Negative eigenvalues of partial transposition of arbitrary bipartite states

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The partial transposition of a two-qubit state has at most one negative eigenvalue and all the eigenvalues lie in $[-1/2,1]$. In this Brief Report, we extend this result by Sanpera *et al.* [A. Sanpera, R. Tarrach, and G. Vidal, Phys. Rev. A **58**, 826 (1998)] to arbitrary bipartite states. We show that partial transposition of an $m \otimes n$ state cannot have more than $(m-1)(n-1)$ number of negative eigenvalues. Low-dimensional states have been studied to show the tightness of this result and explicit examples have been provided for $mn \leqslant 9$. It is also shown that all the eigenvalues of partial transposition lie within $[-1/2,1]$. Some possible applications are also discussed.

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Characterization of entangled states is an important issue in quantum-information theory, from both a theoretical as well as an experimental perspective. Unfortunately, even for the bipartite states, it may be very difficult to decide whether a given state is entangled or not [1,2]. However, there are still many effective ways to detect entanglement, particularly for low-dimensional systems. Undoubtedly, the most useful one is the positive partial transposition (PPT) criteria, introduced by Peres in his seminal work [3].

For an $m \otimes n$ state ρ acting on the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, its partial transposition (PT) with respect to the subsystem A is formally defined by $\rho^{T_A} := (T \otimes I)\rho$, with T being the usual *transposition* map and ρ^{T_B} defined in a similar manner. If $\{|i\rangle\}$ is an orthonormal basis of $\mathcal{H}_A$, then the PT can be computed as

$$\rho^{T_A} = \sum_{i,j} |j\rangle\langle i| \otimes \langle i|\rho|j\rangle. \tag{1}$$

Evidently, ρ^{T_A} depends explicitly on the chosen basis $\{|i\rangle\}$, but its eigenvalues do not. So, while considering properties related to eigenvalues, we use ρ^{Γ} to indicate that the result is independent of the chosen subsystem. If $\rho^{\Gamma} \geqslant 0$, then ρ is called PPT, otherwise nonpositive partial transposition (NPT). It is well known that separable states are PPT and the converse holds only for $mn \leqslant 6$ [3,4]. Also, NPT states are necessarily entangled and the negativity, a well-known measure of mixed state entanglement, is defined as the absolute value of the sum of the negative eigenvalues of ρ^{Γ} [5]. So, the negative eigenvalues of ρ^{Γ} not only certify but also quantify the amount of entanglement in ρ. Thus, it is important and interesting to explore the negative eigenvalues of ρ^{Γ}.

The two-qubit case has been solved by Sanpera *et al.* [6] more than a decade ago. It was shown that the PT of a two-qubit state has at most one negative eigenvalue and all the eigenvalues lie within $[-1/2,1]$ (of course, a two-qubit state has a negative eigenvalue iff it is NPT and hence entangled). Surprisingly, apart from some conjectures, this beautiful result has not been extended to arbitrary states [7,8]. Recently we have shown in Ref. [9] that the PT of a $2 \otimes n$ state can have at most $(n-1)$ number of negative eigenvalues. Examples of such states are provided in Ref. [10]. However, the general $m \otimes n$ case is not yet known. In this Brief Report, we solve this problem. It should be mentioned that based on some numerical findings, the authors of Ref. [8] have conjectured that the PT of an $n \otimes n$ state can have at most $n(n-1)/2$ number of negative eigenvalues. We show that contrary to this, the maximum number of negative eigenvalues of PT could go up to $(n-1)^2$. More generally, for an $m \otimes n$ state, we have the following result.

Theorem 1. Partial transposition of any $m \otimes n$ state cannot have more than $(m-1)(n-1)$ number of negative eigenvalues.

Proof. Here we follow a treatment similar to that of Ref. [6] for the $2 \otimes 2$ case. The main ingredient is Proposition 1.4 from Ref. [11], namely, any subspace of dimension $(m-1)(n-1)+1$ of the space $\mathcal{H}^m \otimes \mathcal{H}^n$ contains at least one (nonzero) product vector.

