Atom-molecule-conversion system subject to phase noises

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The dynamics of an atom-molecule-conversion system subject to dephasing noises is studied in this paper. Using a dephasing master equation and mean-field theory, we derive a Bloch equation for the system, and this equation is compared with the Bloch equation derived by the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy truncation approach. Fixed points of the system are calculated by solving both the Bloch equation and the master equation. Comparison between these two calculations suggests that while at short times the mean-field theory (MFT) is a good approximation for the atom-molecule-conversion system, a high-order hierarchy truncation approach is necessary for the system on a long time scale. Although the MFT cannot predict the fixed points correctly, its predictions for the stability of the fixed points are consistent with the BBGKY theory for a wide range of parameters.

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I. INTRODUCTION

In the realm of ultracold atom-molecule physics, association of ultracold atoms into diatomic molecules is an attractive subject. It inspires much interest due to its applications, ranging from the production of molecular Bose-Einstein condensates (BECs) to the study of chemical reactions and permanent electric dipole moments [1–8]. Coherent oscillations between an atomic BEC and a molecular BEC have been theoretically predicted [9,10] by the use of Gross-Pitaevskii (GP) equations [11–16], the results suggest that the mean-field theory is a good formalism to describe the conversion of atoms to molecules in the absence of noise [17,18].

The noise may come from inelastic collisions between the atoms in the condensates and the noncondensate atoms, and local and nonlocal fluctuations. The noise may also come from random variation of the atom-molecule detuning or magnetic field fluctuations in the Feshbach-resonance setup [17,19–21]. The presence of noise can dephase the Bose-Einstein condensates and strongly limit the validity of the Gross-Pitaevskii equations. There have been several theoretical studies going beyond the GP equations; for example, based on time-dependent field theory, the dynamics of an atom-molecule-conversion system was studied in [22,23], where the noise comes from nonlocal fluctuations due to the time-dependent pair correlations; and within the two-mode approximation, the authors in Refs. [19,24] explored the master equation to investigate the atom-molecule-conversion system.

Earlier studies on a bimodal decoherence-free condensate have shown that the mean-field theory (MFT) may fail near a dynamical instability [25,26]; this inspires us to explore whether the MFT is valid for an atom-molecule-conversion system with noise (dissipation and dephasing). The effect of dissipation on the dynamics of an atom-molecule-conversion system was studied in [24]. In this paper we will focus on the effect of dephasing within the two-mode approximation. We show that the dynamics of a dephasing atom-molecule-conversion system is well treated by the MFT on the short time scale, but it fails to give a correct prediction about the system at the long time scale. This suggests that we use a high order of the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy truncation [25,26] to explore the atom-molecule-conversion system subject to dephasing noises.

The remainder of the paper is organized as follows. In Sec. II, we introduce the dephasing master equation and derive a Bloch equation for the system; the solution of the Bloch equation without dephasing is presented and discussed. In Sec. III, we calculate the fixed points of the system with MFT and compare these fixed points with those obtained by analytically solving the master equation. The Bloch equation derived from the BBGKY hierarchy equation is presented in Sec. IV. In Sec. V, we discuss the stability and the features of the fixed points from both the MFT and the BBGKY hierarchy truncation. Discussion and conclusions are given in Sec. VI.

II. MODEL

We consider the simplest model for the atom-molecule-conversion system. By the two-mode approximation, the model Hamiltonian can be written as [11,19,27]

$$\hat{H} = \frac{\varepsilon}{2}\hat{a}^\dagger\hat{a} + \frac{g}{2}(\hat{a}^\dagger\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}\hat{a}), \tag{1}$$

where $\hat{a}$ and $\hat{b}$ represent annihilation operators for the atom and molecule, respectively, g denotes the strength of the atom-molecule conversion, and ε is the atomic binding energy.

The master equation taking only the dephasing noise into account may be written in the following form [19,28]:

$$\dot{\rho} = -i[\hat{H},\rho] - \Gamma[\hat{\ell},[\hat{\ell},\rho]], \tag{2}$$

where $\hat{\rho}$ is the density matrix of system, Γ is the dephasing rate, and the Lindblad operator $\hat{\ell}$ is the population difference,

$$\hat{\ell} = 2\hat{b}^\dagger\hat{b} - \hat{a}^\dagger\hat{a}. \tag{3}$$

The total atom number operator $\hat{N} = 2\hat{b}^\dagger\hat{b} + \hat{a}^\dagger\hat{a}$ is conserved since $\frac{\partial\langle\hat{N}\rangle}{\partial t} = 0$, so the total atom number N is a constant that does not change with time in the dynamics. Defining

$$\hat{L}_x = \sqrt{2}\frac{\hat{a}^\dagger\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}\hat{a}}{N^{3/2}}, \quad \hat{L}_y = \sqrt{2}i\frac{\hat{a}^\dagger\hat{a}^\dagger\hat{b} - \hat{b}^\dagger\hat{a}\hat{a}}{N^{3/2}}, \quad \hat{L}_z = \frac{2\hat{b}^\dagger\hat{b} - \hat{a}^\dagger\hat{a}}{N}, \tag{4}$$

