Two-photon ladder climbing and transition to autoresonance in a chirped oscillator

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The two-photon ladder climbing (successive two-photon Landau-Zener-type transitions) in a chirped quantum nonlinear oscillator and its classical limit (subharmonic autoresonance) are discussed. An isomorphism between the chirped one- and two-photon resonances in the system is used in calculating the threshold for the phase-locking transition in both the classical and quantum limits. The theory is tested by solving the Schrödinger equation in the energy basis and illustrated via the Wigner function in phase space.

The transition between the quantum and classical descriptions of dynamical systems played a pivotal role in the foundation of quantum mechanics. In this context, the correspondence principle addressed the classical limit of a quantum system for large quantum numbers [1], such that the classical equations of motion describe the average wave packet [2]. Since these early works, studying the subtleties of the quantum-classical crossover still comprises a field of active research (e.g., Refs. [3–6]). An instructive framework for the theoretical and experimental investigation of this correspondence is the ac-driven nonlinear oscillator. Recent studies in the field involved nonlinear resonators in a nanoelectromechanical system [7], parametrically modulated oscillators [8], and chirped-driven Josephson junctions [9,10]. Here we focus on the quantum-classical transition in an oscillator exhibiting the classical *subharmonic autoresonance* phenomenon, i.e., a continuing phase-locking with a driving perturbation slowly passing, say, half the natural frequency of the oscillator.

Autoresonance (AR) is a continuing phase-locking between a classical nonlinear oscillatory system and a chirped frequency driving perturbation. The phenomenon was first utilized in relativistic particle accelerators [11]. In the last two decades, AR was recognized as a robust method of the excitation and control of nonlinear systems, ranging from atoms [12] and molecules [13] through plasmas [14,15] and fluids [16] to nonlinear optics [17]. The most recent applications involved the antihydrogen project at CERN [18] and superconducting Josephson junctions [9,10,19]. The salient feature of the AR is a sharp threshold for capture into resonance by passage through the fundamental linear resonance [20]. The width of this threshold depends on the temperature of the initial state [21], while in the low-temperature limit, this width saturates to a finite value associated with the zero-point fluctuations of the quantum ground state [9,22].

The quantum counterpart of the AR is the ladder climbing (LC), characterized by continuing successive two-level Landau-Zener [23] transitions. This process was studied by Marcus *et al.* [24,25] in application to driven molecules, where chirped frequency laser radiation resonantly interacts with successive energy gaps of the molecule. In addition, the LC was studied in the context of Morse oscillator [26] and more recently in Josephson junctions [10] and Rydberg atoms [27]. The transition between the classical AR and the quantum LC was studied in [22,24].

The classical subharmonic autoresonance (SHAR) is the phase-locked response of a nonlinear oscillator to a chirped driving force passing through a rational fraction of the fundamental linear frequency. This phenomenon was studied in classical nonlinear oscillators [28] and plasmas [29]. On the other hand, quantum multiphoton processes were studied both experimentally and theoretically via adiabatic Floquet analysis in association with atomic systems [30–33], but the issue of the quantum counterpart of the classical SHAR in a driven chirped nonlinear oscillator was not addressed previously. These processes may be important in such applications as quantum Josephson circuits and nanomechanical systems.

Here we discuss this problem and show that the quantum counterpart of this process is indeed the multiphoton ladder climbing (MPLC). We will use the isomorphism between the fundamental and the subharmonic 1 : 2 ARs (the generalization to the 1 : n resonances can be obtained similarly) to estimate the chirped subharmonic (SH) resonant capture probability in both the classical and quantum limits and compare our predictions with numerical simulations.

We focus on a driven weakly nonlinear oscillator governed by the dimensionless Hamiltonian

$$H = \tfrac{1}{2}(p^2 + x^2) + \tfrac{1}{3}\lambda x^3 + \tfrac{1}{4}\beta x^4 + \varepsilon x \cos\varphi_d, \quad (1)$$

