

## Detecting quantum non-Gaussianity via the Wigner function

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We introduce a family of criteria to detect quantum non-Gaussian states of a harmonic oscillator, that is, quantum states that cannot be expressed as a convex mixture of Gaussian states. In particular, we prove that for convex mixtures of Gaussian states, the value of the Wigner function at the origin of phase space is bounded from below by a nonzero positive quantity, which is a function only of the average number of excitations (photons) of the state. As a consequence, if this bound is violated, then the quantum state must be quantum non-Gaussian. We show that this criterion can be further generalized by considering additional Gaussian operations on the state under examination. We then apply these criteria to various non-Gaussian states evolving in a noisy Gaussian channel, proving that the bounds are violated for high values of losses, and thus also for states characterized by a positive Wigner function.

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### I. INTRODUCTION

Several criteria to detect nonclassicality of quantum states of a harmonic oscillator have been introduced, mostly based on phase-space distributions [1–11], ordered moments [12–14], or on information-theoretic arguments [15–21]. At the same time, an ongoing research line addresses the characterization of quantum states according to their Gaussian or non-Gaussian character [22–31], and a question arises as to whether those two different hierarchies are somehow linked to each other.

As a matter of fact, if we restrict our attention to pure states, Hudson’s theorem [32,33] establishes that the border between Gaussian and non-Gaussian states coincides exactly with the one between states with positive and negative Wigner functions. However, if we move to mixed states, the situation gets more involved. Attempts to extend Hudson’s theorem have been made by looking at upper bounds on non-Gaussianity measures for mixed states having positive Wigner function [34]. In this framework, by focusing on states with positive Wigner function, one can define an additional border between states in the *Gaussian convex hull* and those in the complementary set of *quantum non-Gaussian states*, that is, states that cannot be expressed as mixtures of Gaussian states. The situation is summarized in Fig. 1: the definition of the Gaussian convex hull generalizes the notion of Glauber’s nonclassicality [35], with coherent states replaced by generic pure Gaussian states, i.e., squeezed coherent states.

Quantum non-Gaussian states with positive Wigner function are not useful for quantum computation [36,37] and are not necessary for entanglement distillation, e.g., the non-Gaussian entangled resources used in [38] are mixtures of Gaussian states. On the other hand, they are of fundamental interest for quantum information and quantum optics. In particular, since no negativity of the Wigner function can be detected for optical losses higher than 50% [39] (or, equivalently, for detector efficiencies below 50%), criteria able to detect quantum non-Gaussianity are needed in order to certify that a *highly nonlinear* process (such as Fock state generation, Kerr interaction, photon addition/subtraction operations, or conditional photon-number detections) has been implemented

in a noisy environment, even if no negativity can be observed in the Wigner function.

Different measures of non-Gaussianity for quantum states have been proposed [22–24], but these cannot discriminate between quantum non-Gaussian states and mixtures of Gaussian states. An experimentally friendly criterion for quantum non-Gaussianity, based on photon-number probabilities, has been introduced [26] and then employed in different experimental settings to prove the generation of quantum non-Gaussian states, such as heralded single-photon states [27], squeezed single-photon states [28], and Fock states from a semiconductor quantum dot [29].

In this paper, we introduce a family of criteria which are able to detect quantum non-Gaussianity for single-mode quantum states of a harmonic oscillator based on the Wigner function. As we already pointed out, according to Hudson’s theorem, the only pure states having a positive Wigner function are Gaussian states. One can then wonder if any bound exists on the values that the Wigner function of convex mixtures of Gaussian states can take. By following this intuition, we present several bounds on the values of the Wigner function for convex mixtures of Gaussian states, consequently defining a class of sufficient criteria for quantum non-Gaussianity.

In the next section, we will introduce some notation and the preliminary notions needed for the rest of the paper. In Sec. III, we will prove and discuss our Wigner-function-based criteria for quantum non-Gaussianity, and in Sec. IV, we will prove their effectiveness by considering different families of non-Gaussian states evolving in a lossy (Gaussian) channel. We will conclude the paper in Sec. V with some remarks.

### II. PRELIMINARY NOTIONS

Throughout the paper, we will use the quantum optical terminology, where excitations of a quantum harmonic oscillator are called photons. All the results can be naturally applied to any bosonic continuous-variable (CV) system. We will consider a single mode described by a mode operator  $a$ , satisfying the commutation relation  $[a, a^\dagger] = \mathbb{1}$ . A quantum state  $\rho$  is fully described by its characteristic

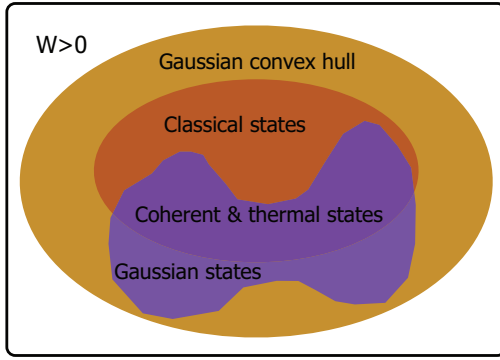


FIG. 1. (Color online) Venn diagram description for continuous-variable quantum states with positive Wigner function. The quantum states can be divided in two sets: *quantum non-Gaussian* states and states belonging to the Gaussian convex hull. The latter trivially includes (Glauber) classical states and Gaussian states.

function [39]

$$\chi[\varrho](\gamma) = \text{Tr}[\varrho D(\gamma)], \quad (1)$$

where  $D(\gamma) = \exp\{\gamma a^\dagger - \gamma^* a\}$  represents the displacement operator. In addition, the quantum state  $\varrho$  can be fully described by the Fourier transform of the characteristic function, i.e., the Wigner function [39]

$$W[\varrho](\alpha) = \int \frac{d^2\gamma}{\pi^2} e^{\gamma^* \alpha - \gamma \alpha^*} \chi[\varrho](\gamma). \quad (2)$$

A state is defined to be *Gaussian* if and only if its Wigner function (or, equivalently, its characteristic function) is Gaussian. All single-mode Gaussian states can be expressed as

$$\varrho = D(\alpha) S(\xi) \nu_\beta S^\dagger(\xi) D^\dagger(\alpha),$$

where  $S(\xi) = \exp\{\frac{1}{2}\xi(a^\dagger)^2 - \frac{1}{2}\xi^* a^2\}$  is the squeezing operator, and  $\nu_\beta = e^{-\beta a^\dagger a} / \text{Tr}[e^{-\beta a^\dagger a}]$  is a thermal state ( $\alpha, \xi \in \mathbb{C}$  and  $\beta > 0$ ). Pure Gaussian states can be written as  $|\psi_G\rangle = D(\alpha) S(\xi) |0\rangle$ , and, according to Hudson's theorem [32,33], are the only pure states having a positive Wigner function. Together with that of a Gaussian state, one can define the concept of *Gaussian map*: a quantum (completely positive) map is defined as Gaussian if and only if it transforms Gaussian states into Gaussian states. All unitary Gaussian maps can be expressed as  $U_G = \exp\{-i H_{\text{bil}} t\}$ , and they correspond to Hamiltonian operators  $H_{\text{bil}}$  at most bilinear in the mode operators. Similarly, a generic Gaussian map can be decomposed as a Gaussian unitary acting on the system plus an ancilla (the latter prepared in a Gaussian state), followed by partial tracing over the ancillary mode [40].

