# Muonic-hydrogen Lamb shift: Dispersing the nucleon-excitation uncertainty with a finite-energy sum rule

Mikhail Gorchtein,[1,*] Felipe J. Llanes-Estrada,[2] and Adam P. Szczepaniak[3]

[1]*Institut für Kernphysik, Universität Mainz, 55128 Mainz, Germany*
[2]*Departamento de Fisica Teorica I, Universidad Complutense de Madrid, Madrid 28040, Spain*
[3]*Department of Physics and Center for Exploration of Energy and Matter, Indiana University, Bloomington, Indiana 47403, USA*

We assess the two-photon exchange contribution to the Lamb shift in muonic hydrogen with forward dispersion relations. The subtraction constant $\bar{T}(0, Q^2)$ that is necessary for a dispersive evaluation of the forward doubly virtual Compton amplitude, through a finite-energy sum rule, is related to the fixed $J = 0$ pole generalized to the case of virtual photons. We evaluated this sum rule using excellent virtual photoabsorption data that are available. We find that the "proton polarizability correction" to the Lamb shift in muonic hydrogen is $-(40 \pm 5)$ $\mu$eV. We conclude that nucleon structure-dependent uncertainty by itself is unlikely to resolve the large (300 $\mu$eV) discrepancy between direct measurement of the Lamb shift in $\mu$H and expectations based on conventional hydrogen measurements.

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## I. INTRODUCTION

An ongoing controversy surrounding the proton size originates from the large discrepancy between the recent measurement of the Lamb shift in muonic hydrogen and earlier measurement based on conventional hydrogen as well as electron scattering (see, for example, the review [1]). The advantage of using muonic hydrogen over the conventional one is that due to a larger reduced mass, the Lamb shift in the former is by an order of magnitude more sensitive to the proton radius. The Lamb shift $\Delta E_{2P-2S}$ in muonic hydrogen depends on the proton charge radius, $R_E$, through [1–5]

$$\Delta E_{2P-2S} \text{ (meV)} = 206.0579(60) - 5.22713 \, R_E^2, \quad (1)$$

where the numerical coefficients include effects up to the order of $O(\alpha^6)$ and $O[\alpha^6 \ln(\alpha)]$. The value of the Lamb shift predicted using $R_E$ quoted by the Committee on Data for Science and Technology (CODATA) [6],

$$R_E = 0.8775(51) \text{ fm}, \quad (2)$$

which is based primarily on the electronic-hydrogen Lamb shift measurement, or using the value of $RE$ extracted from the most recent electron-scattering data [7],

$$R_E = 0.879(8) \text{ fm}, \quad (3)$$

differs by $7\sigma$ from the measurement of the muonic hydrogen Lamb shift by Pohl *et al.* [8,9]. The latter requires a significantly smaller charge radius,

$$R_E = 0.840\,87(39) \text{ fm}. \quad (4)$$

In terms of the Lamb shift, the discrepancy amounts to some 300 $\mu$eV that by far exceeds the experimental sensitivity of the muonic experiment [8]. The first term in Eq. (1), which represents, up to $O(\alpha^5)$, all QED effects associated with the leptonic current, is almost three orders of magnitude larger than the observed discrepancy. This may lead to the conclusion that

a slight adjustment in one of those terms could resolve the whole puzzle. These higher-order QED corrections, however, have been known for a long time and are well established. The reader is referred to the recent reviews which assess the full body of the relevant QED corrections [2–5]. A nonperturbative numeric evaluation is also available [10,11] and yields a similar result, and so does the analysis based on the effective nonrelativistic expansion of QED [12,13].

An exotic possibility is a substantial nonuniversality of the lepton-proton interaction, which has not been observed before; but a more plausible explanation is that higher-order terms in the expansion in $\alpha$ are responsible for the discrepancy. Since QED corrections have a solid founding, attention has been focused on higher-order, nucleon structure-dependent effects. To the lowest order, $O(\alpha^5)$, these arise through a two-photon exchange process and potentially bear significant uncertainty because they involve the complete nucleon-excitation spectrum.