where φ_d is the driving phase, such that the driving frequency $\omega_d(t) = d\varphi_d/dt$ is a slowly varying function of time. The classical *fundamental* AR and the corresponding quantum LC processes in the problem are associated with the case, when the driving frequency passes through the fundamental linear frequency of the oscillator, e.g., $\omega_d(t) = 1 + \alpha t$, α being the chirp rate. This problem was studied quantum mechanically by the authors of Refs. [22,24]. The analysis was based on the expansion of the wave function of the oscillator $|\psi\rangle = \sum_n c_n |\psi_n\rangle$ in the energy basis $|\psi_n\rangle$ of the undriven Hamiltonian, i.e., $H(\varepsilon = 0)|\psi_n\rangle = E_n|\psi_n\rangle$, where $\langle\psi_k|\psi_n\rangle = \delta_{k,n}$. In this basis, the dimensionless ($\hbar = 1$) Schrödinger equation yields

$$i\frac{dc_n}{dt} = E_n c_n + \varepsilon \sum_k c_k \langle\psi_k|\hat{x}|\psi_n\rangle \cos\varphi_d. \quad (2)$$

The energy levels in Eq. (2) for sufficiently small n can be approximated as [34]

$$E_n \approx n + \tfrac{1}{2} + \gamma(n^2 + n) + \tfrac{3}{16}\beta - \tfrac{11}{72}\lambda^2, \quad (3)$$

$n = 0, 1, 2, \ldots,$ and $\gamma = \frac{3}{8}\beta - \frac{5}{12}\lambda^2$. The linear approximation

$$\langle \psi_k | \hat{x} | \psi_n \rangle \approx \frac{\sqrt{n}\delta_{k,n-1} + \sqrt{n+1}\delta_{k,n+1}}{\sqrt{2}} \equiv K_{kn}^L \qquad (4)$$

for the coupling terms in Eq. (2) was used by the authors of Ref. [22] in analyzing the passage though the fundamental resonance in the problem. One can define three characteristic times in the problem of passage through the fundamental resonance [22,24], i.e. $T_{NL} = 2\gamma/\alpha$ (the time of passage through the nonlinear frequency shift between the first two transitions on the energy ladder), $T_R = \sqrt{2}/\varepsilon$ (the inverse Rabi frequency), and $T_S = 1/\sqrt{\alpha}$ (the frequency sweep time scale). These three times yielded two dimensionless parameters $P_1 = T_S/T_R = \varepsilon/\sqrt{2\alpha}$ (measuring the strength of the drive) and $P_2 = T_{NL}/T_S = 2\gamma/\sqrt{\alpha}$ (characterizing the nonlinearity). It was shown that $P_{1,2}$ fully characterizes the phase-locking transition in the fundamental resonance case [22]. To address the problem of two-photon LC (slow passage through the 1 : 2 resonance), we use

$$\omega_d(t) = \frac{1}{2}(1 + \alpha t). \qquad (5)$$

The numerical solutions of Eq. (2) in this case with the coupling terms of Eq. (4) show no two-photon transition. This different process requires the inclusion of additional higher-order coupling terms associated with the nonlinearity. Consequently, we replace Eq. (4) by

$$\langle \psi_k | \hat{x} | \psi_n \rangle \approx K_{kn}^L + \lambda Q_{kn} + \beta R_{kn}, \qquad (6)$$

where, by standard perturbation theory [34]

$$Q_{kn} = \frac{1}{6}[-3(2n+1)\delta_{k,n} + \sqrt{(n+1)(n+2)}\delta_{k,n+2}$$
$$+ \sqrt{n(n-1)}\delta_{k,n-2}]$$

and

$$R_{kn} = \frac{1}{24\sqrt{2}}[3\sqrt{(n+1)(n+2)(n+3)}\delta_{k,n+3}$$
$$- 2(2n+3)\sqrt{(n+1)}\delta_{k,n+1} - 2(2n+1)\sqrt{n}\delta_{k,n-1}$$
$$+ 3\sqrt{n(n-1)(n-2)}\delta_{k,n-3}].$$

