psi| + e^{i\theta_2}|\psi^\perp\rangle\langle\psi^\perp| \\ &= e^{i\theta_1}(|\psi\rangle\langle\psi| + e^{i(\theta_2 - \theta_1)}(\mathbb{I} - |\psi\rangle\langle\psi|)) \\ &= e^{i\theta_1}(e^{i\theta}\mathbb{I} + (1 - e^{i\theta})|\psi\rangle\langle\psi|), \end{aligned}$$

where $|\psi\rangle, |\psi^\perp\rangle$ are eigenvectors of U and $\theta = \theta_2 - \theta_1$. Thus, ignoring the irrelevant global phase, we can rewrite the NoQ as

$$\begin{aligned} Q_{\mathcal{N}}^A(\rho_{AB}) &= \min_V \max_{\theta,|\psi\rangle} |\langle 1|V^\dagger \Lambda(e^{i\theta}\mathbb{I} \\ &\quad + (1 - e^{i\theta})|\psi\rangle\langle\psi|)V|0\rangle|/2 \\ &\stackrel{(i)}{=} \min_V \max_{|\psi\rangle} |\langle 1|V^\dagger \Lambda[|\psi\rangle\langle\psi|]V|0\rangle| \\ &\stackrel{(ii)}{=} \min_V \max_{|\psi\rangle} |\langle \frac{\sigma_x + i\sigma_y}{2}, V^\dagger \Lambda[|\psi\rangle\langle\psi|]V \rangle| \\ &= \min_V \max_{|\psi\rangle} \sqrt{\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2}/2, \tag{25} \end{aligned}$$

where $\langle\sigma_x\rangle$ is the expectation value of σ_x for the state $V^\dagger \Lambda(|\psi\rangle\langle\psi|)V$, i.e., $\langle\sigma_x, V^\dagger \Lambda(|\psi\rangle\langle\psi|)V\rangle$ and similarly $\langle\sigma_y\rangle$ is its expectation value of σ_y. For the equalities, in (i) we used the fact that any Pauli channel is unital and that $\max_\theta |1 - e^{i\theta}| = 2$, while (ii) is due to the relation $|0\rangle\langle 1| = (\sigma_x + i\sigma_y)/2$.

As already mentioned, there is a one-to-one correspondence between Pauli channels and Bell diagonal states, and a Pauli channel with corresponding Bell diagonal state described by $\{R_{ii}\}$ acts on a Bloch vector of components $n_i = \text{Tr}(\sigma_i \rho)/\sqrt{2}$, $i = x,y,z$ (see also Appendix F), according to

$$(n_x, n_y, n_z) \mapsto (R_{11}n_x, -R_{22}n_y, R_{33}n_z).$$

Therefore, its action on the Bloch sphere S^2 results in an ellipsoid of equatorial radii (R_{11}, R_{22}, R_{33}). Now rename $|R_{11}|, |R_{22}|, |R_{33}|$ according to their size as $\lambda_1,\lambda_2,\lambda_3$ with $1 \geqslant \lambda_3 \geqslant \lambda_2 \geqslant \lambda_1 \geqslant 0$. Considering that $\sqrt{\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2}$ represents the Euclidean distance from the origin of the projection of the state $V^\dagger \Lambda(|\psi\rangle\langle\psi|)V$ on the xy plane, one can easily imagine that $Q_{\mathcal{N}}^A(\rho_{AB})$ is achieved when V aligns the largest equatorial radius with the z axis and choosing $|\psi\rangle$ to be the state mapped in the direction of the second-largest equatorial radius (see Fig. 1). It is clear that this choice provides an upper bound $Q_{\mathcal{N}}^A(\rho_{AB}) \leqslant \lambda_2/2$. In the following we prove that this bound is saturated.

First, observe that an ellipsoid with equatorial radii $(\lambda_1,\lambda_2,\lambda_3)$ that are aligned with the coordinate axes can be expressed as

$$\{\underline{x} = (x_1,x_2,x_3) : \underline{x}^T A \underline{x} = 1\},$$

where $A = \text{diag}(1/\lambda_1^2, 1/\lambda_2^2, 1/\lambda_3^2)$. Therefore, an arbitrary ellipsoid with equatorial radii $(\lambda_1,\lambda_2,\lambda_3)$ can be expressed as

$$\{\underline{y} = (y_1,y_2,y_3) : \underline{y}^T B \underline{y} = 1\},$$

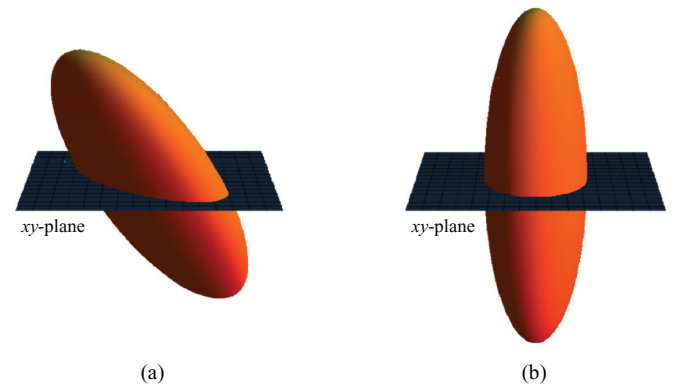|	
xy-plane	*xy*-plane
(a)	(b)

FIG. 1. (Color online) The problem of calculating the one-sided NoQ $Q_{\mathcal{N}}^A$ of Bell diagonal states cast in geometrical terms. (a) The surface of the ellipsoid is the set of states of the form $V^\dagger \Lambda(|\psi\rangle\langle\psi|)V$ with some unitary V for the set of all pure qubit states $|\psi\rangle$. Here we want to find the optimal unitary V such that it minimizes the maximum distance from the origin to its projection onto the xy plane. (b) Clearly choosing the unitary which rotates the ellipsoid such that it aligns the longest equatorial radii along z axis minimizes the maximum distance from the origin to its projection on the xy plane.