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where $\hat{L}_z$ denotes the number difference between the atoms and the molecules in the system, $\hat{L}_x$ and $\hat{L}_y$ can be used to characterize the coherence of atom-molecule conversion. It is easy to prove that

$$[\hat{L}_z,\hat{L}_x]=\frac{4i}{N}\hat{L}_y,\quad [\hat{L}_z,\hat{L}_y]=-\frac{4i}{N}\hat{L}_x,$$
$$[\hat{L}_x,\hat{L}_y]=\frac{i}{N}(1-\hat{L}_z)(1+3\hat{L}_z)+\frac{4i}{N^2}. \tag{5}$$

Notice that $\hat{L}_x,\hat{L}_y,\hat{L}_z$ are not the SU(2) generators, because their commutation relations contain quadratic terms in $\hat{L}_z$. Nevertheless, in the limit of small atom-molecule-number difference and large N ($N\to\infty$), $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ really form a sphere since they satisfy

$$(\hat{L}_x)^2+(\hat{L}_y)^2=\frac{1}{2}(1+\hat{L}_z)(1-\hat{L}_z)^2+\frac{2}{N}(1-\hat{L}_z)+\frac{4}{N^2}\hat{L}_z. \tag{6}$$

We will call this sphere the generalized Bloch sphere even when the system is far from the limits. With these definitions, the Hamiltonian becomes $\hat{H}=-\frac{\varepsilon}{4}N\hat{L}_z+\frac{g}{2\sqrt{2}}N^{3/2}\hat{L}_x$, and the master equation can be rewritten as

$$\dot{\rho}=-i[\hat{H},\rho]-\Gamma N^2[\hat{L}_z,[\hat{L}_z,\rho]]. \tag{7}$$

From this master equation, the expectation values defined by $F_i=\langle\hat{L}_i\rangle=\mathrm{Tr}(\rho\hat{L}_i)$, $i=x,y,z$ are as follows:

$$\begin{aligned}\frac{\partial F_x}{\partial t}&=\varepsilon F_y-16\Gamma F_x,\\ \frac{\partial F_y}{\partial t}&=-\varepsilon F_x-\Delta F_z+\frac{3}{2}\Delta\langle\hat{L}_z^2\rangle-16\Gamma F_y-R,\\ \frac{\partial F_z}{\partial t}&=2\Delta F_y,\end{aligned} \tag{8}$$

where $\Delta=g\sqrt{\frac{N}{2}}$, and $R=\frac{1}{2}\Delta+\frac{2\Delta}{N}$. The lowest-order truncation of Eq. (8) is acquired by approximating the second-order expectation values $\langle\hat{L}_i\hat{L}_j\rangle$ as products of the first-order expectations $\langle\hat{L}_i\rangle$ and $\langle\hat{L}_j\rangle$ [25], namely,

$$\langle\hat{L}_i\hat{L}_j\rangle\approx\langle\hat{L}_i\rangle\langle\hat{L}_j\rangle. \tag{9}$$

With this approximation, Eq. (8) reduces to

$$\begin{aligned}\frac{\partial F_x}{\partial t}&=\varepsilon F_y-16\Gamma F_x,\\ \frac{\partial F_y}{\partial t}&=-\varepsilon F_x-\Delta F_z+\frac{3}{2}\Delta F_z^2-16\Gamma F_y-R,\\ \frac{\partial F_z}{\partial t}&=2\Delta F_y.\end{aligned} \tag{10}$$

Next we discuss the situation with zero dephasing rate, $\Gamma=0$, when Eq. (10) becomes

$$\begin{aligned}\frac{\partial F_x}{\partial t}&=\varepsilon F_y,\\ \frac{\partial F_y}{\partial t}&=-\varepsilon F_x-\Delta F_z+\frac{3}{2}\Delta F_z^2-R,\\ \frac{\partial F_z}{\partial t}&=2\Delta F_y.\end{aligned} \tag{11}$$

Defining $a=\varepsilon^2+g^2N$, $b=-\frac{3}{2}g^2N$, and $c=\frac{g^2N}{2}+2g^2-\varepsilon^2F_{z0}$ with F_{z0} the initial value of F_z, the solution of Eq. (11) can be obtained by solving

$$\frac{\partial^2F_z}{\partial^2t}+aF_z+bF_z^2+c=0. \tag{12}$$

We notice that b must not be zero here, otherwise $g=0$, which would result in $\Delta=0$ leading to $\dot{F}_z(t)=0$, when $F(t)_z\equiv F_{z0}$, i.e., the state of the system remains unchanged. The solution of Eq. (12) is

$$F_z=u_2-(u_2-u_3)\mathrm{cn}^2(k(t-t_0),m)-\frac{a-A}{2b}, \tag{13}$$

where $\mathrm{cn}(k(t-t_0),m)$ is the Jacobi elliptic cosine function. $u_1>u_2>u_3$ and $u_1=n\cos\theta-\frac{A}{2B}$, $u_2=n\cos(\theta+\frac{4\pi}{3})-\frac{A}{2B}$, $u_3=n\cos(\theta+\frac{2\pi}{3})-\frac{A}{2B}$, $n=\frac{A}{B}$, $\cos(3\theta)=-\frac{1}{2}\quad d[(\frac{2B}{A})^3+2)]$, $A=\sqrt{a^2-4bc}, B=b$, $d=-u_0^3-\frac{3A}{2B}u_0^2$, and $u_0=F_{z0}+\frac{a-A}{2b}$.