Another complete description of a CV quantum state  $\varrho$  may be given in terms of the so-called *P* function  $P[\varrho](\alpha)$  [39], defined implicitly via the formula

$$\varrho = \int d^2\alpha P[\varrho](\alpha) |\alpha\rangle\langle\alpha|, \quad (3)$$

where  $|\alpha\rangle = D(\alpha)|0\rangle$  represents a coherent state. According to Glauber, a state is *nonclassical* if and only if its *P* function is not a proper probability distribution, e.g., the *P* function is more singular than a Dirac- $\delta$  function. Note that the negativity of the Wigner function is a more restrictive

definition of nonclassicality: there exist nonclassical states having a positive Wigner function (e.g., squeezed states), while all the states having a nonpositive Wigner function are nonclassical according to Glauber.

In a similar spirit to Glauber's approach to nonclassicality, in this paper we study the concept of *quantum non-Gaussian* states. These are defined as follows. The Gaussian convex hull is the set of states

$$\mathcal{G} = \left\{ \varrho \in \mathcal{H} \mid \varrho = \int d\lambda p(\lambda) |\psi_G(\lambda)\rangle\langle\psi_G(\lambda)| \right\}, \quad (4)$$

where  $\mathcal{H}$  denotes the Hilbert space of continuous-variable quantum states,  $p(\lambda)$  is a proper probability distribution, and  $|\psi_G(\lambda)\rangle$  are pure Gaussian states, i.e., in the single-mode case, squeezed coherent states identified by the set of parameters  $\lambda \equiv \{\alpha, \xi\}$ . Since Gaussian states do not form a convex set, the set in Eq. (4) includes states which are not Gaussian. Moreover, any mixed Gaussian state can be written as a weighted sum of pure Gaussian states, and hence the set above also includes convex mixtures of mixed Gaussian states.

The definition of *quantum non-Gaussianity* naturally follows:

*Definition.* A quantum state  $\varrho$  is quantum non-Gaussian if and only if it is not possible to express it as a convex mixture of Gaussian states, that is, if and only if  $\varrho \notin \mathcal{G}$ .

As illustrated in Fig. 1, the border here defined dividing quantum non-Gaussian states and mixtures of Gaussian states falls in between the border dividing classical and nonclassical states, and the one which divides states with positive and nonpositive Wigner functions. The importance of such a further distinction is evident if we note that all states in  $\mathcal{G}$  can be prepared through a combination of Gaussian operations and classical randomization. On the contrary, if  $\varrho \notin \mathcal{G}$ , then a highly *nonlinear* process (due to a non-Gaussian operation or measurement) had necessarily taken place in the generation of the quantum state  $\varrho$ . While the negativity of the Wigner function is always sufficient to certify it, more elaborated criteria, such as those elaborated in this paper, are needed in order to detect such a characteristic when quantum states exhibit a positive Wigner function.

### III. CRITERIA TO DETECT QUANTUM NON-GAUSSIANITY

In order to find criteria for the detection of quantum non-Gaussian states, we follow the intuition given by Hudson's theorem for pure Gaussian states. We will focus on lower bounds on the values taken by the Wigner function of states which belong to the Gaussian convex hull  $\mathcal{G}$ . In this section, we present our main findings as one lemma leading to two final propositions and two additional corollaries. The "quantum non-Gaussianity criteria," derived directly from these results, are presented at the end of the section.

*Lemma 1.* [Lower bound on the Wigner function at the origin of phase space for a pure Gaussian state] For any given pure single-mode Gaussian state  $|\psi_G\rangle$ , the value of the Wigner function at the origin of the phase space is bounded from below as

$$W[|\psi_G\rangle\langle\psi_G|](0) \geq \frac{2}{\pi} \exp\{-2n(1+n)\}, \quad (5)$$

where  $n = \langle\psi_G|a^\dagger a|\psi_G\rangle$ .

*Proof.* A generic pure single-mode Gaussian state can be always written as  $|\psi_G\rangle = D(\alpha)S(\xi)|0\rangle$ , where  $\alpha = |\alpha|e^{i\theta}$  and  $\xi = re^{i\phi}$  ( $r > 0$ ) are two complex numbers. We can thus write the Wigner function evaluated in zero as

$$W[|\psi_G\rangle\langle\psi_G|](0) = \frac{2}{\pi} \exp\{-2|\alpha|^2[\cosh 2r - \cos(2\theta + \phi) \sinh 2r]\}. \quad (6)$$

Our goal is to minimize the value of the Wigner function or, equivalently, to maximize the function

$$f(\alpha, \xi) = 2|\alpha|^2[\cosh 2r - \cos(2\theta + \phi) \sinh 2r]. \quad (7)$$

A first maximization is obtained by considering

$$2\theta + \phi = \pi + 2k\pi \quad \text{with } k \in \mathbb{N}, \quad (8)$$

which yields

$$f(\alpha, \xi) \leq 2|\alpha|^2 e^{2r} = 2n_d[2n_s + 1 + 2\sqrt{n_s(1+n_s)}]. \quad (9)$$

In the last equation, we introduced the displacement and squeezing photon numbers,  $n_d = |\alpha|^2$  and  $n_s = \sinh^2 r$ , and we used the formula  $\operatorname{arcsinh}(x) = \log(x + \sqrt{1+x^2})$ . Note that these two parameters obey

$$n = \langle\psi_G|a^\dagger a|\psi_G\rangle = n_d + n_s,$$

where  $n$  is the average photon number of the state  $|\psi_G\rangle$ . We can thus express the right-hand side (rhs) of Eq. (9) in terms of  $n$  and  $n_s$ , obtaining

$$f(\alpha, \xi) \leq 2(n - n_s)[2n_s + 1 + 2\sqrt{n_s(1+n_s)}]. \quad (10)$$

For a given average photon number  $n$ , the above function is maximized with regard to the parameter  $n_s$  by choosing

$$n_s = \frac{n^2}{1+2n}, \quad (11)$$

and obtaining

$$f(\alpha, \xi) \leq 2n(1+n). \quad (12)$$

This leads to

$$W[|\psi_G\rangle\langle\psi_G|](0) \geq \frac{2}{\pi} \exp\{-2n(1+n)\}. \quad (13)$$

By looking at the proof, we remark that the bound obtained is tight: given a fixed energy  $n$ , by choosing the phases according to condition (8) and the squeezing energy according to (11), it is always possible to find a family of pure Gaussian states saturating the inequality. In particular, the maximization obtained via condition (8) simply corresponds, at fixed  $n_d$  and  $n_s$ , to displace the state along the direction of the squeezed quadrature. The condition (11) shows that for small values of  $n$ , the minimum of the Wigner function is obtained by using the energy in displacement, while for larger values of  $n$ , the optimal squeezing fraction  $n_s$  tends to an asymptotic value  $n_s^{(\text{as})} = n/2$ . Let us now generalize the bound obtained to a generic convex mixture of Gaussian states.