with $B = R^T A R$ for some rotation $R \in O(3)$. In particular, B has the same—positive—eigenvalues as A. We then find

$$
\begin{aligned}
\max_{\underline{y}:\, \underline{y}^T B \underline{y}=1} & \sqrt{y_1^2 + y_2^2} \\
&\overset{(i)}{\geqslant} \max\{\|\underline{y}\| : \underline{y} = (y_1, y_2, 0), \underline{y}^T B \underline{y} = 1\} \\
&\overset{(ii)}{=} \max\{(\underline{y}'^T B \underline{y}')^{-1/2} : \underline{y}' = (y_1', y_2', 0), \|\underline{y}'\| = 1\} \\
&= 1 \Big/ \sqrt{\min\{(\underline{y}'^T B \underline{y}') : \underline{y}' = (y_1, y_2, 0), \|\underline{y}'\| = 1\}} \\
&\overset{(iii)}{\geqslant} 1 \Big/ \sqrt{1/\lambda_2^2} = \lambda_2,
\end{aligned}
$$

where (i) holds because the cross section of an arbitrary ellipsoid with the xy plane is a subset of its projection onto the xy plane; (ii) holds because for any $\underline{y}$ satisfying $\underline{y}^T B \underline{y} = 1$ we can consider a normalized $\underline{y}' = \underline{y}/\|\underline{y}\|$, and for any normalized $\underline{y}'$ we can consider $\underline{y}/\sqrt{\underline{y}^T B \underline{y}}$ that satisfies by construction $\underline{y}^T B \underline{y} = 1$; finally, (iii) holds because Corollary III.1.2 of [43] implies

$$
\frac{1}{\lambda_2^2} = \max_{\substack{\mathcal{M} \subset \mathbb{R}^3 \\ \dim \mathcal{M}=2}} \min_{\substack{\underline{x} \in \mathcal{M} \\ \|\underline{x}\|=1}} \underline{x}^T B \underline{x}, \tag{26}
$$

where $\mathcal{M}$ is a two-dimensional subspace of $\mathbb{R}^3$.

Therefore, $Q_\mathcal{N}^A(\rho_{AB}) = \frac{\lambda_2}{2}$. ∎

With this result for the one-sided NoQ we can also easily solve the two-sided case.

Corollary VII.1: Two-sided NoQ for Bell diagonal states. For a Bell diagonal state ρ_{AB}, its total NoQ is $Q_\mathcal{N}(\rho_A) = \frac{\lambda_2}{2}$.

Proof. Since local unitaries acts as $O(3)$ elements on the correlation matrix of ρ_{AB}, one can always transform it to

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \lambda_1 & 0 & 0 \\
0 & 0 & \lambda_2 & 0 \\
0 & 0 & 0 & \lambda_3
\end{pmatrix}
$$

and a simple calculation shows that the $\|\cdot\|_{l_1}$-norm—in the computational basis—of a state with such correlation matrix is $1 + \lambda_2$. Since $Q_\mathcal{N}^A(\rho_{AB}) \leqslant Q_\mathcal{N}(\rho_{AB})$ [22], this is the best we can achieve; i.e., $Q_\mathcal{N}(\rho_{AB}) = \lambda_2/2$. ∎

2. Extension of the analysis for Bell diagonal states

For the analysis of Bell diagonal states, we made use of the fact that each Bell diagonal state can be expressed as a state generated by the action of a unique Pauli channel on the maximally entangled state. However, there is no reason to restrict the channel to be a Pauli channel and NoQ can be computed in a similar manner for the state generated by the action of a general qubit channel (CPTP map) on the maximally entangled state.

Theorem VII.2. Let ρ_{AB} be a two-qubit state where $\rho_A = \text{Tr}_B[\rho_{AB}]$ is maximally mixed. Let $\mathbf{R}$ be the correlation matrix of the state ρ_{AB} with elements $R_{\mu\nu} = \langle \sigma_\mu \otimes \sigma_\nu, \rho_{AB} \rangle$ and denote its 3×3 submatrix as $\hat{\mathbf{R}} := [R_{ij}]_{i,j \in \{1,2,3\}}$. Denote the singular values (including zeros) of $\hat{\mathbf{R}}$ in descending order as $s_1(\hat{\mathbf{R}}) \geqslant s_2(\hat{\mathbf{R}}) \geqslant s_3(\hat{\mathbf{R}})$.

Then $Q_\mathcal{N}^A(\rho_{AB}) = s_2(\hat{\mathbf{R}})/2$.

Proof. To apply the techniques from the analysis for the case of Bell diagonal states (Theorem VII.1), it is important to notice the relation between the matrix $\mathbf{T} \in \mathbb{R}^{4 \times 4}$ representing a Hermiticity-preserving linear map Ω (see Appendix G) and the correlation matrix $\mathbf{R}$ of the corresponding operator $\rho_{AB} = (\mathbb{I} \otimes \Omega)[\phi_{AB}]$, with ϕ_{AB} the standard maximally entangled state. This relation can be expressed as $R_{\mu\nu} = \frac{1}{2}(-1)^{\delta_{\mu 2}} T_{\mu\nu}$, with $\delta_{\mu 2}$ a Kronecker δ. Moreover, local unitaries acting on ρ_{AB} corresponds to the action of unitaries on the input and output of the channel. Namely, let $\mathbf{R}'$ be the correlation matrix of $(U_A \otimes U_B)\rho_{AB}(U_A \otimes U_B)^\dagger$, where U_A and U_B are local unitaries on system A and B. Then the matrix $\mathbf{T}'$ defined via $T'_{\mu\nu} = 2(-1)^{\delta_{\mu 2}} R'_{\mu\nu}$ represents the action of $W_{U_B} \circ \Omega \circ W_{U_A^T}$, where W_U stands for the conjugation by U, i.e., $W_U[X] = U X U^\dagger$. Also, it is easy to see that the singular values of the 3×3 submatrix $\hat{\mathbf{R}} = [R_{ij}]_{i,j \in \{1,2,3\}}$ and those of the submatrix of $[T_{ij}]_{i,j \in \{1,2,3\}}$ of $\mathbf{T}$ are equal up to the constant $1/2$.