In Sec. II, we assess this two-photon exchange contribution to the Lamb shift using forward dispersion relations. Section III deals with the specific feature of our approach where we use the finite-energy sum rule (FESR) to relate the value of the subtraction function that arises in the dispersive calculation to the contribution from the fixed $J = 0$ Regge pole. Section IV is dedicated to the numerical analysis. Discussion of the results and comparison with the existing calculations is summarized in Sec. V.

## II. DISPERSION RELATIONS FOR COMPTON SCATTERING

The $O(\alpha^5)$ contribution to the Lamb shift sensitive to the proton structure enters through the matrix element of the two-photon exchange (TPE) between the lepton and the nucleon integrated over the atomic wave function. This can be seen as the virtual excitation and deexcitation of the proton by the successive photons, and thus all the complexity of the excited nucleon states is affecting a precision atomic physics computation. Adopting the standard approach for computing

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*gorshtey@kph.uni-mainz.de

bound-state corrections in atomic physics, which expresses nucleon current effects in terms of the atomic wave function at the origin, the TPE contribution to the Lamb shift is then given by Refs. [14,15]

$$E = 4\pi i \frac{\phi_n^2(0)}{2m_l} e^2 \int \frac{d^4q}{(2\pi)^4} \frac{(q^2 + 2\nu^2)T_1 - (q^2 - \nu^2)T_2}{q^4[(q^2/2m_l)^2 - \nu^2]},$$
(5)

where $m_l = m_e, m_\mu$ is the lepton mass in conventional and muonic hydrogen, respectively. The wave function at the origin is given by $\phi_n^2(0) = (\alpha m_r)^3/n^3\pi$; $\alpha = e^2/4\pi$ is the fine-structure constant, and $m_r \equiv m_l M/(m_l + M)$ is the reduced mass, with $M$ the proton mass. The scalar functions $T_{1,2} = T_{1,2}(\nu, q^2)$, with $\nu = (pq)/M$, are the standard amplitudes that parametrize the spin-independent hadronic tensor of doubly virtual forward Compton scattering $\gamma^*(q) + N(p) \rightarrow \gamma^*(q) + N(p)$, and are given by

$$T^{\mu\nu} = \frac{i}{8\pi M} \int d^4x e^{iqx} \langle N|T[J^\mu(x), J^\nu(0)]|N\rangle$$
$$= \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) T_1(\nu, q^2)$$
$$+ \frac{1}{M^2}\left(p^\mu - \frac{pq}{q^2}q^\mu\right)\left(p^\nu - \frac{pq}{q^2}q^\nu\right) T_2(\nu, q^2). \quad (6)$$

The hadronic tensor can be measured in a restricted kinematic range of the variables $\nu$ and $Q^2$ and needs to be extrapolated outside the physical range to compute the integral in Eq. (5). The extrapolation is based on analytical continuation. Specifically, the functions $T_{1,2}$ are discontinuous along the real axis in the complex energy plane $\nu$ with discontinuities (equal to $2i$ times imaginary parts) related to the inclusive virtual-photon cross sections,

$$\text{Im}T_1(\nu, q^2) = \frac{e^2}{4M} F_1, \quad \text{Im}T_2(\nu, q^2) = \frac{e^2}{4\nu} F_2. \quad (7)$$

As customary in dispersive approaches, we make use of the complex $\nu = (s - u)/(4M)$ plane. Since this variable is crossing-symmetric, upon applying Cauchy's theorem, the left and right cut can be combined in the same integral, yielding a relatively simple forward dispersion relation [16],

$$\text{Re}\, T_1(\nu, Q^2) = T_1(0, Q^2) + \frac{\nu^2 e^2}{2\pi M} \mathcal{P} \int_{\nu_{tr}}^\infty d\nu' \frac{F_1(\nu', Q^2)}{\nu'(\nu'^2 - \nu^2)},$$

$$\text{Re}\, T_2(\nu, Q^2) = \frac{e^2}{2\pi} \mathcal{P} \int_{\nu_{tr}}^\infty d\nu' \frac{F_2(\nu', Q^2)}{(\nu'^2 - \nu^2)}. \quad (8)$$