Now, if possible, let the partially transposed state ρ^{T_A} have $(m-1)(n-1)+1$ number of negative eigenvalues λ_i with corresponding eigenvectors $|\psi_i\rangle$. Then the hyperplane generated by these $|\psi_i\rangle$'s must contain at least one product vector, say $|e,f\rangle$. Therefore, expanding the product vector as $|e,f\rangle = \sum c_i|\psi_i\rangle$, we get

$$\langle e,f|\rho^{T_A}|e,f\rangle = \sum_{i=1}^{(m-1)(n-1)+1} \lambda_i|c_i|^2 < 0.$$

But this would imply $\langle e^*,f|\rho|e^*,f\rangle < 0$, which is impossible as ρ is positive semidefinite. ■

We note that, by Schmidt decomposition, any $m \otimes n$ pure state can be written as

$$|\psi\rangle = \sum_{i=1}^{d\leqslant\min\{m,n\}} \lambda_i|ii\rangle, \quad \lambda_i > 0, \quad \sum_i \lambda_i^2 = 1. \tag{2}$$

Clearly, its PT is given by

$$F := \sum_{i,j=1}^{d} \lambda_i\lambda_j|ij\rangle\langle ji|.$$

It could be easily checked that $|ii\rangle$ and $|ij\rangle \pm |ji\rangle$ are the eigenvectors of F with the corresponding eigenvalues

$$\begin{aligned} &\lambda_i^2, && \forall i = 1,2,\ldots,d,\\ &\pm\lambda_i\lambda_j, && \forall 1 \leqslant i < j \leqslant d. \end{aligned} \tag{3}$$

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Thus, for any pure state $P = |\psi\rangle\langle\psi|$, its PT, P^Γ, has $d(d-1)/2$ number of negative eigenvalues. We also observe that, due to the restriction $\sum \lambda_i^2 = 1$, the following inequality holds,

$$-\tfrac{1}{2} \leqslant \lambda_{\min}(P^\Gamma) \leqslant \lambda_{\max}(P^\Gamma) \leqslant 1. \quad (4)$$

The bound for $\lambda_{\min}(P^\Gamma)$ could be easily derived using Lagrange's multiplier method or by setting $x = \lambda_i^2$ and noting that the maximum value of

$$f(x) = x(1 - x - c) \quad (5)$$

over $0 \leqslant x \leqslant 1$ and $c \geqslant 0$ is $1/4$. The bound for $\lambda_{\max}(P^\Gamma)$ follows trivially.

This observation about pure states immediately leads to the following general result.

Theorem 2. All eigenvalues of PT of any $m \otimes n$ state always lie within $[-1/2,1]$.

Proof. Let the spectral decomposition of ρ be given by

$$\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k| := \sum_k p_k P_k. \quad (6)$$

Then we have

$$\begin{aligned}\lambda_{\min}(\rho^\Gamma) &\geqslant \sum_k p_k \lambda_{\min}\left(P_k^\Gamma\right)\\ &\geqslant \sum_k p_k \left(-\frac{1}{2}\right)\\ &= -\frac{1}{2},\end{aligned} \quad (7)$$

where in the first inequality we have used the fact that, for Hermitian matrices A_i, $\lambda_{\min}(\sum A_i) \geqslant \sum \lambda_{\min}(A_i)$. Similarly, utilizing the dual inequality for $\lambda_{\max}$, we have

$$\begin{aligned}\lambda_{\max}(\rho^\Gamma) &\leqslant \sum_k p_k \lambda_{\max}\left(P_k^\Gamma\right)\\ &\leqslant \sum_k p_k 1\\ &= 1.\end{aligned} \quad (8)$$