Now we consider a state ρ_{AB} maximally mixed on A, which can always—and uniquely—be represented as $\rho_{AB} = (\mathbb{I}_A \otimes \Lambda)[\phi_{AB}]$, with Λ a channel. Now as explained in Appendix G, one can find unitaries U_A and U_B such that $\Lambda = W_{U_B} \circ \tilde{\Lambda} \circ W_{U_A}$ with the matrix representation $\mathbf{T}$ of $\tilde{\Lambda}$ in the canonical form

$$
\mathbf{T} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
t_1 & \lambda_1 & & \\
t_2 & & \lambda_2 & \\
t_3 & & & \lambda_3
\end{pmatrix}.
$$

Following the same steps taken in Eq. (24), one finds

$$
\begin{aligned}
Q_\mathcal{N}^A(\rho_{AB}) = \min_{\{|a_0\rangle, |a_1\rangle\}} & (\|\langle a_0|(\mathbb{I}_A \otimes \Lambda_B[\phi_{AB}])|a_1\rangle\|_1 \\
& + \|\langle a_1|(\mathbb{I}_A \otimes \Lambda_B[\phi_{AB}])|a_0\rangle\|_1)/2 \\
= \min_V \max_U & |\langle 1|_B V^\dagger \Lambda_B^\dagger[U] V |0\rangle_B|/2,
\end{aligned}
$$

where we have taken into account that now in general the channel λ is not self-dual, so that $\Lambda^\dagger \neq \Lambda$.

From $\Lambda = W_{U_B} \circ \tilde{\Lambda} \circ W_{U_A}$, so that $\Lambda^\dagger = W_{U_A^\dagger} \circ \tilde{\Lambda}^\dagger \circ W_{U_B^\dagger}$, and following step (i) of Eq. (25), we arrive at

$$
Q_\mathcal{N}^A(\rho_{AB}) = \min_V \max_{|\psi\rangle} |\langle 0|1|, V^\dagger \tilde{\Lambda}^\dagger[|\psi\rangle\langle\psi|] V\rangle|, \tag{27}
$$

having used the fact that for any channel Λ, the dual map $\Lambda^\dagger$ is unital, i.e., $\tilde{\Lambda}^\dagger[\mathbb{I}] = \mathbb{I}$.

Now, the action of $\tilde{\Lambda}^\dagger$ on a state with Bloch coordinates $(1, w_1, w_2, w_3)$ is

$$
\begin{aligned}
\tilde{\Lambda}^\dagger \left[\frac{1}{2} \left(\mathbb{I} + \sum_i w_i \sigma_i \right) \right] \\
= \frac{1}{2} \left[\left(1 + \sum_i t_i w_i \right) \mathbb{I} + \sum_i \lambda_i w_i \sigma_i \right], \tag{28}
\end{aligned}
$$

since the matrix representation of $\tilde{\Lambda}^\dagger$ is $\mathbf{T}^T$ (see Appendix G). While as soon as some t_i is different from zero the map $\tilde{\Lambda}^\dagger$ is not trace-preserving, we see that this is irrelevant, as the first term on the right-hand side of Eq. (28) effectively does not contribute to the right-hand side of Eq. (27). Hence, we can follow the proof of Theorem VII.1 as if we were dealing with a Pauli channel fully characterized by $\lambda_1, \lambda_2, \lambda_3$. ∎

B. Werner states

Here we present the formula for the one-sided and two-sided NoQ for general Werner states [35].

Definition VII.1: Werner states. Let A and B be d-dimensional quantum systems. Then a Werner state ρ_{AB} is a bipartite state of the form [35]

$$\rho_{AB} = \frac{\mathbb{I}_{AB} + \beta W}{d^2 + d\beta},$$

where $\beta \in \mathbb{R}$ satisfies $|\beta| \leqslant 1$, $W = \sum_{ij} |i\rangle\langle j| \otimes |j\rangle\langle i|$ is the swap operator and $\mathbb{I}_{AB}$ is the identity operator on AB. Werner states are UU invariant, i.e., $U_A \otimes U_B \rho_{AB} U_A^\dagger \otimes U_B^\dagger = \rho_{AB}$, for all U.