While this suffices to reconstruct $T_2$ from knowledge of the dispersive part, $T_1$ requires an additional input in the form of a subtraction constant at each $Q^2$, i.e., the function $T_1(0, Q^2)$. This is due to divergence of the unsubtracted dispersive integral at large energies, as dictated by the high-energy asymptotic properties of the $F_1$ structure function. At the real photon point $Q^2 = 0$, the subtraction term is fixed by the well-known Thomson-scattering limit, $T_1(0,0) = -\alpha/M$. For virtual photons, however, existing estimates carry large uncertainties. They are based on the not-so-well-determined polarizability and the $Q^2$ dependence of elastic form factors.

The $F_i$ structure functions measured with virtual photons receive a contribution from the single nucleon pole (Born

terms) at $\nu_{tr} = \nu_N = \pm Q^2/2M$, and from the unitarity cut due to opening of particle production thresholds, which start with pion production at $\nu_{tr} = \nu_\pi(Q^2) = \pm[(M + m_\pi)^2 - M^2 + Q^2]/2M$ (with $m_\pi$ being the pion mass). Following [14], we divide the contribution to the Lamb shift into three physically distinct terms that originate from the subtraction term $T_1(0, Q^2)$, the nucleon pole, and finally all excited intermediate states that may couple to $\gamma N$, respectively,

$$\Delta E = \Delta E^{\text{subt}} + \Delta E^{\text{el}} + \Delta E^{\text{inel}}, \quad (9)$$

with

$$\Delta E^{\text{subt}} = \frac{\alpha}{m_l} \phi_n^2(0) \int_0^\infty \frac{dQ^2}{Q^2} \frac{\gamma_1(\tau_l)}{\sqrt{\tau_l}} T_1(0, Q^2),$$

$$\Delta E^{\text{el}} = -\frac{\alpha^2 m_l}{M(M^2 - m_l^2)} \phi_n^2(0) \int_0^\infty \frac{dQ^2}{Q^2}$$

$$\times \left\{ \left[\frac{\gamma_2(\tau_p)}{\sqrt{\tau_p}} - \frac{\gamma_2(\tau_l)}{\sqrt{\tau_l}}\right] \frac{G_E^2 + \tau_p G_M^2}{\tau_p(1 + \tau_p)} \right.$$

$$\left. - \left[\frac{\gamma_1(\tau_p)}{\sqrt{\tau_p}} - \frac{\gamma_1(\tau_l)}{\sqrt{\tau_l}}\right] G_M^2 \right\}, \quad (10)$$

$$\Delta E^{\text{inel}} = -\frac{2\alpha^2}{m_l M} \phi_n^2(0) \int_0^\infty \frac{dQ^2}{Q^2} \int_{\nu_\pi}^\infty \frac{d\nu}{\nu}$$

$$\times \left[\tilde{\gamma}_1(\tau, \tau_l) F_1(\nu, Q^2) + \frac{M\nu}{Q^2} \tilde{\gamma}_2(\tau, \tau_l) F_2(\nu, Q^2)\right],$$

$\tau_l = Q^2/(4m_l^2)$, $\tau_p = Q^2/(4M^2)$, $\tau = \nu^2/Q^2$, and the auxiliary functions defined by

$$\gamma_1(\tau) \equiv (1 - 2\tau)\sqrt{1 + \tau} + 2\tau^{3/2},$$

$$\gamma_2(\tau) \equiv (1 + \tau)^{3/2} - \tau^{3/2} - \frac{3}{2}\sqrt{\tau},$$

$$\tilde{\gamma}_1(\tau, \tau_l) \equiv \frac{\sqrt{\tau_l}\gamma_1(\tau_l) - \sqrt{\tau}\gamma_1(\tau)}{\tau_l - \tau}, \quad (11)$$

$$\tilde{\gamma}_2(\tau, \tau_l) \equiv \frac{1}{\tau_l - \tau}\left[\frac{\gamma_2(\tau)}{\sqrt{\tau}} - \frac{\gamma_2(\tau_l)}{\sqrt{\tau_l}}\right].$$