*Proposition 1.* [Lower bound on the Wigner function at the origin for a convex mixture of Gaussian states] For any single-mode quantum state  $\varrho$  which belongs to the Gaussian

convex hull  $\mathcal{G}$ , the value of the Wigner function at the origin is bounded by

$$W[\varrho](0) \geq \frac{2}{\pi} \exp\{-2\bar{n}(1+\bar{n})\}, \quad (14)$$

where  $\bar{n} = \operatorname{Tr}[\varrho a^\dagger a]$ .

*Proof.* The multi-index  $\lambda$ , which labels every Gaussian state in the convex mixture  $|\psi_G(\lambda)\rangle = D(\alpha)S(\xi)|0\rangle$ , contains the information about the squeezing  $\xi$  and displacement  $\alpha$ . We can then equivalently consider as variables  $\lambda = \{n, n_s, \theta, \phi\}$ . By exploiting the linearity property of the Wigner function, we obtain

$$W[\varrho](0) = \int d\lambda p(\lambda) W[|\psi_G(\lambda)\rangle\langle\psi_G(\lambda)|](0) \geq \frac{2}{\pi} \int d\lambda p(\lambda) \exp\{-2n(1+n)\}, \quad (15)$$

where inequality (5) has been used. By defining

$$\tilde{p}(n) = \int_0^n dn_s \int_0^{2\pi} d\phi \int_0^{2\pi} d\theta p(\lambda), \quad (16)$$

which is a valid probability distribution with respect to the variable  $n$ , Eq. (15) becomes

$$W[\varrho](0) \geq \frac{2}{\pi} \int_0^\infty dn \tilde{p}(n) \exp\{-2n(1+n)\}. \quad (17)$$

Studying the second derivative of

$$B_{\min}(n) = \frac{2}{\pi} \exp\{-2n(1+n)\}, \quad (18)$$

we conclude that the function is *convex* in the whole *physical* region (i.e.,  $n \geq 0$ ). As a consequence,

$$\int_0^\infty dn \tilde{p}(n) B_{\min}(n) \geq B_{\min} \left[ \int_0^\infty dn \tilde{p}(n) n \right] = B_{\min}(\bar{n}), \quad (19)$$

where  $\bar{n} = \int_0^\infty dn \tilde{p}(n) n = \operatorname{Tr}[\varrho a^\dagger a]$ . From the last inequality, we obtain straightforwardly the thesis

$$W[\varrho](0) \geq \frac{2}{\pi} \exp\{-2\bar{n}(1+\bar{n})\}. \quad (20)$$

The following proposition generalizes the bound obtained above.

*Proposition 2.* For any single-mode quantum state  $\varrho \in \mathcal{G}$ , and for any given Gaussian map  $\mathcal{E}_G$  (or, alternatively, a convex mixture thereof), the following inequality holds:

$$W[\mathcal{E}_G(\varrho)](0) \geq \frac{2}{\pi} \exp\{-2\bar{n}_\mathcal{E}(1+\bar{n}_\mathcal{E})\}, \quad (21)$$

where  $\bar{n}_\mathcal{E} = \operatorname{Tr}[\mathcal{E}_G(\varrho) a^\dagger a]$ .

*Proof.* Given a quantum state  $\varrho$  which can be expressed as a mixture of a Gaussian state and a Gaussian map  $\mathcal{E}_G$  (or a convex mixture thereof), the output state

$$\varrho' = \mathcal{E}_G(\varrho) \quad (22)$$

can still be expressed as a mixture of Gaussian states. As a consequence, we can apply to the state  $\varrho'$  the result in Proposition 1 obtaining the thesis. ■

Proposition 2 leads to two corollaries that will be used in the rest of the paper.

*Corollary 1.* For any single-mode quantum state  $\varrho \in \mathcal{G}$ , the following inequality holds:

$$W[\varrho](\beta) \geq \frac{2}{\pi} \exp\{-2\bar{n}_\beta(1 + \bar{n}_\beta)\}, \quad \forall \beta \in \mathbb{C}, \quad (23)$$

where  $\bar{n}_\beta = \text{Tr}[\varrho D(\beta)a^\dagger a D^\dagger(\beta)]$ .

*Proof.* The proof is straightforward from Proposition 2 with the Gaussian map  $\mathcal{E}_G(\varrho) = D(-\beta)\varrho D^\dagger(-\beta)$ . We also use the property of the Wigner function

$$W[\varrho](\beta) = W[D^\dagger(\beta)\varrho D(\beta)](0),$$

and  $D^\dagger(\beta) = D(-\beta)$ .  $\blacksquare$

*Corollary 2.* For any single-mode quantum state  $\varrho$  belonging to the Gaussian convex hull  $\mathcal{G}$ , the following inequality holds:

$$W[\varrho](0) \geq \max_{\xi \in \mathbb{C}} \left( \frac{2}{\pi} \exp\{-2\bar{n}_\xi(1 + \bar{n}_\xi)\} \right), \quad (24)$$

where  $\bar{n}_\xi = \text{Tr}[\varrho S^\dagger(\xi)a^\dagger a S(\xi)]$ .

*Proof.* The proof follows from Proposition 2 by considering the Gaussian map  $\mathcal{E}_G(\varrho) = S(\xi)\varrho S^\dagger(\xi)$ . Moreover, since the value of the Wigner function at the origin is invariant under any squeezing operation, i.e.,

$$W[S(\xi)\varrho S^\dagger(\xi)](0) = W[\varrho](0), \quad (25)$$

one can maximize the rhs of inequality (21) with regard to the squeezing parameter  $\xi$ .  $\blacksquare$

The violation of any of the inequalities presented in the last two propositions and two corollaries provides a *sufficient* condition to conclude that a state is quantum non-Gaussian. We formalize this by reexpressing the previous results in the form of two criteria for the detection of quantum non-Gaussianity.