Theorem VII.3: NoQ of Werner states. For any Werner state $\rho_{AB} = \frac{\mathbb{I}_{AB} + \beta W}{d^2 + d\beta}$ the partial and the total NoQ are both equal to

$$Q_\mathcal{N}^A(\rho_{AB}) = Q_\mathcal{N}^{AB}(\rho_{AB}) = \frac{|\beta|(d-1)}{2(d+\beta)}.$$

Proof. First we prove the case for one-sided NoQ. Recall from Eq. (17), $Q_\mathcal{N}^A(\rho_{AB}) = \min_{\mathcal{B}(A)} \frac{1}{2}(\sum_{i,j} \|\rho_{ij}\|_1 - 1)$, where $\rho_{ij} = \rho_{ij}^B = \langle a_i|\rho_{AB}|a_j\rangle$ for $|a_i\rangle = U|i\rangle$ elements of the basis $\mathcal{B}(A)$, for some unitary U. It is clear that each $\|\rho_{ij}\|_1$ is invariant unitaries on B. We can then use the UU invariance of Werner states as follows:

$$\|\langle a_i|_A \rho_{AB}|a_j\rangle_A\|_1$$
$$= \|\langle i|_A U_A^\dagger \rho_{AB} U_A|j\rangle_A\|_1$$
$$= \|\langle i|_A U_A^\dagger (U_A \otimes U_B)\rho_{AB}(U_A \otimes U_B)^\dagger U_A|j\rangle_A\|_1$$
$$= \|U_B \langle i|_A \rho_{AB}|j\rangle_A U_B^\dagger\|_1$$
$$= \|\langle i|_A \rho_{AB}|j\rangle_A\|_1.$$

This proves that there is no need to perform the optimization to calculate the NoQ and one can use the computational basis on A. A straightforward calculation proves then the claim.

One can then calculate the two-sided NoQ checking that local measurements in the computational basis are optimal, since they reach the lower bound constituted by the one-sided value. ∎

C. Isotropic states

Isotropic states [36] are another class of bipartite states, amenable to an exact quantumness analysis.

Definition VII.2: Isotropic states. Isotropic states are bipartite quantum state of the following form [36]:

$$\rho_{AB} = \lambda\Phi + \frac{1-\lambda}{d^2 - 1}(\mathbb{I} - \Phi),$$

where d is the dimension of A and B, $0 \leqslant \lambda \leqslant 1$, and $\Phi = |\phi\rangle\langle\phi|$ with $|\phi\rangle = 1/\sqrt{d}\sum_{i=1}^{d} |i\rangle_A |i\rangle_B$ is the d-dimensional maximally entangled state. Isotropic states are UU^* invariant; i.e., $U_A \otimes U_B^* \rho_{AB} U_A^\dagger \otimes U_B^T = \rho_{AB}$, for all U.

Theorem VII.4. The one-sided and two-sided NoQ of an isotropic state are both equal to

$$Q_\mathcal{N}^A(\rho_{AB}) = Q_\mathcal{N}^{AB}(\rho_{AB}) = \frac{|\lambda d^2 - 1|}{d+1}.$$

Proof. The proof is essentially identical to the one for Werner states. One can use the UU^* invariance to prove that the optimization in the calculation of the one-sided NoQ in unnecessary. Then a straightforward calculation considering the computational basis of A leads to $Q_\mathcal{N}^A(\rho_{AB}) = \frac{|\lambda d^2 - 1|}{d+1}$ (similarly for B). The result for the two-sided NoQ is obtained matching this lower bound by considering measurements in the two local computational bases. ∎

VIII. CONCLUSIONS

In this paper we have quantitatively investigated a general notion of quantumness of correlations in bipartite and multipartite states [5]. Such quantumness can be related, for instance, to the disturbance induced on quantum states by local projective measurements.

We have reviewed several approaches to reveal and quantify the quantumness of correlations, proving that several of them are, in fact, equivalent in the general case of bipartite systems where local measurements act on a two-dimensional subsystem. We focused our analysis on a measure of quantumness of correlations defined as the minimum entanglement (measured by the negativity) created with a set of measurement apparatuses during the action of local measurements, following the so-called "activation" paradigm for nonclassical correlations [19–22]. The ensuing quantumness measure, known as NoQ, turns out to have very interesting properties. In particular, when the measured subsystem is a qubit it reduces to the minimum disturbance as measured by the trace distance or, alternatively, to the minimum trace distance to states that are classical on the qubit. We clarified the mechanism of the activation protocol for negativity and proved a bound on the negativity of arbitrary bipartite states conjectured in [34].

We finally presented a number of examples on which the NoQ can be computed exactly. These include relevant families of states such as Werner, isotropic, and two-qubit states that have one maximally mixed marginal. The latter class not only includes all Bell diagonal states, but also all the states isomorphic to an arbitrary qubit channel Λ via $(\mathbb{I}_A \otimes \Lambda_B)[\phi_{AB}]$, with ϕ_{AB} the standard maximally entangled state of two qubits. Given the hierarchical relation

$$Q_\mathcal{N}^A(\rho_{AB}) \geqslant \mathcal{N}(\rho_{AB})$$

of [22], the closed formula of this paper allows, e.g., a consistent study and comparison of the evolution of entanglement—as measured by negativity—and quantumness of correlations—as measured by the one-sided NoQ—under the action of a family of qubit channels, e.g., a semigroup. We remark that, while $Q_\mathcal{N}^A$ could increase under actions on A, both $Q_\mathcal{N}^A$ and $\mathcal{N}$ can only decrease under actions on B.

We believe the unveiled connections between apparently unrelated approaches to define and quantify general nonclassical correlations might inspire further research into the rationale of quantum measurements, possibly bringing a better understanding of the most essential features which mark a departure from a classical description of nature. The NoQ is an inherently relevant marker of nonclassicality, as it embodies the most general notion of quantum coherence for composite systems. From an applicative perspective, the NoQ has been already linked to the performance of certain quantum

information tasks. In particular, the authors of Ref. [44] have studied the primitive of remote state preparation in realistic models of one-way quantum computation based on noisy cluster states. They considered two possible sources of noise affecting the computation, either phase flip errors or white noise. The result was that the fidelity of remote state preparation in such a context does not relate either with entanglement or with the conventional measure of quantum discord for the considered mixed resource states, but it turns out to be a monotonic function of the NoQ (thereby referred to as "minimum entanglement potential") [44]. This links our measure with the quality of a quantum computation protocol in a specific practical setting. Other interesting features of the NoQ in open system dynamical evolutions have been investigated theoretically and experimentally in very recent works [45–47].