Note that generally, in addition to the integral over the muon continuum that is represented in the above equations, a sum over the discrete spectrum must be taken. The latter contributes to the Lamb shift at the order of $O(\alpha^6)$ and is dropped from our considerations. Using these formulas, in Ref. [14] the inelastic contribution $\Delta E^{\text{inel}}$ was evaluated using the photoabsorption cross-section parametrization of Ref. [17] for the resonance region complemented with the high-energy parametrization of Ref. [18]. Their elastic (nucleon-pole) contribution $\Delta E^{\text{el}}$ was computed using three different phenomenological parametrizations of nucleon electromagnetic form factors [7,19,20]. Here we also give an independent evaluation of the two contributions. For $\Delta E^{\text{inel}}$, we use a recent parametrization of inclusive structure functions [21] that also uses the parametrization of the resonance region from Ref. [17], but it uses a modified Regge-inspired background that is fitted to the total photoabsorption cross section of [22]. The $Q^2$ dependence is introduced as in Ref. [23]. For $E^{\text{el}}$, we

use the parametrization from Ref. [19] to finally obtain

$$\Delta E^{\rm el} = -30.1 \pm 1.2 \ \mu{\rm eV}, \quad \Delta E^{\rm inel} = -13.0 \pm 0.6 \ \mu{\rm eV}. \tag{12}$$

Within errors, these agree with the recent computation reported in Ref. [14],

$$\Delta E^{\rm el} = -29.5 \pm 1.3 \ \mu{\rm eV}, \quad \Delta E^{\rm inel} = -12.7 \pm 0.5 \ \mu{\rm eV}, \tag{13}$$

and with the older calculation of Ref. [2],

$$\Delta E^{\rm el} = -28 \pm 1 \ \mu{\rm eV}, \quad \Delta E^{\rm inel} = -12 \pm 2 \ \mu{\rm eV}. \tag{14}$$

In this last equation only, and following the discussion of Ref. [14], we subtracted the nonpole elastic part from $\Delta E^{\rm el}$ and effectively added it to the subtraction term. We emphasize that we do not advocate the subtraction of this term from the finite result, as Ref. [14] does, nor will we do it in our computations below, but rather exclude it from Eq. (14) for the sake of a meaningful comparison with our Eq. (12).

We will discuss the subtraction term in more detail, in relation with various calculations, in Sec. V.

## III. EVALUATION OF THE SUBTRACTION TERM

### A. Finite-energy sum rules

While previous analyses concentrate on the low-energy constraints for the subtraction term, here we focus on implications of the high-energy behavior for constraining the subtractions. This is done by exploiting the finite-energy sum rule (FESR) for the Compton amplitude. The subtraction term in the dispersion relation (DR) for $T_1$ arises because the high-energy photoabsorption cross section does not vanish asymptotically. It can be well described by a Regge-theory-inspired parametrization,

$$\sigma_T \rightarrow \sigma_T^R(\nu,0) = c_P(0)\left(\frac{\nu}{\nu_0}\right)^{\alpha_P-1} + c_R(0)\left(\frac{\nu}{\nu_0}\right)^{\alpha_R-1}, \tag{15}$$

with the effective Pomeron and leading Regge trajectory intercepts given by $\alpha_P = 1.097$ and $\alpha_R = 0.5$, respectively. The remaining parameters were found to be [24] $c_P(0) = 68.0 \pm 0.2 \ \mu{\rm b}$ and $c_R(0) = 99.0 \pm 1.2 \ \mu{\rm b}$, with $\nu_0 = 1$ GeV.