$F_z(t)$ is a periodic function of time with period $T=\frac{2K(m)}{k}, k=\sqrt{\frac{-B(u_1-u_3)}{6}}$, and $K(m)=\int_0^{\pi/2}\frac{1}{\sqrt{1-m^2\sin^2(\varphi)}}d\varphi$ is the Legendre complete elliptic integral of the first kind. t_0 denotes the time when F_z becomes F_{z0} which can be determined by solving Eq. (13). Equation (11) describes a rotation of the Bloch vector $\mathbf{F}$; obviously the norm $|\mathbf{F}|$ is conserved in the MFT when the dephasing rate is zero.

In Fig. 1, we plot the ratio of N_a to N as a function of time. Two results are presented, one coming from Eq. (13), and the other obtained by solving the master equation with $\Gamma=0$ numerically. We find that at a short time scale, the two results are in good agreement; however, at a long time scale, the two results are evidently different. This suggests that the MFT is a good approximation to describe the dynamics of the atom-molecule-conversion system at a short time scale. In addition, the binding energy of the atom can turn the system from the self-trapping regime [Fig. 1(a)] to the tunneling regime

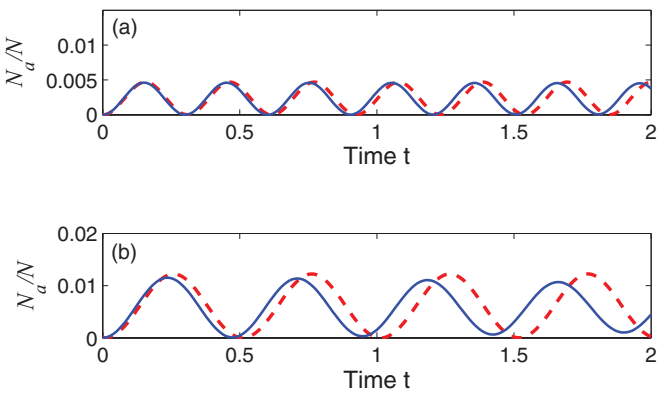

FIG. 1. (Color online) The number of atoms in the atomic mode as a function of time. The red dashed line represents Eq. (13), which is an analytical solution of the Bloch equation with $\Gamma=0$ based on the mean-field theory. In contrast, the numerical simulation of the Liouville equation (2) with $\Gamma=0$ is shown by the blue solid line with $F_{z0}=1, F_{x0}=F_{y0}=0$ at time $t=0$. Here and hereafter, ε and Γ are rescaled in units of g, and t is then in units of $1/g$. Hence all parameters are dimensionless. $N=100$, $g=1$. (a) and (b) are for different ε. (a) $\varepsilon=25$, and (b) $\varepsilon=19$.

[Fig. 1(b)]. This can be understood as a conversion blockage due to the energy difference between the atoms and molecules. Note that the binding energy of the molecules is zero.

III. STEADY STATE AND FIXED POINTS

The fixed point of the system is defined by

$$\dot{F}_x = \dot{F}_y = \dot{F}_z = 0. \tag{14}$$

By this definition, we can obtain the fixed points in the MFT,

$$F_{xf} = F_{yf} = 0, F_{zf} = \frac{1}{3}\left(1 - \sqrt{1 + 3\,(1 + 4/N)}\right). \tag{15}$$

On the other hand, we can obtain the steady state by analytically solving the master equation (2). Once we have the steady state of the system, the fixed points can be calculated by the definition of F_j. The steady state ρ_s satisfies the following equation:

$$\dot{\rho}_s = -i[\hat{H}, \rho_s] + \Gamma(2\hat{\ell}\rho_s\hat{\ell} - \hat{\ell}\hat{\ell}\rho_s - \rho_s\hat{\ell}\hat{\ell}) = 0. \tag{16}$$

It is easy to prove that the off-diagonal elements of the density matrix vanish in the steady state due to the dephasing. The proof is as follows. Defining the Fock states $|n\rangle \equiv |N - 2n, n\rangle$ denoting $(N - 2n)$ atoms and n molecules $(n = 0, 1, 2, \ldots, N/2)$, we have the following equation for the off-diagonal elements of the density matrix:

$$\frac{\partial \rho_{mn}}{\partial t} + i(a_m - a_n)\rho_{mn} + 16\Gamma(m - n)^2\rho_{mn} + \xi(t) = 0, \tag{17}$$

where $\xi(t) = i(b_m\rho_{m-1n} + c_m\rho_{m+1n} - b_n\rho_{mn-1} - c_n\rho_{mn+1})$, $a_n = \frac{\varepsilon}{2}(N - 2n)$, $b_n = \frac{g}{2}[\sqrt{(N - 2n + 1)(N - 2n + 2)(n)}]$, and $c_n = \frac{g}{2}[\sqrt{(N - 2n)(N - 2n - 1)(n + 1)}]$. The formal solution of Eq. (17) is