We can expect more general frameworks to be defined in the near future where the quantumness of correlations, possibly measured by the NoQ, can emerge as a resource to beat classical strategies for some particular task. Quantum communication and metrology seem to be fertile grounds for such an expectation to grow into practice. We finally remark that the NoQ can be bounded by experimentally accessible witnesses [48]. An experimental demonstration of the activation of nonclassical correlations into entanglement during local measurements, featuring an experimental estimation of the NoQ in a particular class of two-qubit states, will appear elsewhere [49].

Note added. Recently, we became aware of other works investigating a distance-based measure of quantumness based on trace norm, which is equivalent to the NoQ for qubit states [47,48,50–53]. In particular, Paula *et al*. [50] aim at computing the minimum distance between a two-qubit Bell-diagonal state and any classical-quantum state. Paula *et al*. make the assumption that the closest classical-quantum state is itself in the class of Bell-diagonal states and provide evidence for the correctness of this assumption via a numerical investigation. That the assumption is correct follows from Theorem V.5 in this work. More in general, Theorems V.4 and V.5 taken together imply that Approach 1, Approach 3, and Approach 2 are all equivalent when the system whose classicality is under scrutiny is a qubit, with the choice of trace-norm as distance and negativity as entanglement measure. It follows that the results of [50] must necessarily coincide with those of our Theorem VII.1, which we further generalize in Theorem VII.2.

APPENDIX A: PROPERTIES OF THE SCHATTEN *p*-NORMS

As *Schatten p-norms* are defined as

$$\|A\|_p := \left(\sum_i \sigma_i^p(A) \right)^{1/p}, \qquad (A1)$$

where $\sigma_i(A)$'s are the singular values of a matrix A, they are invariant under the action of unitary matrices, namely $\|A\|_p = \|UAV\|_p$ for any unitary matrices U and V.

If A can be partitioned into orthogonal block-diagonal matrices, i.e., if there exists mutually orthogonal projections $P_1, \ldots, P_k$ such that $P_1 \oplus \cdots \oplus P_k = \mathbb{I}$ and $A = \sum_{i=1}^k P_i A P_i$, then $\|A\|_p = \sum_{i=1}^k \|P_i A P_i\|_p$. This can be easily derived by the pinching inequality [43] and by the fact that equality can be achieved trivially.

Several transformations on the Schatten 1-norm which we use throughout the paper can be derived by these two properties. For example,

$$\left\| \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \right\|_1 = \left\| \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \right\|_1$$
$$= \left\| \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \right\|_1$$
$$= \|X\|_1 + \|Y\|_1.$$

There is an alternative expression for the Schatten 1-norm by its dual norm. The dual norm of $\| \cdot \|$, denoted as $\| \cdot \|^\circ$, is defined for a matrix A as

$$\|A\|^\circ := \sup_{\|B\|=1} |\langle B, A \rangle|.$$

It is known that the dual norm of the Schatten *p*-norm for $1 \leqslant p \leqslant \infty$ is the Schatten *q*-norm with $\frac{1}{p} + \frac{1}{q} = 1$ [43]. For the Schatten 1-norm,

$$\|A\|_1 = \sup_{\|B\|_\infty=1} |\langle B, A \rangle| = \max_{V \in \mathbb{U}(n)} |\langle V, A \rangle| = \max_{V \in \mathbb{U}(n)} |\mathrm{Tr}(VA)|,$$

for a matrix $A \in \mathbb{M}(n)$.

APPENDIX B: PROPERTIES OF THE l_1-NORM

The l_1-norm (sometimes called taxicab metric) of a vector is defined as the sum of absolute values of all entries. For $\underline{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$,

$$\|\underline{x}\|_{l_1} = \sum_{i=1}^n |x_i|.$$

We will be interested in applying this kind of norm to matrices $A \in \mathbb{M}(n)$, where $\mathbb{M}(n)$ is the set of $n \times n$ matrices with complex entries, so that

$$\|A\|_{l_1} = \sum_{i,j=1}^n |A_{ij}|. \qquad (B1)$$

One advantage of the norm (B1) is its ease of calculation, but it has some drawbacks. First, it is not submultiplicative; i.e., it does not respect $\|AB\|_{l_1} \leqslant \|A\|_{l_1} \|B\|_{l_1}$. Moreover, this norm is not invariant under conjugation by unitaries, which implies a matrix takes a different l_1-norm value depending on the basis

chosen for its representation. To take this into account, the basis with respect to which the l_1-norm is calculated is indicated as superscript when needed.

One notable feature of the l_1-norm which is used in the paper is the following.

Lemma B.1. For any $A \in \mathbb{M}(n)$ it holds

$$\|A\|_{l_1}^{\mathcal{B}} \geqslant \|A\|_1 \tag{B2}$$

independently of the basis $\mathcal{B}$ in which the l_1-norm is calculated.

If A is normal, i.e., $AA^\dagger = A^\dagger A$, then there is a choice of basis $\bar{\mathcal{B}}$ such that the minimum in (B2) is achieved, i.e.,

$$\|A\|_{l_1}^{\bar{\mathcal{B}}} = \|A\|_1. \tag{B3}$$

Proof. For a fixed choice of basis $\mathcal{B}$—the one in which we calculate $\|\cdot\|_{l_1}^{\mathcal{B}}$—consider the subset of matrices $\Omega^{\mathcal{B}}$ whose entries have modulus equal or smaller than 1, i.e., $\Omega^{\mathcal{B}} = \{[b_{ij}]_{i,j \in \{1,2,\dots,n\}} \in \mathbb{M}(n) ; |b_{ij}| \leqslant 1 , \forall i,j \in \{1,2,\dots,n\}\}$. The l_1-norm of A in the basis $\mathcal{B}$ can be written as

$$\|A\|_1^{\mathcal{B}} = \max_{V \in \Omega} |\operatorname{Tr}(VA)|. \tag{B4}$$

On the other hand, the 1-norm can be written as

$$\|A\|_1 = \max_{V \in \mathbb{U}(n)} |\operatorname{Tr}(VA)|, \tag{B5}$$

where $\mathbb{U}(n)$ is the set of $n \times n$ unitary matrices. Observe that $\mathbb{U}(n) \subset \Sigma$, because the rows and columns of a unitary matrix form a set of orthonormal vectors. Therefore, Eq. (B2) holds.