The tightness of Eq. (7) follows from the fact that PT of the pure state

$$|\psi\rangle = \sqrt{\frac{1}{2}}|00\rangle + \sqrt{\frac{1}{2} - \epsilon}|11\rangle + \sqrt{\frac{\epsilon}{m-1}}\sum_{k=2}^{m}|kk\rangle$$

has an eigenvalue $-\sqrt{(1/2)(1/2-\epsilon)}$, where ϵ could be chosen to vanish. Similarly, the tightness of Eq. (8) follows from the fact that PT of the (separable) state

$$\rho = (1-\epsilon)|00\rangle\langle 00| + \frac{\epsilon}{m}\sum_{k=1}^{m}|kk\rangle\langle kk|$$

has an eigenvalue $(1-\epsilon)$. Actually, for all pure product states, equality holds in Eq. (8) and no state can saturate both the bounds. ■

It is clear from Eq. (2) and Eq. (3) that the PT of any $n \otimes n$ pure state with n nonzero Schmidt coefficients will have $n(n-1)/2$ number of negative eigenvalues. This gives the intuition that the maximum number of negative eigenvalues could go beyond the conjectured number $n(n-1)/2$. We now give several examples to show that this is indeed the case and that the bound given in Theorem is tight.

TABLE I. Eigenvalues of ρ_a^Γ with multiplicities.

Eigenvalues	Multiplicities
-1	$\frac{n(n-1)}{2} - 2$
1	$\frac{n(n+1)}{2} - 4$
$1 \pm \sqrt{2}a$	1
$\frac{1}{2}(1 + a^2 \pm \sqrt{5 - 2a^2 + a^4})$	2

Example 1. A class of $\rho \in \mathbb{C}^n \otimes \mathbb{C}^n$ such that ρ^Γ has $1 + n(n-1)/2$ number of negative eigenvalues.

Let us consider the following one-parameter family of unnormalized states:

$$\begin{aligned}\rho_a &= \sum_{i=1}^{3} |\psi_i\rangle\langle\psi_i|,\\ |\psi_i\rangle &= |0i\rangle - a|i0\rangle, \quad i = 1,2,\\ |\psi_3\rangle &= \sum_{i=0}^{n-1} |ii\rangle.\end{aligned} \quad (9)$$

We list the eigenvalues (with multiplicities) of its PT in Table I. Thus, ρ_a^Γ has $n(n-1)/2 + 1$ number of negative eigenvalues for any $a \in (1/\sqrt{2},1)$.

Example 2. A class of $\rho \in \mathbb{C}^3 \otimes \mathbb{C}^3$ such that ρ^Γ has four negative eigenvalues.

We first note that the class of states given by Eq. (9) qualifies for the $3 \otimes 3$ example. However, for a more constructive example, we generalize the construction of the $2 \otimes n$ example from Ref. [10]. We consider the following family:

$$\begin{aligned}\rho(a,b,c) &= \sum_{i=1}^{3} |\psi_i\rangle\langle\psi_i|,\\ |\psi_1\rangle &= |00\rangle + a_1|11\rangle + a_2|22\rangle,\\ |\psi_2\rangle &= |01\rangle + b_1|12\rangle + b_2|20\rangle,\\ |\psi_3\rangle &= |02\rangle + c_1|10\rangle + c_2|21\rangle.\end{aligned} \quad (10)$$

It could be easily checked that the characteristic polynomial of its PT has three factors of the form

$$\begin{aligned}x^3 &- \left(1 + a_1^2 + b_2^2\right)x^2 + \left(a_1^2 - a_2^2 - b_1^2 + b_2^2 + a_1^2 b_2^2\right.\\ &\left. - c_1^2 c_2^2\right)x a_1^2 a_2^2 + b_1^2 b_2^2 + c_1^2 c_2^2 - a_1^2 b_2^2 - 2a_2 b_1 c_1 c_2 = 0.\end{aligned} \quad (11)$$

Given that all roots are real, the cubic equation $x^3 - p^2 x + qx + r = 0$ always has a positive root. Furthermore, the conditions $q < 0$ and $r < 0$ are necessary and sufficient for two negative roots. Thus we could force one of the three factors to have two negative roots and the other two to have only one. That is, there always exists real a,b,c such that $\rho^\Gamma(a,b,c)$ has four negative eigenvalues. An example is $a_1 = 1/4$, $b_1 = b_2 = 1/3$, $c_1 = 1/2$, and $c_2 = a_2 = 1$. Indeed there are an infinite number of such states.