*Criterion 1.* Let us consider a quantum state  $\varrho$  and define the quantity

$$\Delta_1[\varrho] = W[\varrho](0) - \frac{2}{\pi} \exp\{-2\bar{n}(\bar{n} + 1)\}. \quad (26)$$

Then,

$$\Delta_1[\varrho] < 0 \Rightarrow \varrho \notin \mathcal{G},$$

that is,  $\varrho$  is quantum non-Gaussian.

*Criterion 2.* Let us consider a quantum state  $\varrho$ , a Gaussian map  $\mathcal{E}_G$  (or a convex mixture thereof), and define the quantity

$$\Delta_2[\varrho, \mathcal{E}_G] = W[\mathcal{E}_G(\varrho)](0) - \frac{2}{\pi} \exp\{-2\bar{n}_\mathcal{E}(\bar{n}_\mathcal{E} + 1)\}. \quad (27)$$

Then,

$$\exists \mathcal{E}_G \text{ such that } \Delta_2[\varrho, \mathcal{E}_G] < 0 \Rightarrow \varrho \notin \mathcal{G}.$$

Typically, Criterion 1 can be useful to detect quantum non-Gaussianity of phase-invariant states having the minimum of the Wigner function at the origin of phase space. On the other hand, Criterion 2 is of broader applicability. To give two paradigmatic examples, the latter criterion can be useful if (i) the minimum of the Wigner function is far from the origin, so that one may be able to violate inequality (23) by considering displacement operations, or (ii) the state is not phase invariant and presents some squeezing, and thus one

may be able to violate inequality (24) by using single-mode squeezing operations.

#### IV. VIOLATION OF THE CRITERIA FOR NON-GAUSSIAN STATES EVOLVING IN A LOSSY GAUSSIAN CHANNEL

In this section, we test the effectiveness of our criteria by applying them to typical quantum states that are of relevance to the quantum optics community. We shall consider pure, non-Gaussian states evolving in a lossy channel and test their quantum non-Gaussianity after such evolution. Specifically, we focus on the family of quantum channels associated with the Markovian master equation,

$$\frac{d\varrho}{dt} = \gamma a \varrho a^\dagger - \frac{\gamma}{2} (a^\dagger a \varrho + \varrho a^\dagger a). \quad (28)$$

The resulting time evolution, characterized by the parameter  $\epsilon = 1 - e^{-\gamma t}$ , models both the incoherent loss of photons in a dissipative zero-temperature environment and inefficient detectors with an efficiency parameter  $\eta = 1 - \epsilon$ . The evolved state  $\mathcal{E}_\epsilon(\varrho_0)$  can be equivalently derived by considering the action of a beam splitter with reflectivity  $\epsilon$ , which couples the system to an ancillary mode prepared in a vacuum state. The corresponding average photon number reads

$$\bar{n}_\epsilon = \text{Tr}[\mathcal{E}_\epsilon(\varrho_0)a^\dagger a] = (1 - \epsilon)\bar{n}_0, \quad (29)$$

where  $\bar{n}_0 = \text{Tr}[\varrho_0 a^\dagger a]$  is the initial average photon number.

It is well known that for  $\epsilon > 0.5$  (i.e., for detector efficiencies  $\eta < 0.5$ ), no negativity of the Wigner function can be observed. We will focus then on the violation of our criteria for larger values of  $\epsilon$ , which ensures that the evolved states have a positive Wigner function.

Notice that the quantum map  $\mathcal{E}_\epsilon$  is a Gaussian map. As a consequence, by combining the divisibility property of the map [inherited from the Markovian structure of Eq. (28)] and Criterion 2, if a violation is observed for a given loss parameter  $\bar{\epsilon}$ , then the state is quantum non-Gaussian for any lower value  $\epsilon \leq \bar{\epsilon}$  [41]. For this reason, we will focus on the maximum values of the loss parameter  $\epsilon$  for which a violation of the bounds is observed, i.e.,

$$\epsilon_{\max}^{(1)}[\varrho] = \max\{\epsilon : \Delta_1[\mathcal{E}_\epsilon(\varrho)] \leq 0\}, \quad (30)$$

$$\epsilon_{\max}^{(2)}[\varrho] = \max\{\epsilon : \exists \mathcal{E}_G \text{ such that } \Delta_2[\mathcal{E}_\epsilon(\varrho), \mathcal{E}_G] \leq 0\}. \quad (31)$$

In what follows, we start by focusing on Criterion 1, and thus we will look for negative values of the non-Gaussianity indicator  $\Delta_1[\varrho]$  defined in Eq. (26). We will consider different families of states, namely Fock states, photon-added coherent states, and photon-subtracted squeezed states. In Sec. IV B, we will study how to improve the results obtained by considering the second criterion and thus by studying the non-Gaussianity indicator  $\Delta_2[\varrho, \mathcal{E}_G]$ .

##### A. Violation of the first criterion

###### 1. Fock states

Let us start by considering Fock states  $|m\rangle$ , that is, the eigenstates of the number operator:  $a^\dagger a |m\rangle = m |m\rangle$ . A Fock state evolved in a lossy channel can be written as a mixture of

Fock states as

$$\mathcal{E}_\epsilon(|m\rangle\langle m|) = \sum_{l=0}^m \alpha_{l,m}(\epsilon) |l\rangle\langle l|, \quad (32)$$

with

$$\alpha_{l,m}(\epsilon) = \binom{m}{l} (1-\epsilon)^l \epsilon^{m-l}. \quad (33)$$

We recall here that the Wigner function at the origin is proportional to the expectation value of the parity operator  $\Pi = (-)^{a^\dagger a}$ , that is,

$$W[\varrho](0) = \frac{2}{\pi} \text{Tr}[\varrho \Pi] = \frac{2}{\pi} (P_{\text{even}} - P_{\text{odd}}), \quad (34)$$

where  $P_{\text{even}}$  ( $P_{\text{odd}}$ ) represents the probability of detecting an even (odd) number of photons. By using Eq. (32), one obtains

$$W[\mathcal{E}_\epsilon(|m\rangle\langle m|)](0) = \frac{2}{\pi} (2\epsilon - 1)^m, \quad (35)$$

and thus the non-Gaussianity indicator reads

$$\Delta_1[\mathcal{E}_\epsilon(|m\rangle\langle m|)] = \frac{2}{\pi} \{(2\epsilon - 1)^m - e^{-2(1-\epsilon)m[(1-\epsilon)m+1]}\}. \quad (36)$$

The behavior of  $\Delta_1[\mathcal{E}_\epsilon(|m\rangle\langle m|)]$  as a function of  $\epsilon$  for the first three Fock states is plotted in Fig. 2 (left panel). One can observe that the criterion works very well for the Fock state  $|1\rangle$ , which is proven to be quantum non-Gaussian for all values of  $\epsilon < 1$ . For the Fock states  $|2\rangle$  and  $|3\rangle$ , a nonmonotonous behavior of  $\Delta_1$  is observed as a function of the loss parameter. Still, negative values of the non-Gaussian indicator are observed in the region of interest  $\epsilon > 0.5$ . However, the maximum value of the noise parameter  $\epsilon_{\text{max}}^{(1)}$  decreases monotonically as a function of  $m$ , as shown in Fig. 2 (right panel). By increasing the Fock number  $m$ , it settles to the asymptotic value  $\epsilon_{\text{max}}^{(1)} \rightarrow 0.5$ . As one would expect by looking at the bound in Eq. (14), for high values of the average photon number, the criterion becomes practically equivalent to the detection of negativity of the Wigner function, and thus the maximum noise corresponds to  $\epsilon = 0.5$ .