Equation (B3) is trivial because a normal matrix can be diagonalized by a change of basis. ∎

APPENDIX C: THE EQUALITY OF ENTANGLEMENT AND QUANTUMNESS FOR MCS

Theorem C.1. Let ρ_{AB} be a MCS. Then

$$Q_E^{AB}(\rho_{AB}) = Q_E^A(\rho_{AB}) = E_{A:B}(\rho_{AB}),$$

with Q_E^{AB} and Q_E^A the measures of quantumness (two-sided and one-sided, respectively) defined by means of (6) and $E_{A:B}$ the entanglement between A and B.

Proof. By the result of [22], $Q_E^{AB}(\rho_{AB}) \geqslant Q_E^A(\rho_{AB}) \geqslant E_{A|B}(\rho_{AB})$ holds. Therefore, it is sufficient to prove $E_{A|B}(\rho_{AB}) \geqslant Q_E^{AB}(\rho_{AB})$. By definition $Q_E^{AB}(\rho_{AB}) := \min E_{AB|A'B'}(\tilde{\rho}_{ABA'B'})$, where the minimum is taken over the choice of different pre-measurement state. Now we can choose a particular measurement interaction which acts on the basis of the maximally correlated form, i.e.,

$$\sum_{ij} \rho_{ij} |a_i\rangle\langle a_j| \otimes |b_i\rangle\langle b_j|$$

$$\mapsto \sum_{ij} \rho_{ij} |a_i\rangle\langle a_j| \otimes |b_i\rangle\langle b_j| \otimes |i\rangle\langle j|_{A'} \otimes |i\rangle\langle j|_{B'}.$$

The resultant state is equivalent to the original ρ_{AB} up to local unitary on AB or $A'B'$. Therefore, $E_{A|B}(\rho_{AB}) \geqslant Q_E^{AB}(\rho_{AB})$. ∎

APPENDIX D: PROOF OF THEOREM V.3

Proof. We simply denote τ_{AB} and σ_{AB} as τ, σ. Let $\eta = \sum_{klmn} \lambda_{klmn} |a_k\rangle\langle a_l| \otimes |b_m\rangle\langle b_n|$ be a traceless Hermitian

matrix such that $\xi = \tau + \eta$ is an arbitrary separable state. We want to prove that

$$\|\tau - \sigma\|_{l_1}^{\{|a_i\rangle \otimes |b_j\rangle\}} \leqslant \|\tau - \xi\|_{l_1}^{\{|a_i\rangle \otimes |b_j\rangle\}} = \|\eta\|_{l_1}^{\{|a_i\rangle \otimes |b_j\rangle\}}.$$

Since $\|\tau - \sigma\|_{l_1}^{\{|a_i\rangle \otimes |b_j\rangle\}} = \sum_{i \neq j} |\tau_{ij}|$, it is sufficient to prove that $\sum_{i \neq j} |\tau_{ij}| \leqslant \sum_{klmn} |\lambda_{klmn}|$.

In our argument we make use of the negative eigenvectors of τ^Γ

$$|\phi_{ij}\rangle = \frac{1}{\sqrt{2}}\left(|a_i\rangle \otimes |b_j\rangle - \frac{\tau_{ji}}{|\tau_{ji}|}(|a_j\rangle \otimes |b_i\rangle)\right),$$

where $i \neq j$, with corresponding eigenvalues $-|\tau_{ij}|$ (see Lemma IV.1).

We now consider two cases.

(1) Suppose the diagonal entries of ξ are the same as those of τ, i.e.,

$$\lambda_{ijkl} = 0 \quad \text{for} \quad i = j, \quad k = l.$$

Now consider the partial transpose of ξ, $\xi^\Gamma = \tau^\Gamma + \eta^\Gamma = \tau^\Gamma + \sum_{klmn} \lambda_{klmn} |a_k\rangle\langle a_l| \otimes |b_n\rangle\langle b_m|$. It holds

$$\begin{aligned}
&\langle \phi_{ij}|\xi^\Gamma|\phi_{ij}\rangle \\
&= \langle \phi_{ij}|\tau^\Gamma|\phi_{ij}\rangle + \langle \phi_{ij}| \sum_{klmn} \lambda_{klmn} |a_k\rangle\langle a_l| |b_n\rangle\langle b_m| |\phi_{ij}\rangle \\
&= -|\tau_{ij}| - \frac{1}{2|\tau_{ij}|}(\tau_{ji}^* \lambda_{ijij}^* + \tau_{ji} \lambda_{ijij}) \\
&= -|\tau_{ij}| - \operatorname{Re}\left[\frac{\tau_{ji}}{|\tau_{ij}|}\lambda_{ijij}\right].
\end{aligned}$$

By assumption ξ is a separable state, i.e., the numerical range of ξ^Γ is in a positive real line. Therefore, a necessary condition for ξ to be PPT is $\langle \phi_{ij}|\xi^\Gamma|\phi_{ij}\rangle \geqslant 0$ for all $i \neq j$.