The corresponding contribution to the Compton amplitude $T_1$ of this Regge part is given by

$$\begin{aligned} {\rm Im} T_1^R(\nu,0) &= (\nu/4\pi)\sigma_T^R(\nu,0), \\ {\rm Re} \ T_1^R(\nu,0) &= \frac{\nu^2}{2\pi^2}\mathcal{P}\int_0^\infty d\nu' \frac{\sigma_T^R(\nu')}{\nu'^2 - \nu^2}. \end{aligned} \tag{16}$$

Following [25], we write a dispersion relation for the difference $T_1 - T_1^R$,

$$\begin{aligned} &{\rm Re} \ T_1(\nu,0) - {\rm Re} \ T_1^R(\nu,0) \\ &= -\frac{\alpha}{M} + \frac{\nu^2}{2\pi^2}\mathcal{P}\int_{\nu_\pi}^\infty d\nu' \frac{\sigma_T(\nu',0) - \sigma_T^R(\nu',0)}{\nu'^2 - \nu^2}. \end{aligned} \tag{17}$$

With the large-$\nu$ tail thus removed, the dispersion integral on the right-hand side of Eq. (18) is dominated by energies below a scale $N = O(\nu_0)$, which is discussed below. Removal of the

asymptotic contribution from the dispersive integral introduces a new subtraction, $C_\infty$, defined by

$$C_\infty(0) \equiv \left[{\rm Re} \ T_1(\nu,0) - {\rm Re} \ T_1^R(\nu,0)\right]\big|_{\nu\rightarrow\infty}. \tag{18}$$

With the help of currently available high-energy data, $C_\infty(0)$ has recently been determined with high accuracy [24] and it follows from Eq. (17) that it is related to the high-energy parameters by

$$\begin{aligned} C_\infty(0) &= -\frac{\alpha}{M} - \frac{1}{2\pi^2}\int_{\nu_\pi}^N d\nu' \sigma_T(\nu',0) \\ &+ \frac{\nu_0}{2\pi^2}\sum_{i=P,R} \frac{c_i(0)}{\alpha_i}\left(\frac{N}{\nu_0}\right)^{\alpha_i}. \end{aligned} \tag{19}$$

The resonance contribution given by the integral over the photoabsorption cross section is well established and can be readily evaluated from the low-energy data. The parameter $N$ defines the lowest photon energy above which Regge parametrization suffices to describe the data, which in the analysis of Ref. [24] was taken to be 2 GeV. From this analysis, it follows that $C_\infty(0) = (-0.72 \pm 0.35) \ \mu{\rm b}$ GeV.

For our application to muonic hydrogen, we need to generalize the above dispersion relation for the real Compton amplitude to the virtual-photon case. Using

$$F_1(\nu,Q^2) = \frac{M\nu(1-x)}{\pi e^2}\sigma_T(\nu,Q^2), \tag{20}$$

where $x = Q^2/(2M\nu)$, we may write

$$T_1(\nu,Q^2) = T_1(0,Q^2) + \frac{\nu^2 e^2}{2\pi M}\int_{\nu_\pi(Q^2)}^\infty \frac{d\nu' F_1(\nu',Q^2)}{\nu'(\nu'^2 - \nu^2)}. \tag{21}$$

In analogy to the real photon case, we introduce the Regge-theory-motivated representation for the high-energy data valid for $\nu \geqslant N(Q^2)$,

$${\rm Re} \ T_1^R(\nu,Q^2) = \frac{\nu^2 e^2}{2\pi M}\mathcal{P}\int_0^\infty d\nu' \frac{F_1^R(\nu',Q^2)}{\nu'(\nu'^2 - \nu^2)}, \tag{22}$$

with

$$F_1^R(\nu,Q^2) = \frac{M\nu_0}{\pi e^2}\sum_{i=P,R} c_i(Q^2)\left(\frac{\nu}{\nu_0}\right)^{\alpha_i}. \tag{23}$$

The generalization of Eq. (23) is not unique since, in principle, $\nu_0$ and $\alpha_i$ might be made $Q^2$ dependent. These eventual $Q^2$ dependences for low $Q^2 \lesssim 1$ GeV$^2$ that are of interest here can, however, be absorbed in $c_i(Q^2)$ without loss of generality. The coefficients $c_i(Q^2)$ must reduce to those found for real photons at $Q^2 = 0$ that are listed below Eq. (15). Their $Q^2$ dependence, and that of $N(Q^2)$, is obtained by matching the Regge parametrization of Eq. (23) and $F_1(\nu,Q^2)$ defined by Eq. (20). For $\nu \geqslant N(Q^2)$ and moderate $Q^2 \lesssim 1$ GeV$^2$, we obtain