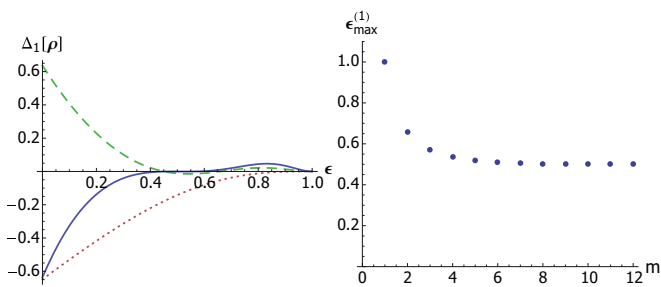


FIG. 2. (Color online) Left: Non-Gaussianity indicator  $\Delta_1[\mathcal{E}_\epsilon(|m\rangle\langle m|)]$  for the first three Fock states. Red dotted line:  $m = 1$ ; green dashed line:  $m = 2$ ; blue solid line:  $m = 3$ . Right: Maximum value of the noise parameter  $\epsilon_{\text{max}}^{(1)}$  such that the bound (14) is violated for the state  $\mathcal{E}_\epsilon(|m\rangle\langle m|)$ , as a function of the Fock number  $m$ .

## 2. Photon-added coherent states

A photon-added coherent (PAC) state is defined as

$$|\psi_{\text{pac}}\rangle = \frac{1}{\sqrt{1+|\alpha|^2}} a^\dagger |\alpha\rangle. \quad (37)$$

The operation of photon addition has been implemented in different contexts [42–45], and in particular non-Gaussianity and nonclassicality of PAC states have been investigated in [25].

Since the quantum non-Gaussianity indicator  $\Delta_1[\varrho]$  is phase insensitive, we can consider  $\alpha \in \mathbb{R}$  without loss of generality. The average photon number can be easily calculated, obtaining

$$\bar{n}_0^{(\text{pac})} = \langle \psi_{\text{pac}} | a^\dagger a | \psi_{\text{pac}} \rangle = \frac{\alpha^4 + 3\alpha^2 + 1}{1 + \alpha^2}, \quad (38)$$

while its Wigner function reads

$$W[|\psi_{\text{pac}}\rangle](\lambda) = \frac{2}{\pi} \frac{e^{-2(\alpha-\lambda)(\alpha-\lambda^*)}}{1 + \alpha^2} \times [-1 + \alpha^2 + 4|\lambda|^2 - 2\alpha(\lambda + \lambda^*)]. \quad (39)$$

The Wigner function of the state after the loss channel  $\mathcal{E}_\epsilon(|\psi_{\text{pac}}\rangle\langle\psi_{\text{pac}}|)$  can be evaluated by means of the formula

$$W[\mathcal{E}_\epsilon(\varrho)](\lambda) = \int d^2\lambda' K_\epsilon(\lambda, \lambda') W[\varrho](\lambda'), \quad (40)$$

where

$$K_\epsilon(\lambda, \lambda') = \frac{2}{\pi\epsilon} \exp\left\{-\frac{2|\lambda - \lambda'\sqrt{1-\epsilon}|^2}{\epsilon}\right\}. \quad (41)$$

The non-Gaussianity indicator  $\Delta_1[\mathcal{E}_\epsilon(|\psi_{\text{pac}}\rangle\langle\psi_{\text{pac}}|)]$  can then be straightforwardly evaluated and is plotted in Fig. 3 (left) as a function of  $\epsilon$  for different values of  $\alpha$ . We note that negative values of the indicator can be observed in an interval for the noise parameter  $\epsilon$ , which decreases with the increase of  $\alpha$ . We can explain this feature by noting that as  $\alpha$  decreases, the PAC state approaches the Fock state  $|1\rangle$ : as a consequence, its quantum non-Gaussianity can be more easily detected via Criterion 1, in particular due to the minimum value of the

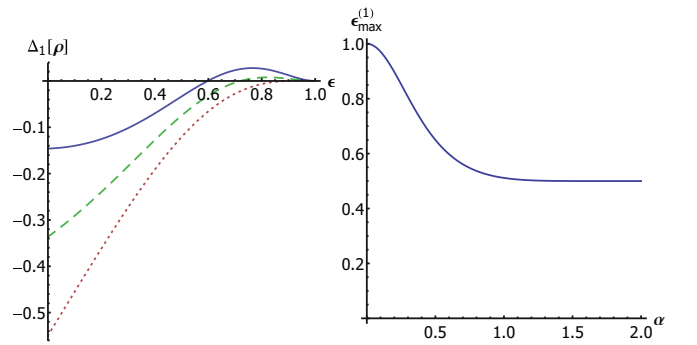


FIG. 3. (Color online) Left: Non-Gaussianity indicator  $\Delta_1[\mathcal{E}_\epsilon(|\psi_{\text{pac}}\rangle\langle\psi_{\text{pac}}|)]$  for PAC states as a function of  $\epsilon$  and for different values of  $\alpha$ . Red dotted line:  $\alpha = 0.2$ ; green dashed line:  $\alpha = 0.4$ ; blue solid line:  $\alpha = 0.6$ . Right: Maximum value of the noise parameter  $\epsilon_{\text{max}}^{(1)}$  such that the bound (14) is violated for the state  $\mathcal{E}_\epsilon(|\psi_{\text{pac}}\rangle\langle\psi_{\text{pac}}|)$ , as a function of the parameter  $\alpha$ .

Wigner function approaching the origin of the phase space. We plotted in Fig. 3 (right) the maximum value  $\epsilon_{\max}^{(1)}$  at which the violation of the bound is observed as a function of  $\alpha$ . Similarly to Fock states, we observe that by increasing the energy, this value tends to the asymptotic value  $\epsilon_{\max}^{(1)} \rightarrow 0.5$ .