$$\rho_{mn} = e^{-[i(a_m - a_n) + 16\Gamma(m-n)^2]t} \times \left[\Xi - \int \xi(t) e^{[i(a_m - a_n) + 16\Gamma(m-n)^2]t} dt\right], \tag{18}$$

where Ξ is a constant determined by the initial condition of ρ_{mn}. We find that when $t \to \infty$, $\rho_{mn} \to 0$ $(m \neq n)$. This gives the steady state,

$$\rho_s = \sum_{n=0}^{N/2} \rho_n |n\rangle \langle n|. \tag{19}$$

For the steady state, it is required that $[\hat{H}, \rho_s] = 0$, from which we obtain $\rho_j = \rho_{j-1}$. From this together with $\mathrm{Tr}\rho_s = 1$, we obtain $\rho_0 = \rho_1 = \rho_2 = \cdots = \rho_{N/2} = \frac{1}{N/2+1}$. Collecting all together, we have

$$\rho_s = \sum_{n=0}^{N/2} \left(\frac{1}{N/2 + 1}\right) |n\rangle \langle n|. \tag{20}$$

The fixed points F_{is} $(i = x, y, z)$ of the system can be given by the steady-state Eq. (20) as $F_{zs} = \mathrm{Tr}(\rho_s \hat{L}_z) = \sum_{n=0}^{N/2} \hat{L}_z(\frac{1}{N/2+1}) |n\rangle \langle n| = 0$. In the same way, $F_{xs} = F_{ys} = 0$. Namely, the fixed point given by solving the master equation is

$$F_{xs} = F_{ys} = F_{zs} = 0. \tag{21}$$

It is easy to see that the fixed points given by the MFT and the master equation are different. This indicates that the MFT is not a good approach to describe the atom-molecule-conversion system at a long time scale. This stimulates us to use the BBGKY hierarchy truncation [25,26] to study the system.

IV. THE BBGKY HIERARCHY OF EQUATIONS OF MOTION

As aforementioned, the differential equation for the Bloch vector up to the first order is not a good treatment at a long time scale. Thus high-order expectation values are required. In this section, we will obtain an improvement to the MFT using the next order of the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy of equations of motion.

Writing $\langle \hat{L}_z^2 \rangle$ in Eq. (8) in terms of the expectation value

$$K_{ij} = \langle \hat{L}_i \hat{L}_j + \hat{L}_j \hat{L}_i \rangle - 2\langle \hat{L}_i \rangle \langle \hat{L}_j \rangle, \quad i, j = x, y, z, \tag{22}$$

and truncating the BBGKY hierarchy of equations of motion for the first- and second-order operators $\hat{L}_i, \hat{L}_i\hat{L}_j$ [25,26],

$$\langle \hat{L}_i \hat{L}_j \hat{L}_k \rangle \approx \langle \hat{L}_i \hat{L}_j \rangle \langle \hat{L}_k \rangle + \langle \hat{L}_i \rangle \langle \hat{L}_j \hat{L}_k \rangle + \langle \hat{L}_i \hat{L}_k \rangle \langle \hat{L}_j \rangle - 2\langle \hat{L}_i \rangle \langle \hat{L}_j \rangle \langle \hat{L}_k \rangle, \tag{23}$$

we get the following set of equations for the first- and second-order moments:

$$\begin{aligned}
\frac{\partial F_x}{\partial t} &= \varepsilon F_y - 16\Gamma F_x, \\
\frac{\partial F_y}{\partial t} &= -\varepsilon F_x - \Delta F_z + \frac{3}{2}\Delta\left(\frac{1}{2}K_{zz} + F_z^2\right) - 16\Gamma F_y - R, \\
\frac{\partial F_z}{\partial t} &= 2\Delta F_y, \\
\frac{\partial K_{xx}}{\partial t} &= 2\varepsilon K_{xy} - 32\Gamma K_{xx} + 32\Gamma K_{yy} + 64\Gamma F_y^2, \\
\frac{\partial K_{yy}}{\partial t} &= -2\varepsilon K_{xy} - 2\Delta K_{yz} + 6\Delta F_z K_{yz} - 32\Gamma K_{yy} \\
&\quad + 32\Gamma K_{xx} + 64\Gamma F_x^2, \\
\frac{\partial K_{zz}}{\partial t} &= 4\Delta K_{yz}, \\
\frac{\partial K_{xy}}{\partial t} &= -\varepsilon K_{xx} - \Delta K_{xz} + 3\Delta F_z K_{xz} + \varepsilon K_{yy} - 64\Gamma K_{xy} \\
&\quad -64\Gamma F_x F_y, \\
\frac{\partial K_{yz}}{\partial t} &= 2\Delta K_{yy} - \varepsilon K_{xz} - \Delta K_{zz} + 3\Delta F_z K_{zz} - 16\Gamma K_{yz}, \\
\frac{\partial K_{xz}}{\partial t} &= 2\Delta K_{xy} + \varepsilon K_{yz} - 16\Gamma K_{xz}.
\end{aligned} \tag{24}$$

Equations (24) have been called Bogoliubov back-reaction (BBR) equations [25,26], because the fluctuations K_{ij} are driven by the mean-field Bloch vector $\mathbf{F}$, which is physically described by the Bogoliubov theory. In turn, the Bloch vector is affected by the fluctuations K_{ij}. This back reaction causes the trajectory of the system to not be confined to the surface of the generalized Bloch sphere, which is reminiscent of the effect of dephasing.