At this stage, we illustrate the MPLC and SHAR in simulations. We have solved Eq. (2) numerically, subject to ground-state initial conditions $c_n(t_0 = -10/\sqrt{\alpha}) = \delta_{n,0}$ for two sets of parameters, in the quantum MPLC (Fig. 1) and the classical SHAR (Fig. 2) regimes. Figure 1(a) corresponds to the set of parameters $\{\alpha, \beta, \lambda, \varepsilon\} = \{10^{-6}, 0.016, 0.05, 0.18\}$ and shows the energy of the system versus the slow time $\tau = \sqrt{\alpha}t$. Taking 40 levels into account was sufficient in this example. One can see that the response of the quantum nonlinear oscillator to the chirped frequency drive is by successive transitions between neighboring energy levels. The (red) line in the figure is the time average over an interval of $\Delta\tau = 0.1$, eliminating fast oscillations in the dynamics, similar to the procedure used in Ref. [28]. The theoretical, perfect energy LC scenario is illustrated in Fig. 1(a) by the solid black line. We also observe that, similar to the fundamental LC, the nonlinearity parameter $P_2 = 2\gamma/\sqrt{\alpha} = 10$ in the MPLC regime is much larger than unity and that the transitions
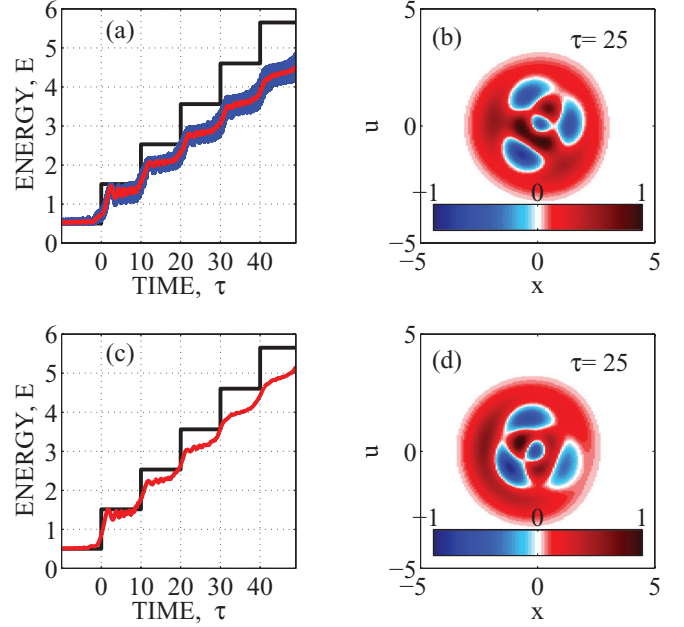

FIG. 1. (Color online) (a,c) The dynamics in the energy basis and (b,d) the corresponding Wigner functions at time $\tau = 25$ in (a,b) the 1 : 2 subharmonic and (c,d) the fundamental quantum ladder climbing regime, with the same $P_2 = 10$, but P_1 divided by $\epsilon\lambda$.

between neighboring levels occur at times $\tau_n = nP_2$ [22]. For further illustration, we have calculated the Wigner function [35] in phase space and show a snapshot of time $\tau = 25$ in Fig. 1(b). The Wigner function exhibits structure characteristic to the $n = 3$ level of the quantum ladder as is expected at this time from Fig. 1(a), while the probability of capture into resonance (total occupation of resonant levels) was 74%.
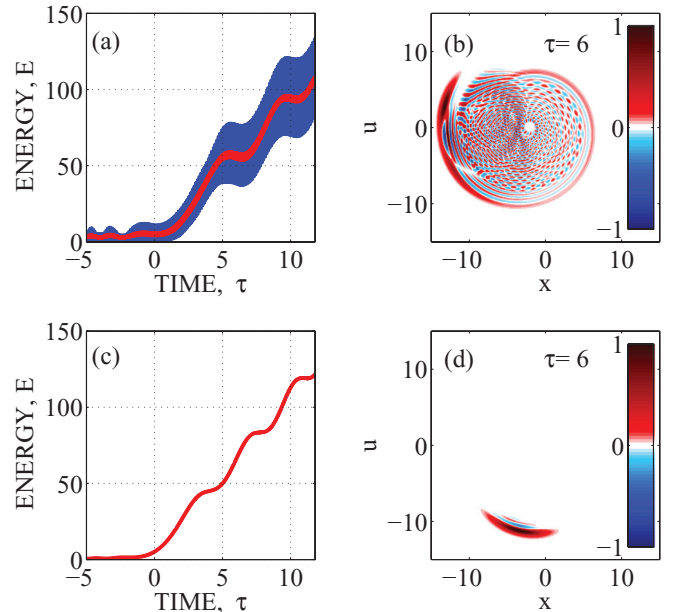

FIG. 2. (Color online) (a,c) The dynamics in the energy basis and (b,d) the corresponding Wigner function at time $\tau = 6$ in (a,b) the 1 : 2 subharmonic and (c,d) the fundamental classical autoresonance regime, with the same $P_2 = 0.1$, but P_1 divided by $\epsilon\lambda$.