In Fig. 1, we have shown the eigenvalues of $\rho^\Gamma(a,b,c)$ when each of a_i, b_i, and c_i takes a value from $\{0,1,2,3,4\}/4$. The situation remains almost same, even if we choose a_i, b_i, and c_i values as randomly generated complex numbers.

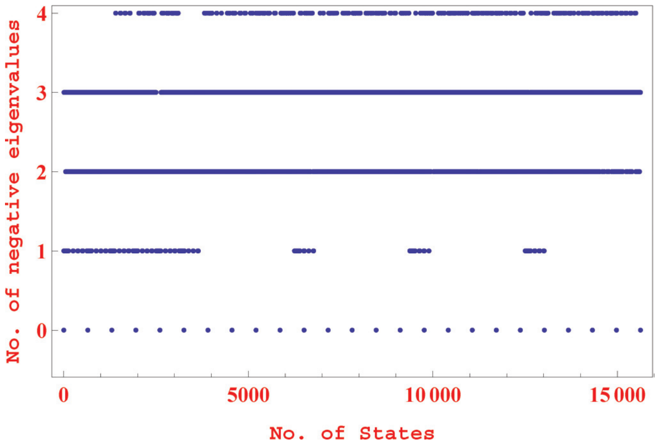

FIG. 1. (Color online) In $3 \otimes 3$, many $\rho^\Gamma(a,b,c)$, PT of the states in Eq. (10), have four negative eigenvalues. This figure is a *list plot*—each point on the horizontal axis represents a state from the family given by Eq. (10) and the vertical axis represents the number of negative eigenvalues of its PT. For example, the first point corresponds to the state with $a_i = 0 = b_i = c_i$ and its PT has no negative eigenvalues (see the written text for details).

Example 3. An example of $\rho \in \mathbb{C}^4 \otimes \mathbb{C}^4$ such that ρ^Γ has eight negative eigenvalues.

Based on Example 2, it is tempting to generalize the construction for arbitrary $\mathbb{C}^n \otimes \mathbb{C}^n$. We note that the characteristic equation of $\rho^\Gamma(a,b,c,d)$ has four factors of the form $x^4 - p^2x^3 + qx^2 + rx + s = 0$. In order for $\rho^\Gamma(a,b,c,d)$ to have nine negative eigenvalues, one of the factors must have three negative roots and each of the others at least two. The set of constraints thus generated is very complicated for analytic calculations. We, therefore, have tried to explore numerically and it looks like each such factor has at least two positive roots; thereby $\rho^\Gamma(a,b,c,d)$ cannot have more than eight negative eigenvalues. Indeed, there are infinitely many $\rho^\Gamma(a,b,c,d)$ having eight negative eigenvalues. In Fig. 2 we show some of such states where a_i,b_i,c_i,d_i takes value from $\{2,4,6,8,10\}/10$. Like the previous case, the parameters could be taken as random complex numbers as well.

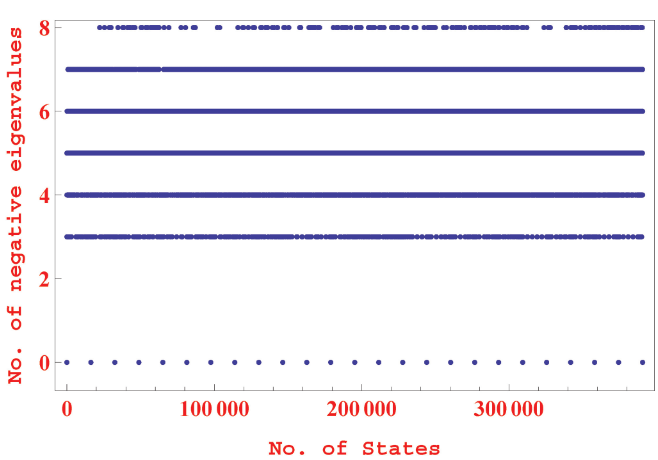

FIG. 2. (Color online) In $4 \otimes 4$, many $\rho^\Gamma(a,b,c,d)$ have eight negative eigenvalues. However, none seems to achieve the maximum number of negative eigenvalues (nine).