We plot the time evolution of the atom number N_a and the fluctuation K_{zz} given by the BBR equations and MFT in Fig. 2 and the time evolution of F_z in Fig. 3. The results from

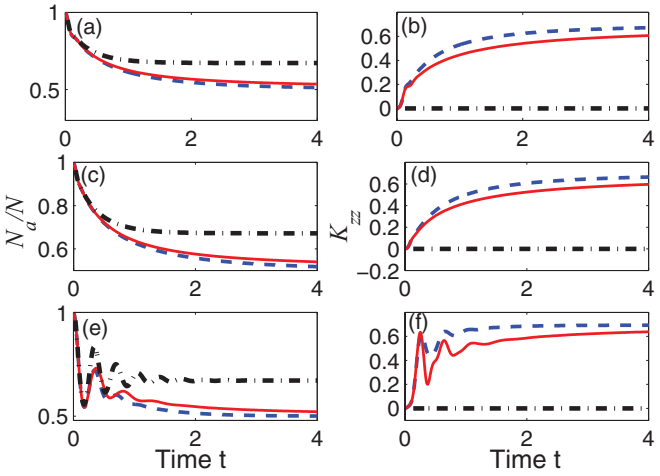

FIG. 2. (Color online) N_a/N and K_{zz} versus time. The results are obtained by the mean-field theory (black dash-dotted line), the Bogoliubov back-reaction equations (red solid line), and the numerical solution of the master equation (blue dashed line). The initial conditions of the system are the same as in Eq. (25). Parameters chosen are $g = 1, N = 100, \varepsilon = 30, \Gamma = 1$ for (a) and (b), $\varepsilon = 40, \Gamma = 1.8$ for (c) and (d), and $\varepsilon = 10, \Gamma = 0.2$ for (e) and (f).

numerically solving the master equation (2) are also presented. To plot the figure, the initial conditions

$$\begin{aligned} F_z &= -1, \\ K_{xx} &= K_{yy} = 4(N-1)/N^2, \\ F_x &= F_y = K_{zz} = K_{xy} = K_{xz} = K_{yz} = 0 \end{aligned} \tag{25}$$

are taken; the corresponding quantum state is the molecular vacuum state $|N,0\rangle$.

We find that the results given by the BBR equations are in good agreement with those obtained by numerically solving the master equation. The results from the MFT are different from those at a long time scale. This difference comes from the fluctuations K_{ij}, which are ignored in the MFT. Noticing

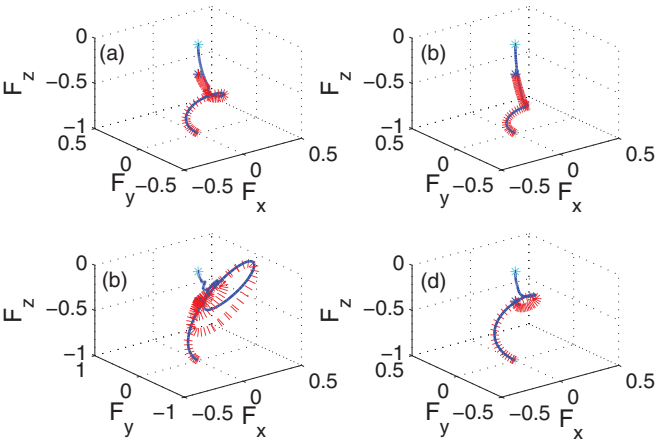

FIG. 3. (Color online) F_z as a function of time from the mean-field theory (red dotted line) and the BBR equations (blue solid line). The black star denotes the fixed point from the MFT in Eq. (10) and the green star denotes the fixed point from the BBR equations in Eq. (24). The parameters of (a), (b), and (c) are the same as in Fig. 2. In (d), $g = 1, \varepsilon = 15, \Gamma = 0.8, N = 100$.

that the fixed points given by the BBR equations are the same as those from the numerical method but different from those obtained by the MFT, we emphasize that the stability of the fixed points from MFT and the BBR equations are the same for a wide range of parameters in the space spanned by F_x, F_y, and F_z; this is due to the linear coupling between the Bloch vector $\mathbf{F}$ and the fluctuations K_{ij} in $\mathbf{F}$ [see the first three equations in Eq. (24)].