It is instructive to compare these results with those for the fundamental LC case presented in Figs. 1(c) and 1(d) and obtained by using the same set of parameters, but ε replaced by $\varepsilon^2\lambda$ (P_1 multiplied by $\varepsilon\lambda$). The resonant capture probability in this case was 84%, while Fig. 1 exhibits a similarity between the simulations results for the fundamental and SH resonances with this choice of parameters.

The second numerical example shown in Figs. 2(a) and 2(b) uses the same initial conditions, but parameters $\{\alpha,\beta,\lambda,\varepsilon\} = \{10^{-4}, 0.0016, 0.0155, 1.9\}$, and the calculation involves 250 quantum levels. Here, $P_2 = 0.1$ describing the classical limit ($P_2 \ll 1$) [22], where the energy does not vary in steps, but grows monotonically with superimposed slow oscillations, as expected from the theory of the classical nonlinear resonance [36]. As above, the thin (red) line in the figure is the time average of the results over a window of $\Delta\tau = 0.1$ for eliminating fast oscillations. A snapshot of the calculated Wigner function in this example at time $\tau = 6$ is shown in Fig 2(b) and the probability of capture into AR was 85%. The figure shows that the most populated part of the phase space is a crescent corresponding to the resonantly trapped phase space area of the oscillator, while the characteristic interference patterns (which can be eliminated by coarse graining) is seen in nonresonant regions of phase space. As in the previous LC example, we compare these results with the corresponding fundamental resonance case shown in Figs. 2(c) (energy evolution) and 2(d) (Wigner function), where P_2 is the same, but P_1 again multiplied by the factor $\varepsilon\lambda$. The capture probability in this case was 99%. One observes again a noticeable similarity between the SH and rescaled fundamental AR cases.

Our theoretical analysis uses the following canonical transformation of the coordinate and momentum

$$x' = e^{-iS} x e^{iS}, \tag{7}$$

$$p' = e^{-iS} p e^{iS}, \tag{8}$$

where $S = \frac{4}{3}\varepsilon (x\sin\varphi_d + p\cos\varphi_d)$. The transformed Hamiltonian in this case becomes

$$H' = e^{-iS} H e^{iS} + \frac{dS}{dt}, \tag{9}$$

where, as before, we set ($\hbar = 1$). The first term in the right-hand side can be calculated via the identity [34]

$$e^{-iS} H e^{iS} = H - i[S,H] - \frac{1}{2}[S,[S,H]] + \cdots . \tag{10}$$

Then one finds that all $O(\varepsilon)$ terms in the transformed Hamiltonian H' vanish. We seek $1:2$ SH resonance in the problem as the driving frequency $\omega_d = \dot\varphi_d \approx \frac{1}{2}$ passes through the two-photon resonance. There exist only one $O(\varepsilon^2)$ two-photon-resonant term in the transformed Hamiltonian, i.e., $\frac{8}{9}\varepsilon^2\lambda x' \cos 2\varphi_d$. After neglecting all other nonresonant and higher-order terms, the transformed Hamiltonian becomes

$$H' = \frac{1}{2}\left(p'^2 + x'^2\right) + \frac{1}{3}\lambda x'^3 + \frac{1}{4}\beta x'^4 + \frac{8}{9}\varepsilon^2\lambda x'\cos 2\varphi_d. \tag{11}$$

One can see that this Hamiltonian with $\omega_d = \dot\varphi_d = \frac{1}{2}(1+\alpha t)$ is the same as the Hamiltonian (1) with $\omega_d = \dot\varphi_d = 1 + \alpha t$

studied by the authors of Ref. [22] for the fundamental resonance case, but with ε replaced by $\frac{8}{9}\varepsilon^2\lambda$. This explains the similarity between the chirped fundamental and the SHAR, illustrated in our numerical examples. The same isomorphism was found in the classical theory of the SHAR [28]. Consequently, the parameter P_1 for the fundumental resonance should be replaced by $\widetilde{P}_1 = \frac{8}{9}\varepsilon\lambda P_1$ for the two-photon resonance case, while P_2 remains unchanged. Note that the classical problem of the fundamental AR is fully controlled by a *single* parameter, $\mu = \frac{1}{2}P_1 P_2^{1/2}$ [22] ($\widetilde{\mu} = \frac{1}{2}\widetilde{P}_1 P_2^{1/2}$ in the SH case). In contrast, the quantum-mechanical counterpart in the problem is characterized by two parameters $(\widetilde{P}_1, P_2)$ due to a new scale associated with $\hbar$.