### 3. Photon-subtracted squeezed states

Let us consider now another important class of non-Gaussian states that can be engineered with current technology. The *photon-subtracted squeezed* (PSS) states are defined as

$$|\psi_{\text{pss}}\rangle = \frac{1}{\sinh r} a S(r) |0\rangle. \quad (42)$$

For low values of squeezing, these states approximate the *Schrödinger kitten* states, that is, superpositions of coherent states  $|\pm\alpha\rangle$  with opposite phase and small amplitude ( $|\alpha| \lesssim 1$ ) [46]. The generation of this kind of state has been demonstrated experimentally [47–50] and it relies on performing conditional photon-number measurements.

Without loss of generality, we shall consider a real squeezing parameter  $r \in \mathbb{R}$ ; the corresponding average photon number of a PSS state reads

$$\bar{n}_0^{(\text{pss})} = 3 \sinh^2 r + 1, \quad (43)$$

while its Wigner function is

$$W[|\psi_{\text{pss}}\rangle](\lambda) = -\frac{2}{\pi} e^{-2|\lambda|^2 \cosh 2r + (\lambda^2 + \lambda^{*2}) \sinh 2r} \times [1 - 4|\lambda|^2 \cosh 2r + 2(\lambda^2 + \lambda^{*2}) \sinh 2r]. \quad (44)$$

As for the PAC states, the Wigner function of the evolved state can be evaluated by means of Eq. (40) and the non-Gaussianity indicator  $\Delta_1[\mathcal{E}_\epsilon(|\psi_{\text{pss}}\rangle\langle\psi_{\text{pss}}|)]$  can be evaluated accordingly. Its behavior as a function of  $\epsilon$  and for different values of the squeezing factor  $r$  is plotted in Fig. 4 (left). In the right panel of Fig. 4, we plot the maximum noise parameter  $\epsilon_{\max}^{(1)}$  as a function of the squeezing parameter  $r$ , observing the same behavior obtained for Fock and PAC states: the value of  $\epsilon_{\max}^{(1)}$  decreases monotonically with the energy of the state, approaching the asymptotic value  $\epsilon_{\max}^{(1)} \rightarrow 0.5$ .

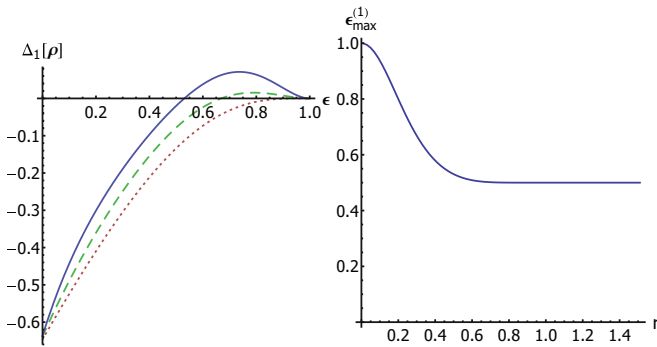


FIG. 4. (Color online) Left: Non-Gaussianity indicator  $\Delta_1[\mathcal{E}_\epsilon(|\psi_{\text{pss}}\rangle\langle\psi_{\text{pss}}|)]$  for PSS states as a function of  $\epsilon$  and for different values of  $r$ . Red dotted line:  $r = 0.1$ ; green dashed line:  $r = 0.3$ ; blue solid line:  $r = 0.5$ . Right: Maximum value of the noise parameter  $\epsilon_{\max}^{(1)}$  such that the bound (14) is violated for the state  $\mathcal{E}_\epsilon(|\psi_{\text{pss}}\rangle\langle\psi_{\text{pss}}|)$ , as a function of the initial squeezing parameter  $r$ .

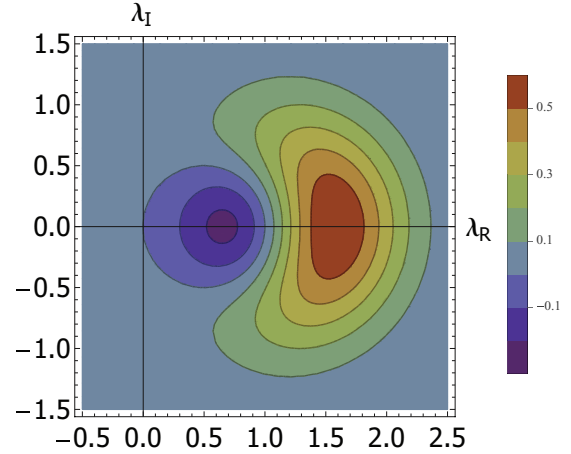


FIG. 5. (Color online) Contour plot of the Wigner function of the photon-added coherent state  $|\psi_{\text{pac}}\rangle$  for  $\alpha = 1$ . The minimum of the Wigner function is not at the origin of the phase space, and the state has nonzero first moments.

### B. Violation of the second criterion

We will now show how the second criterion, which is based on the violation of the inequality (21), can be exploited in order to improve the results shown in the previous section. Since in this case one can optimize the procedure over an additional Gaussian channel, in general one has  $\epsilon_{\max}^{(2)} \geq \epsilon_{\max}^{(1)}$ . The simplest Gaussian maps that one can consider are displacement and squeezing operations; correspondingly, we are going to seek violation of the bounds described by Eqs. (23) and (24). As anticipated in Sec. III, these new criteria are useful for states which are not phase invariant: the paradigmatic examples are states *displaced* in the phase space, that is, having the minimum of the Wigner function outside the origin, or states that exhibit squeezing in a certain quadrature. Due to this fact, the bounds based on Eqs. (23) and (24) cannot help in optimizing the results we obtained for Fock states. We will focus, then, on the other classes of states we introduced, that is, PAC and PSS states.

#### 1. Photon-added coherent states

By looking at the PAC state Wigner function in Fig. 5, one observes that its minimum is not at the origin of the phase space. Moreover, these states have nonzero first moments, implying that one can decrease their average photon number by applying an appropriate displacement. Both observations suggest that it is possible to decrease the value of the quantum non-Gaussianity indicator defined in Eq. (27),

$$\Delta_{\text{pac}}(\beta) = \Delta_2[\mathcal{E}_\epsilon(|\psi_{\text{pac}}\rangle\langle\psi_{\text{pac}}|), \mathcal{D}_\beta], \quad (45)$$

by means of a displacement operation  $\mathcal{D}_\beta(\varrho) = D(\beta)\varrho D(\beta)^\dagger$ . To evaluate  $\Delta_{\text{pac}}(\beta)$  according to Eq. (45), one has simply to evaluate the Wigner function of the state  $\varrho = \mathcal{E}_\epsilon(|\psi_{\text{pac}}\rangle\langle\psi_{\text{pac}}|)$  in a displaced point in the phase space, i.e.,  $W[\varrho](-\beta)$ , and its average photon number

$$\bar{n}^{(\text{pac})}(\beta) = (1 - \epsilon)n_0^{(\text{pac})} + |\beta|^2 + \sqrt{1 - \epsilon}(\beta^* \langle a \rangle_0 + \beta \langle a^\dagger \rangle_0), \quad (46)$$