We find

$$\begin{aligned}
\sum_{i \neq j} |\tau_{ij}| &\leqslant \sum_{i \neq j} -\operatorname{Re}\left[\frac{\tau_{ji}}{|\tau_{ij}|}\lambda_{ijij}\right] \\
&\leqslant \sum_{i \neq j} |\lambda_{ijij}| \leqslant \sum_{klmn} |\lambda_{klmn}|,
\end{aligned}$$

as claimed.

(2) Consider an arbitrary η, i.e., no conditions are imposed on the coefficients λ_{klmn} except that they lead to a separable state ξ. Then there are more terms in the expression for $\langle \phi_{ij}|\xi^\Gamma|\phi_{ij}\rangle$ than those encountered in the previous calculation. Namely,

$$\begin{aligned}
\langle \phi_{ij}|\xi^\Gamma|\phi_{ij}\rangle &= -|\tau_{ij}| - \frac{1}{2|\tau_{ij}|}(\tau_{ji}^* \lambda_{ijij}^* + \tau_{ji}\lambda_{jjii}) \\
&\quad + \frac{1}{2}(\lambda_{iijj} + \lambda_{jjii}^*) \\
&= -|\tau_{ij}| - \operatorname{Re}\left[\frac{\tau_{ji}}{|\tau_{ij}|}\lambda_{ijij}\right] + \frac{1}{2}(\lambda_{iijj} + \lambda_{jjii}^*).
\end{aligned}$$

Nonetheless, by imposing $\langle \phi_{ij} | \xi^\Gamma | \phi_{ij} \rangle \geqslant 0$ for all $i \neq j$ we find

$$\sum_{i \neq j} |\tau_{ij}| \leqslant \sum_{i \neq j} \left(-\operatorname{Re}\left[\frac{\tau_{ji}}{|\tau_{ij}|} \lambda_{ijij} \right] + \frac{1}{2}(\lambda_{iijj} + \lambda_{jjii}^*) \right)$$

$$\leqslant \sum_{i \neq j} (|\lambda_{ijij}| + |\lambda_{iijj}|)$$

$$\leqslant \sum_{klmn} (|\lambda_{klmn}|).$$

Therefore, for both cases, $\sum_{i \neq j} |\tau_{ij}|$ is the smallest possible value and σ is one of the closest separable state. ∎

APPENDIX E: THE ISOMETRIC MAPPING FOR QUANTUM RELATIVE ENTROPY

Relative entropy of entanglement is an entanglement measure defined as the "distance"—in terms of quantum relative entropy (see Sec. II)—from the closest separable state [31]:

$$E_R(\rho_{AB}) = \min_{\sigma_{AB} \in \mathcal{SEP}} S(\rho_{AB} || \sigma_{AB}). \quad (E1)$$

The measure of nonclassical correlation based on quantum relative entropy is called relative entropy of discord [23] and is defined as

$$E_R(\rho_{AB}) = \min_{\sigma_{AB} \in \mathcal{CC}} S(\rho_{AB} || \sigma_{AB}), \quad (E2)$$

and in [23] its equivalence to zero-way quantum deficit (8) was proved. Namely,

$$\min_{\sigma_{AB} \in \mathcal{CC}} S(\rho_{AB} || \sigma_{AB}) = \min_{\Pi_A \otimes \Pi_B} S(\rho_{AB} || \Pi_A \otimes \Pi_B[\rho_{AB}]). \quad (E3)$$

Since quantum relative entropy is invariant under linear isometry,

$$\min_{\Pi_A \otimes \Pi_B} S(\rho_{AB} || \Pi_A \otimes \Pi_B[\rho_{AB}])$$
$$= \min_{\Pi_A \otimes \Pi_B} S(\tilde{\rho}_{ABA'B'} || \Pi_A \otimes \Pi_B[\tilde{\rho}_{ABA'B'}]), \quad (E4)$$

where $\tilde{\rho}_{ABA'B'}$ is the premeasurement state constructed by the measurement interaction on the basis $\Pi_A \otimes \Pi_B$. Finally, by the exact expression of relative entropy of entanglement for MCS given in [32], one can deduce that $\Pi_A \otimes \Pi_B[\tilde{\rho}_{ABA'B'}]$ is indeed one of the closest separable states for $\rho_{ABA'B'}$.

Equation (E5) summarizes the relations and equivalences just explained:

$$\min_{\Pi_A \otimes \Pi_B} S(\tilde{\rho}_{ABA'B'} || (\Pi_A \otimes \Pi_B)[\tilde{\rho}_{ABA'B'}]) =_{(ii)} \min_{\eta \in \mathcal{SEP}} S(\tilde{\rho}_{ABA'B'} || \eta)$$

$$=_{(i)}$$

$$\min_{\Pi_A \otimes \Pi_B} S(\rho_{AB} || (\Pi_A \otimes \Pi_B)[\rho_{AB}]) \quad =_{(iii)} \quad \min_{\sigma \in \mathcal{CC}} S(\rho_{AB} || \sigma). \quad (E5)$$

In Eq. (E5), (i) is because the measurement interaction for the basis $\Pi_A \otimes \Pi_B$ is isometric, (ii) is because the closest separable state of MCS is its diagonal part, and (iii) is because the closest $\mathcal{CC}$ state is again its diagonal part.