The aforementioned isomorphism allows to apply all the results of the theory of the fundamental chirped resonance to the SH scenario by replacing $P_1 \to \widetilde{P}_1$. For instance, in the fundamental resonance case, it was found that the separator between the classical and the quantum regimes in the (P_1,P_2) parameter space is the line $P_2 = P_1 + 1$ [22]. Hence, we conclude that in the SH case, the classicality condition is $P_2 < \widetilde{P}_1 + 1$. Furthermore, the probability of capture into fundamental resonance depends on the parameters $P_{1,2}$ and, thus, in the SH case, this probability is fully described by the parameters $\widetilde{P}_1, P_2$. For example, the threshold for the phase-locking transition by passage through the fundamental resonance is a line in the parameter space $P_{1cr} = f(P_2)$ defined as the value of P_1 for which the capture probability is 50% [22]. Therefore, in the SH case the equivalent threshold line is $\widetilde{P}_{1cr} = f(P_2)$. Figure 3 compares these predictions with the results of the numerical solution of Eq. (2) for different values of P_2. The dots in the figure show the numerically found threshold, while the dashed lines correspond to the appropriately rescaled theoretical predictions in the classical SHAR, i.e., $\widetilde{P}_{1cr} = 0.82/\sqrt{P_2}$ (dashed doted line), and in the
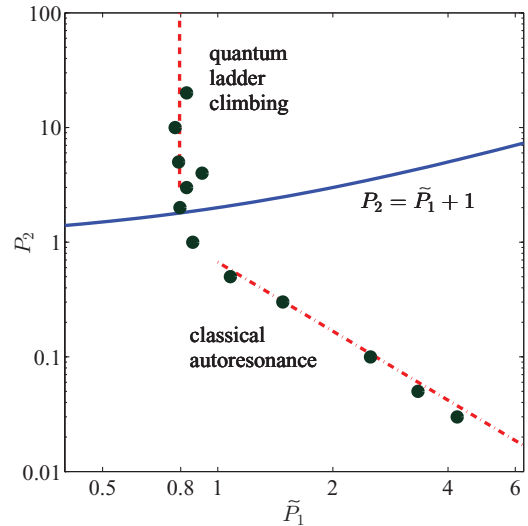

FIG. 3. (Color online) Different regimes of the phase-locking transition in the chirped, $1:2$ subharmonic resonance. The dots show the location of the threshold for the phase-locking transition. The dashed and dashed-dotted lines represent the theoretical thresholds in the quantum MPLC and classical SHAR regimes, respectively, while the solid line separates these regimes.

quantum MPLC, $\widetilde{P}_{1cr} = 0.79$ (dashed line) limits rescaled from the fundamental resonance theory [22]. One observes a very good agreement in both dynamical limits, including the characteristic transition between these two limits near the rescaled theoretical separator $P_2 = \widetilde{P}_1 + 1$ (solid line).

In conclusion, we have studied the problem of passage through two-photon nonlinear resonance and identified the quantum counterpart of the classical SHAR in the nonlinear oscillator, i.e., the quantum two-photon LC. We have used the isomorphism between the quantum fundamental and the two-photon chirped resonance phenomena. A similar isomorphism for stationary ($\alpha = 0$) quantum resonance exists as a special case of the chirped resonance. The calculation can be generalized to similar $n > 2$ photon processes. The theory shows that all the results in the chirped fundamental resonance process in both the quantum and classical limits can be extended to the SH resonance case by simply rescaling the driving parameter. The engineering and control of a desired quantum state of the oscillator via the LC process can be achieved by passage through both the fundamental and SH resonances. However, in the 1 : 2 subharmonic chirped resonance case, a desired state is achieved with just half of the driving frequency bandwidth, as compared to the same final state reached via passage through the fundamental resonance.

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