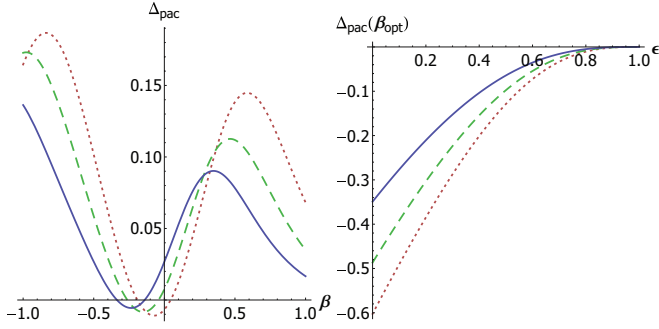


FIG. 6. (Color online) Left: Non-Gaussianity indicator  $\Delta_{\text{pac}}(\beta)$  as a function of the additional displacement parameter  $\beta$ , for  $\epsilon = 0.8$  and for different values of the initial parameter  $\alpha$ . Red dotted line:  $\alpha = 0.2$ ; green dashed line:  $\alpha = 0.4$ ; blue solid line:  $\alpha = 0.6$ . Right: Optimized non-Gaussianity indicator  $\Delta_{\text{pac}}(\beta_{\text{opt}})$  as a function of  $\epsilon$  and for different values of  $\alpha$ , where the displacement parameter  $\beta_{\text{opt}}$  has been chosen as in Eq. (48). Red dotted line:  $\alpha = 0.2$ ; green dashed line:  $\alpha = 0.4$ ; blue solid line:  $\alpha = 0.6$ .

where  $\langle A \rangle_0 = \langle \psi_0 | A | \psi_0 \rangle$ , and for  $|\psi_0\rangle = |\psi_{\text{pac}}\rangle$ ,

$$\langle a \rangle_0 = \langle a^\dagger \rangle_0 = \frac{\alpha(2 + \alpha^2)}{1 + \alpha^2}. \quad (47)$$

Our goal is then to minimize  $\Delta_{\text{pac}}(\beta)$  over the possible displacement parameters  $\beta$ .

In Fig. 6 (left), we plot  $\Delta_{\text{pac}}(\beta)$  as a function of  $\beta$  for different values of the coherent-state parameter  $\alpha$  and for  $\epsilon = 0.8$ . We observe that while for  $\beta = 0$  the bound is not always violated, it is possible to find values such that  $\Delta_{\text{pac}}(\beta) < 0$  and thus prove that the state is quantum non-Gaussian. Unfortunately, the optimal value  $\beta_{\text{opt}}$ , which minimizes  $\Delta_{\text{pac}}(\beta)$ , cannot be obtained analytically. However, we observed that for large values of  $\epsilon$  and for  $\alpha \gtrsim 1.5$ , one can approximate it as

$$\beta_{\text{opt}} \simeq -\alpha\sqrt{1 - \epsilon} = -\alpha e^{-\gamma t/2}. \quad (48)$$

The behavior of  $\Delta_{\text{pac}}(\beta_{\text{opt}})$  as a function of  $\epsilon$  is shown in Fig. 6 (right) for different values of  $\alpha$  and fixing  $\beta_{\text{opt}}$  as in Eq. (48). If we compare this with Fig. 3, not only do we observe an improvement in our capacity to witness quantum non-Gaussianity for these states, but we also see that  $\Delta_{\text{pac}}(\beta_{\text{opt}})$  remains negative for all values of  $\epsilon$ . Indeed, numerical investigations seem to suggest that  $\epsilon_{\text{max}}^{(2)} \simeq 1$  for all the possible values of  $\alpha$ : we indeed conjecture that any initial PAC state remains quantum non-Gaussian during the lossy evolution induced by Eq. (28), and that this feature can be captured by our second criterion. However, as one can observe from Fig. 6 (right), the non-Gaussianity indicator approaches zero quite fast with both  $\alpha$  and  $\epsilon$ , and thus it may be more challenging to detect its negativity in an actual experiment for states with a high average photon number and for large losses.

## 2. Photon-subtracted squeezed states

Like PAC states inherit a displacement in phase space from the initial coherent states, PSS states inherit squeezing, as we can observe by looking at the Wigner function in Fig. 7. This motivates us to make use of Corollary 2, and thus optimize the

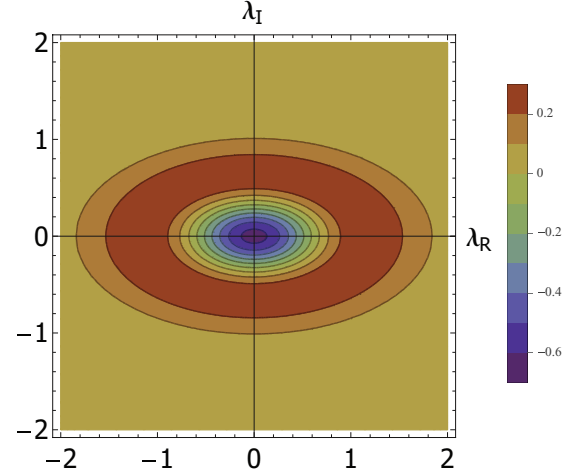


FIG. 7. (Color online) Contour plot of the Wigner function of the photon-subtracted squeezed state  $|\psi_{\text{pss}}\rangle$  for  $r = 0.3$ . The minimum of the Wigner function is at the origin of the phase space, and the state exhibits squeezing in one of the quadratures.

non-Gaussianity indicator in Eq. (27) as

$$\Delta_{\text{pss}}(s) = \Delta_2[\mathcal{E}_\epsilon(|\psi_{\text{pss}}\rangle\langle\psi_{\text{pss}}|, \mathcal{S}_s)], \quad (49)$$

that is, by considering an additional squeezing operation  $\mathcal{S}_s(\varrho) = S(s)\varrho S^\dagger(s)$  on the evolved state  $\varrho = E(|\psi_{\text{pss}}\rangle\langle\psi_{\text{pss}}|)$ . As pointed out in the proof of inequality (24), the Wigner function at the origin is invariant under squeezing operations. Hence, the optimal value  $s_{\text{opt}}$  that minimizes  $\Delta_{\text{pas}}(s)$  coincides with the value which minimizes the average photon number of  $\mathcal{S}_s = S(s)\varrho S^\dagger(s)$ ,

$$\bar{n}^{(\text{pss})}(s) = (1 - \epsilon)[n_0^{(\text{pss})}(\mu_s^2 + \nu_s^2) + \mu_s \nu_s (\langle a^2 \rangle_0 + \langle a^{\dagger 2} \rangle_0)] + \nu_s^2, \quad (50)$$

where  $\mu_t = \cosh t$ ,  $\nu_t = \sinh t$ , and for an initial PSS state (with a real squeezing parameter  $r$ ),

$$\langle a^2 \rangle_0 = \langle a^{\dagger 2} \rangle_0 = 3\mu_r \nu_r. \quad (51)$$