TABLE II. The bound of Theorem 1 and its tightness.

Dimensions	The bound $(m-1)(n-1)$	Maximum achieved
$2 \otimes n$	$n-1$	$n-1$
$3 \otimes 3$	4	4
$3 \otimes 4$	6	5
$3 \otimes 5$	8	6
$4 \otimes 4$	9	8

We have explored (both numerically and analytically) other small-dimensional states as well. Unfortunately, however, we are unable to settle the question of the tightness of the bound $(m-1)(n-1)$ beyond two qutrits. In Table II we summarize our findings.

As mentioned earlier, the main ingredient in the proof of Theorem 1 was the result of maximal dimension of entangled subspace from Ref. [11] and thus the proof is not constructive. However, it appears that the problem could be solved completely using only matrix theoretic techniques. But, the question about the tightness of the bound is yet to be explored.

Although the main motivation for this study was the curiosity of extending the result of two qubits to arbitrary states, nonetheless let us mention some possible applications of this upper bound. Indeed prior to this work, the exact number of negative eigenvalues of PT was applied to get interesting results about small-dimensional systems. For example, the result of the $2 \otimes 2$ system has been used to show that all separable states can be expressed as a mixture of at most four pure product states. The two-qubit case being so special, this result, coupled with the fact that NPT is equivalent to a full rank of PT, implies that a $2 \otimes 2$ state ρ is entangled iff $\det \rho^\Gamma < 0$. Clearly, this condition, though always sufficient, is not necessary for separability beyond two qubits. The pure state after Eq. (8) also shows that, to estimate negativity, not the number of negative eigenvalues of PT but rather the the one with maximum modulus is significant. Thus, apparently this generic bound, contrary to small-dimensional systems, may have less direct physical significance for higher-dimensional systems.

Apart from its close connection with the maximal dimension of *completely entangled subspaces* [11,12], the present bound for arbitrary bipartite states, albeit mostly a mathematical result, may also have some possible applications in quantum-information theory. For example, similar to Ref. [9], this bound could readily be applied to give a semianalytical proof that squared negativity may exceed geometric discord in higher-dimensional states as well and the number of such states will increase with the dimension. It is mentioned in Ref. [9] that, due to lack of knowledge about this generic bound (and also lack of analytic formula for geometric discord), only $2 \otimes n$ states were considered. In view of the bound derived here, the said result (about geometric discord and negativity) can be easily arrived at by following exactly the proof of Theorem 2 therein. For unnecessary repetitions, we skip the details. In another direction, following Ref. [7], the result may have some applications in the study of the dynamics of entanglement.

To conclude, extending a decade-old result for two-qubits, we have shown that the partial transposition of a generic

$m \otimes n$ state cannot have more than $(m-1)(n-1)$ number of negative eigenvalues. Besides giving some explicit examples of tightness in small dimension, we have shown that all the eigenvalues always lie within $[-1/2,1]$. Some consequences of this bound have been discussed; in particular, two possible applications of the results have been mentioned. However, the question of tightness of this bound beyond two qutrits remains open.

Note added in proof. Recently we found a work [13] describing an interpretation of the number of negative eigenvalues of ρ^Γ. It has been shown that, if for any mixed state ρ, ρ^Γ has $K+1$ number of negative eigenvalues ($K \geqslant 1$), then for any K product state $|\psi_k\rangle$, the state $\rho + \sum_{k=1}^{K} \lambda_k |\psi_k\rangle\langle\psi_k|, \lambda_k \neq 0$ will always remain NPT. Addition of one more pure product state to ρ may lead to PPT (both separable and entangled) as well as NPT states.

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