APPENDIX F: BELL DIAGONAL STATES AND PAULI CHANNELS

Since the set of pure qubit states corresponds to the complex projective space $\mathbb{C}\mathbf{P}^1$ and there is an isomorphism between the unit sphere $\mathbf{S}^2 \subset \mathbb{R}^3$ and $\mathbb{C}\mathbf{P}^1$, the states of a qubit can be represented as points in a unit ball $\mathbf{B}^2$. Namely, for a qubit state with a density matrix ρ, define a vector $\underline{n} = (n_1, n_2, n_3) \in \mathbb{R}^3$ as $n_i = \operatorname{Tr}[\sigma_i \rho]$, where $\sigma_1 = \sigma_x, \sigma_2 = \sigma_y, \sigma_3 = \sigma_z$ are the Pauli matrices (see also Sec. II). The vector $\underline{n}$ is called the Bloch vector of the qubit state ρ. A pure ρ corresponds to a unit vector $\underline{n}$, while a mixed state has $||\underline{n}|| < 1$. That is, pure states correspond to $\mathbf{S}^2$ and mixed states to its interior. Conversely, one can recover the density matrix associated to a Bloch vector $\underline{n}$ via

$$\rho = \frac{1}{2}(\mathbb{I} + \underline{n} \cdot \underline{\sigma}),$$

where $\underline{n} \cdot \underline{\sigma} = \sum_{i=1}^3 n_i \sigma_i$.

This representation allows us to geometrically analyze qubit states. For example, when a Pauli channel of the form $\Lambda[\rho] = \sum_{i=0}^3 p_i \sigma_\mu \rho \sigma_\mu$ with $\{p_i\}$ a probability vector and $\sigma_0 = \mathbb{I}$ acts on a qubit sate ρ, the Bloch vector $\underline{n}$ of ρ transforms as

$$(n_1, n_2, n_3) \mapsto [(p_0 + p_1 - p_2 - p_3)n_1,$$
$$(p_0 - p_1 + p_2 - p_3)n_2,$$
$$(p_0 - p_1 - p_2 + p_3)n_3].$$

Bell diagonal states have some notable properties. First, a Bell diagonal state can be expressed as the action of a Pauli channel on a Bell state. More precisely, there is a one-to-one relation between the set of Pauli channels and the set of Bell diagonal states,

$$p_0 |\phi^+\rangle\langle\phi^+| + p_1 |\psi^+\rangle\langle\psi^+| + p_2 |\psi^-\rangle\langle\psi^-| + p_3 |\phi^-\rangle\langle\phi^-|$$
$$= (\Lambda \otimes \mathbb{I})[|\phi^+\rangle\langle\phi^+|],$$

where $\mathbb{I}$ indicates here the identity channel on a qubit and $\Lambda[\rho] = \sum_{\mu=0}^3 p_i \sigma_\mu \rho \sigma_\mu$ as defined before. Second, the correlation matrix of Bell diagonal states have diagonal form. A simple algebra shows that the correlation matrix of Bell diagonal states satisfies the following relations:

$$R_{00} = 1, \quad R_{ij} = 0 \quad \text{for} \quad i \neq j,$$
$$R_{11} + R_{22} + R_{33} \leqslant 1, \quad 1 + R_{11} + R_{22} \geqslant R_{33},$$
$$1 + R_{11} + R_{33} \geqslant R_{22}, \quad 1 + R_{22} + R_{33} \geqslant R_{11}.$$

Indeed the restrictions above force the vector (R_{11}, R_{22}, R_{33}) to be within a tetrahedron in $\mathbb{R}^3$ [54]. The two different pictures of Bell diagonal states, the Pauli channel representation and the

correlation matrix representation, are related in the following way:

$$
\begin{pmatrix} R_{00} \\ R_{11} \\ R_{22} \\ R_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}.
$$

APPENDIX G: CANONICAL FORM OF QUBIT CHANNELS

1. Matrix representation of qubit channels

The Bloch parametrization $n_\mu = \mathrm{Tr}(\sigma_\mu X)$, $\mu = 0, \dots, 3$, of a Hermitian matrix $X = X^\dagger \in \mathbb{M}(2)$ allows us to represent such Hermitian matrices—and in particular states, for which $n_0 = 1$—as vectors in $\mathbb{R}^4$. Any Hermiticity-preserving map $\Omega : \mathbb{C}^{2 \times 2} \to \mathbb{C}^{2 \times 2}$ can then be represented as matrix $\mathbf{T} \in \mathbb{R}^{\mathbf{4 \times 4}}$, with $T_{ij} = \langle \sigma_i, \Omega[\sigma_j] \rangle$. If Ω is a channel, so that in particular it preserves trace, then $T_{01} = T_{02} = T_{03} = 0$. Also, it is easy to check that the matrix representing the dual of Ω (see Sec. I) is given by the transpose $\mathbf{T}^T$ of the original matrix representation $\mathbf{T}$ of Ω.

2. The canonical form of qubit channels [55]

Let Γ be an arbitrary Hermiticity- and trace-preserving linear map on qubits represented by $\mathbf{T} \in \mathbb{R}^{\mathbf{4 \times 4}}$. Then one can find suitable local unitaries U_A and U_B such that the $\mathbf{T}$ representation of the new channel $\Gamma' = W_{U_A} \circ \Gamma \circ W_{U_B}$, with W_U acting by conjugation, i.e., $W_U[X] = U X U^\dagger$, has the canonical form

$$
\begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & \lambda_1 & & \\ t_2 & & \lambda_2 & \\ t_3 & & & \lambda_3 \end{pmatrix},
$$

where the λ_i's are the singular values of the 3×3 real submatrix $[T_{ij}]_{ij}$, $i,j = 1,2,3$. The conjugate channel of Γ' is $\Gamma'^\dagger = W_{U_B^\dagger} \circ \Gamma^\dagger \circ W_{U_A^\dagger}$ and is represented by the matrix

$$
\begin{pmatrix} 1 & t_1 & t_2 & t_3 \\ 0 & \lambda_1 & & \\ 0 & & \lambda_2 & \\ 0 & & & \lambda_3 \end{pmatrix}.
$$

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