V. STABILITY OF THE FIXED POINTS WITH $\varepsilon = 0$

In this section, we will discuss the stability of the fixed points obtained from both the MFT and the BBGKY hierarchy. For simplicity, let us consider the situation of zero atomic binding energy, $\varepsilon = 0$. In this case, Eq. (10) reduces to

$$\begin{aligned} \frac{\partial F_y}{\partial t} &= -\Delta F_z + \frac{3}{2}\Delta F_z^2 - 16\Gamma F_y - R, \\ \frac{\partial F_z}{\partial t} &= 2\Delta F_y. \end{aligned} \tag{26}$$

Using the Jacobian matrix defined by

$$J = \begin{pmatrix} \frac{\partial P}{\partial F_y} & \frac{\partial P}{\partial F_z} \\ \frac{\partial Q}{\partial F_y} & \frac{\partial Q}{\partial F_z} \end{pmatrix}_{(F_{xf}, F_{yf}, F_{zf})}, \tag{27}$$

we can study the stability of the fixed points in the MFT. Here $P = -\Delta F_z + \frac{3}{2}\Delta F_z^2 - 16\Gamma F_y - R$, $Q = 2\Delta F_y$. The eigenvalues of the Jacobi matrix J determine the stability of the fixed points, which can be given by simple calculations: $\lambda_\pm = \frac{1}{2}[-16\Gamma \pm \sqrt{256\Gamma^2 - 4Ng^2(1-3F_{zf})}] = \frac{1}{2}[-16\Gamma \pm \sqrt{256\Gamma^2 - 4Ng^2\sqrt{1+3(1+4/N)}}]$. If

$$64\Gamma^2 \geqslant Ng^2\sqrt{1+3\,(1+4/N)} \tag{28}$$

is satisfied, the Jacobi matrix J has two negative roots, and the fixed point (15) is a stable junction fixed point. When

$$64\Gamma^2 \leqslant Ng^2\sqrt{1+3\,(1+4/N)}, \tag{29}$$

the Jacobi matrix J has two conjugate complex roots; in this case the fixed point (15) is a stable focus fixed point.

Now we turn our discussion to the fixed points given by the BBGKY hierarchy of equations of motion. To compare the stability of fixed points obtained by the BBGKY hierarchy with the prediction by the MFT, we restrict the discussion to the space spanned by F_x, F_y, and F_z. This means that the fluctuations which drive the system away from the fixed points (steady state) occur only in F_x, F_y, and F_z. We start with the fixed points in nine-dimensional space. By the same definition as in the MFT, we obtain the fixed point in the BBGKY equation (24) ($\varepsilon = 0$),

$$\begin{aligned} F_{xB} &= F_{yB} = F_{zB} = K_{xzB} = K_{xyB} = K_{yzB} = 0, \\ K_{zzB} &= \frac{2}{3} + \frac{8}{3N}, \\ K_{xxB} &= K_{yyB} = \frac{1}{2}K_{zzB}. \end{aligned} \tag{30}$$

As mentioned, we discuss the case where the fluctuations are only in F_x, F_y, and F_z. For $\varepsilon = 0$, F_x decouples from F_y and F_z, and the discussion reduces to discussing fluctuations only

in F_y and F_z,

$$F_y \to F_{yB} + \delta f_y,$$
$$F_z \to F_{zB} + \delta f_z. \tag{31}$$

Substituting Eq. (31) into Eq. (24), we have

$$\frac{\partial \delta f_y}{\partial t} = -\varepsilon \delta f_x - \Delta \delta f_z + \frac{3}{2}\Delta\left(\frac{1}{2}K_{zzB} + 2F_{zB}\delta f_z\right) - 16\Gamma \delta f_y - R,$$
$$\frac{\partial \delta f_z}{\partial t} = 2\Delta \delta f_y. \tag{32}$$

By the same discussion as for Eq. (27), the Jacobian matrix in this case is

$$J' = \begin{pmatrix} \frac{\partial P'}{\partial \delta f_y} & \frac{\partial P'}{\partial \delta f_z} \\ \frac{\partial Q'}{\partial \delta f_y} & \frac{\partial Q'}{\partial \delta f_z} \end{pmatrix}_{(F_{xB}, F_{yB}, F_{zB})}, \tag{33}$$

where

$$P' = -\varepsilon \delta f_x - \Delta \delta\ f_z + \frac{3}{2}\Delta\left(\frac{1}{2}K_{zzB} + 2\ F_{zB}\delta f_z\right) - 16\Gamma \delta f_y$$

and $Q' = 2\Delta \delta f_y$, and the eigenvalues of the Jacobi matrix J' are $\lambda'_{\pm} = \frac{1}{2}[-16\Gamma \pm \sqrt{256\Gamma^2 - 4Ng^2(1-3F_{zB})}] = \frac{1}{2}(-16\Gamma \pm \sqrt{256\Gamma^2 - 4Ng^2})$. If

$$64\Gamma^2 \geqslant Ng^2, \tag{34}$$

all $\lambda'_{\pm}$ are negative, and the fixed point (30) is a stable junction fixed point. Otherwise, if

$$64\Gamma^2 \leqslant Ng^2, \tag{35}$$

$\lambda'_{\pm}$ are complex and their real parts are negative, and the fixed point (30) is then a stable focus fixed point.

When the parameters satisfy simultaneously Eqs. (28) and (34), the stability of the fixed points is the same in the MFT and the BBGKY hierarchy, i.e., the fixed points are stable junction points; see Fig. 4. In this situation, the system approaches the fixed points straightforwardly. When the parameters satisfy both Eqs. (29) and (35), the stability of the fixed points in the MFT and the BBGKY hierarchy is also the same. The fixed points in this case are stable focus fixed points (Fig. 5). The system goes to the fixed points in a spiral pattern.