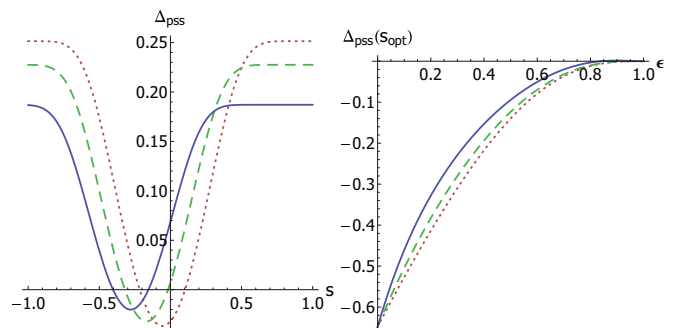


FIG. 8. (Color online) Left: Non-Gaussianity indicator  $\Delta_{\text{pss}}(s)$  as a function of the additional squeezing parameter  $s$ , for  $\epsilon = 0.7$  and for different values of the initial parameter  $r$ . Red dotted line:  $r = 0.1$ ; green dashed line:  $r = 0.3$ ; blue solid line:  $r = 0.5$ . Right: Optimized non-Gaussianity indicator  $\Delta_{\text{pss}}(s_{\text{opt}})$  as a function of  $\epsilon$  and for different values of  $r$ , where the squeezing parameter is given by the following: red dotted line:  $r = 0.1$ ; green dashed line:  $r = 0.3$ ; blue solid line:  $r = 0.5$ .

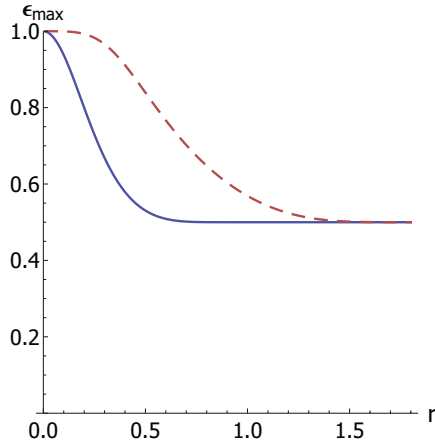


FIG. 9. (Color online) Maximum values of the noise parameter  $\epsilon_{\max}^{(2)}$  and  $\epsilon_{\max}^{(1)}$ , obtained, respectively, by means of the optimized and nonoptimized criteria, for the state  $\mathcal{E}_\epsilon(|\psi_{\text{PSS}}\rangle\langle\psi_{\text{PSS}}|)$ , as a function of the initial squeezing parameter  $r$ . Red dotted line:  $\epsilon_{\max}^{(2)}$ ; blue solid line:  $\epsilon_{\max}^{(1)}$ .

The behavior of  $\Delta_{\text{pas}}(s)$  as a function of the additional squeezing  $s$  is plotted in Fig. 8. As we observed in the previous case, the optimized criterion works in cases where the bound (14) (corresponding to  $s = 0$ ) was not violated.

Moreover, the optimal squeezing value can be evaluated analytically, yielding

$$s_{\text{opt}} = -\text{arccosh}(\mu_{\text{opt}}), \quad (52)$$

$$\mu_{\text{opt}} = \frac{1}{\sqrt{2}} \left[ 1 + \frac{6(1-\epsilon)\mu_r^2 + 4\epsilon - 3}{\sqrt{(4\epsilon - 3)^2 + 12(1-\epsilon)\epsilon\mu_r^2}} \right]^{1/2}. \quad (53)$$

The optimized quantum non-Gaussianity indicator  $\Delta_{\text{PSS}}(s_{\text{opt}})$  is plotted in Fig. 8 (right), where we observe that negative values are obtained for large values of losses. However, while for PAC states we had evidence that the maximum value of losses is  $\epsilon_{\max}^{(2)} \simeq 1$  for all the possible initial states, this is no longer true for PSS states. The behavior of  $\epsilon_{\max}^{(2)}$  as a function of  $r$  is plotted in Fig. 9, together with the previously obtained  $\epsilon_{\max}^{(1)}$ . We can notice the big improvement in our detection capability, obtained by exploiting Corollary 2; however, for large values of  $r$ , we still observe that  $\epsilon_{\max}^{(2)}$  decreases towards the same limiting value  $\epsilon_{\max}^{(2)} \rightarrow 0.5$ . Moreover, as it can be observed in Fig. 8 (right), the indicator  $\Delta_{\text{pas}}(s_{\text{opt}})$  approaches zero by increasing  $r$  and  $\epsilon$ , and thus also in this case it can become challenging to witness quantum non-Gaussianity with our methods, in experiments with large values of the initial squeezing  $r$  and large losses.

## V. CONCLUSIONS

We have presented a set of criteria to detect quantum non-Gaussian states, that is, states that cannot be expressed as mixtures of Gaussian states. The first criterion is based on seeking the violation of a lower bound for the values that the Wigner function can take at the origin, depending only on the average photon number of the state. To verify the effectiveness of the criterion, we considered the evolution of non-Gaussian pure states in a lossy Gaussian channel, looking for the maximum value of the noise where such bound is violated. We observed that the criterion works well, detecting quantum non-Gaussianity in the nontrivial region of the noise parameters where no negativity of the Wigner function can be observed.

We have also shown how the criterion can be generalized and improved by optimizing over additional Gaussian operations applied to the states of interest. Notice that in a possible experimental implementation, one does not need to perform such additional Gaussian operations, such as displacement or squeezing, in the actual experiment. Indeed, it suffices to use the data obtained on the state itself, and then apply suitable postprocessing to evaluate the optimized non-Gaussianity indicator.

Our criterion, which expresses a sufficient condition for quantum non-Gaussianity, shares some similarities with Hudson's theorem for pure Gaussian states, in the sense that it establishes a relationship between the concept of Gaussianity (combined with classical mixing) and the possible values that a Wigner function can take. The successful implementation of our criteria corresponds to the measurement of the Wigner function at the origin of the phase space which, in turn, corresponds to the (photon) parity of the state under investigation. This may be obtained with current technology by direct parity measurement [51], or by reconstruction of the photon distribution either by tomographic reconstruction or by the on/off method [52–65]. When the criterion is satisfied, one can confirm that the quantum state at disposal has been generated by means of a highly nonlinear process, even in the cases where, perhaps due to inefficient detectors or other types of noise, negativity of the Wigner function cannot be detected.

## ACKNOWLEDGMENTS

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