When the parameters fall in the range of

$$Ng^2 < 64\Gamma^2 < Ng^2\sqrt{1+3\,(1+4/N)}, \tag{36}$$

the system in the BBGKY theory Eq. (24) would go to a stable junction fixed point, but by the MFT, the system would approach a stable focus fixed point. We plot the time evolution of F_z in Fig. 6. From the figure, we can see that the population difference F_z in BBGKY theory increases monotonically as t increases (the blue dashed line), but it increases first, then decreases, and finally reaches the stable state in the MFT. In addition, comparing Figs. 6(a) and 6(b), we learn that in (a) F_z changes slowly, while in (b) the change is faster; this is due to the difference of the dephasing rate Γ.

Before concluding the paper, we present a discussion of the time-dependent many-body theory [22,23] and the master equation approach in the two-mode approximation. We start with the many-body description for the photoassociation in a uniform Bose-Einstein condensate [23]. In the two-body case, the system model reduces to a set of coupled modes; two of them are atoms in the condensate and molecules. The other modes represent the noncondensate atom pairs. This treatment is very similar to the master equation description, when the noncondensate atom pairs are treated as an environment. Then the elimination of the modes of noncondensate atom pairs in the two-body theory would lead to equations of

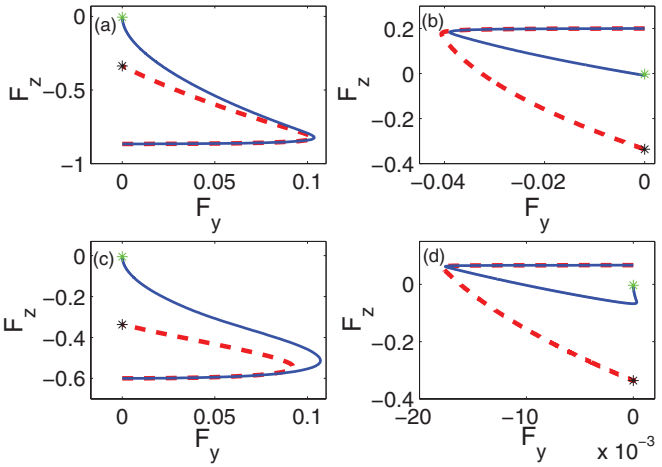

FIG. 4. (Color online) The fixed points and the approach of the system to the fixed points. The black and green stars denote the location of the stable junction fixed point obtained by the MFT and by the BBR equations [see Eq. (30)], respectively. The red dashed line (MFT) and the blue solid line (BBR) show how the system approaches the fixed points. The initial state of the system is $|\psi_0\rangle = |n\rangle$. Parameters chosen are $g = 1, \Gamma = 10, N = 300, |\psi_0\rangle = |10\rangle$ for (a), $\Gamma = 12, |\psi_0\rangle = |90\rangle$ for (b), $\Gamma = 4, |\psi_0\rangle = |30\rangle$ for (c), and $\Gamma = 24, |\psi_0\rangle = |80\rangle$ for (d).

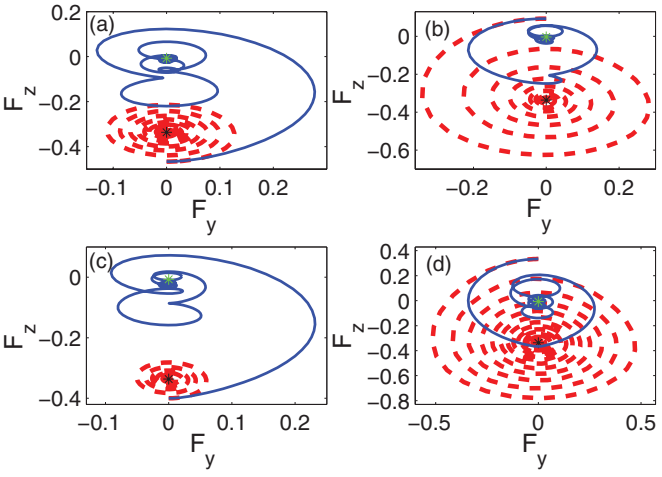

FIG. 5. (Color online) Black star and green star denote the location of the stable focus fixed points predicted by MFT and the BBR equations, respectively. The red dashed (MFT) and the blue solid (BBR) lines show the trajectories for the system from the initial state to the fixed points. Parameters chosen are $g = 1, \Gamma = 0.12, N = 300, |\psi_0\rangle = |40\rangle$ for (a), $\Gamma = 0.2, |\psi_0\rangle = |82\rangle$ for (b), $\Gamma = 0.16, |\psi_0\rangle = |45\rangle$ for (c), and $\Gamma = 0.1, |\psi_0\rangle = |100\rangle$ for (d).

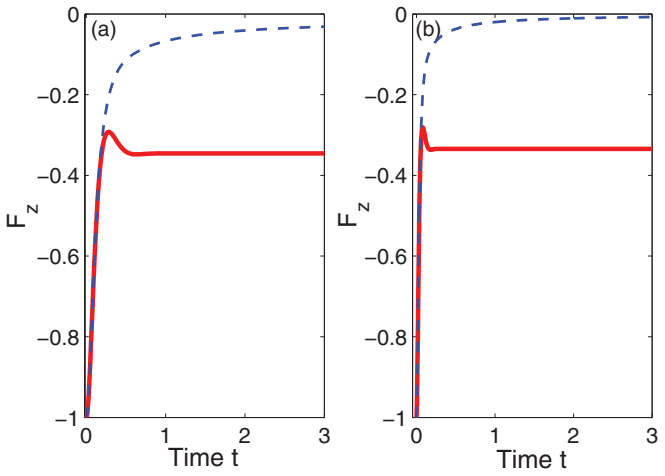

FIG. 6. (Color online) Red thick line shows how the system goes to the stable focus fixed points in MFT, while the blue dashed line shows how the system goes to the stable junction fixed point by the BBGKY hierarchy in Eq. (24). Parameters chosen satisfying Eq. (36) are $g = 1, N = 80, \Gamma = 1.1192, |\psi_0\rangle = |0\rangle$ for (a), and $\Gamma = 3.9568, N = 1000, |\psi_0\rangle = |0\rangle$ for (b).

motion (almost) equivalent to those in the master equation description.

To be specific, we take the photoassociation of a Bose-Einstein condensate [22] as an example. The equation of motion of the system reads

$$\begin{aligned}\dot{\alpha} &= i\frac{\Omega}{\sqrt{2}}\alpha^*\beta,\\ \dot{\beta} &= i\delta\beta + i\frac{\Omega^*}{\sqrt{2}}\alpha^2 + i\int d\epsilon\,\xi(\epsilon)\,c_\epsilon,\\ \dot{c}_\epsilon &= -i\epsilon\,c_\epsilon + i\xi^*(\epsilon)\beta,\end{aligned} \tag{37}$$

where $\alpha = a/\sqrt{N}$, $\beta = \sqrt{2/N}b$, and c_ϵ represent the c-number atomic, molecular, and noncondensate atom pair amplitudes, respectively. Formally integrating the third equation of Eqs. (37) and substituting it into the second, we have

$$\begin{aligned}\dot{\beta} = i\delta\beta + i\frac{\Omega^*}{\sqrt{2}}\alpha^2 + i\int d\epsilon\xi(\epsilon)e^{-i\epsilon(t-t_0)}c_\epsilon(t_0)\\ - \int d\epsilon|\xi(\epsilon)|^2\int_{t_0}^{t}\beta(t')e^{-i\epsilon(t-t')}dt'.\end{aligned} \tag{38}$$

In the Weisskopf-Wigner approximation [29], $|\xi(\epsilon)|^2$ is set to be a constant around $\epsilon = 0$, so we can replace ϵ by 0 in $|\xi(\epsilon)|^2$. This together with the initial condition $c_\epsilon(t_0) = 0$ gives

$$\dot{\beta} = i\delta\beta + i\frac{\Omega^*}{\sqrt{2}}\alpha^2 - |\xi(0)|^2\int_{t_0}^{t}dt'\beta(t')\int d\epsilon\,e^{-i\epsilon(t-t')}. \tag{39}$$

Finally, we arrive at

$$\begin{aligned}\dot{\alpha} &= i\frac{\Omega}{\sqrt{2}}\alpha^*\beta,\\ \dot{\beta} &= i\delta\beta + i\frac{\Omega^*}{\sqrt{2}}\alpha^2 - \Gamma\beta,\end{aligned} \tag{40}$$

where $\Gamma = \pi|\xi(0)|^2$. On the other hand, under the mean-field approximation, the coupled equations of α and β can be derived from a master equation with a dissipation part,

$$\frac{\Gamma}{2}(2b\rho b^\dagger - \rho b^\dagger b - b^\dagger b\rho).$$

Although the descriptions based on the master equation and the many-body theory yield a very similar equation of motion for the condensed atoms and molecules in the photoassociation, the master equation loses (almost all) information about the noncondensate atoms, as it is traced out as an environment. The benefit we gain from the master equation description is that it reduces the calculation's complexity. Nevertheless, eliminating the environmental degrees of freedom in the many-body theory in the mean-field approximation cannot give a mixed state for the reduced system.

VI. CONCLUSION

In this paper, the dynamics of an atom-molecule-conversion system subject to dephasing noises has been explored. We find that the fixed points given by the mean-field theory and by numerically solving the master equation are different; this indicates that the mean-field theory is not a good treatment at a long time scale for the atom-molecule-conversion system. We further develop the BBGKY hierarchy truncation approach to study the atom-molecule-conversion system, and fixed points are calculated and the stability around the fixed points is discussed. We observe that for a wide range of parameters the stability around the fixed points is the same in the MFT and the BBGKY hierarchy truncation approach. The dynamics of the atom-molecule-conversion system is also explored, and the results suggest that the second order of the BBGKY hierarchy is a good approach for the atom-molecule-conversion system.

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