$$c_P(Q^2) = c_P(0),$$

$$c_R(Q^2) = c_R(0) - (20 \pm 10) \ \mu{\rm b}\left(\frac{Q}{\rm GeV}\right)^2, \tag{24}$$

and

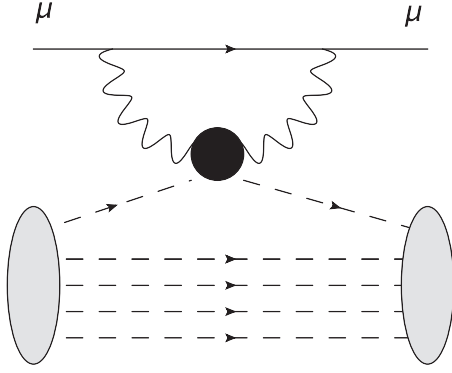$$N(Q^2) \approx 5 \ {\rm GeV} + \frac{Q^2}{2M}. \tag{25}$$

FIG. 1. The residual term for the high-energy Compton amplitude corresponding to a fixed pole at $J = 0$ in the complex angular momentum plane. It corresponds to Compton scattering on a pointlike quark at instant light-cone time.

Note that the presence of the factor $1 - x = 1 - Q^2/2M\nu$ in the relation between $\sigma_T$ and $F_1$, given by Eq. (20), requires a value of $N(Q^2)$ larger than that found for real photons $N(0)$. In any case, the resulting FESR will not be sensitive to the value of $N$, as long as the Regge amplitude correctly represents the data for all $\nu > N$. The values $c_P(0), c_R(0)$ are fixed by very precise fit to real photoabsorption data, and $c_P(Q^2)$ is, moreover, fixed to its real photon value (for low $Q^2 \lesssim 1$ GeV$^2$ only) to ensure that asymptotically $\sigma_T - \sigma_T^R$ vanishes, which is the assumption that is crucial for the FESR method. This effectively leaves the $Q^2$ slope of the coefficient $c_R(Q^2)$ (which we take as a linear function) as the only parameter that has an uncertainty, and we assign a generous 50% uncertainty thereto. The analog of Eq. (19) at finite $Q^2$,

$$C_\infty(Q^2) \equiv \left[ \mathrm{Re}\, T_1(\nu, Q^2) - \mathrm{Re}\, T_1^R(\nu, Q^2) \right]\big|_{\nu \to \infty}, \quad (26)$$

satisfies now

$$C_\infty(Q^2) = T_1(0, Q^2) - \frac{e^2}{2\pi M} \int_{\nu_\pi(Q^2)}^{N(Q^2)} \frac{d\nu'}{\nu'} F_1(\nu', Q^2)$$
$$+ \frac{\nu_0}{2\pi^2} \sum_i \frac{c_i(Q^2)}{\alpha_i} \left[ \frac{N(Q^2)}{\nu_0} \right]^{\alpha_i}. \quad (27)$$

We expect a finite $C_\infty(Q^2)$ at high $Q^2$. It represents a light-cone instantaneous, two-photon interaction on a pointlike quark [26], as depicted in Fig. 1. This causes no problem in the first of Eqs. (10) for $E^{\mathrm{subt}}$ that is convergent upon substitution of a constant contribution to $T_1(0, Q^2)$. The constant $C_\infty(Q^2)$ is related to the virtual Compton amplitude $T_1(0, Q^2)$ through Eq. (27) and enters the Lamb shift though $E^{\mathrm{subt}}$.

To evaluate the integral on the right-hand side of Eq. (27), we need a parametrization of the virtual photon-proton cross section to substitute in Eq. (20). We use the form obtained in Ref. [21] that reproduces the electroproduction data in the resonance region and above,

$$\sigma_T(W^2, Q^2) = \sum_a BW_a(W^2) F_a^2(Q^2)$$
$$+ \left[ 1 - e^{\frac{(M+m_\pi)^2 - W^2}{M^2}} \right] \sigma_{\mathrm{tot}}^R(W^2, 0) F_B(Q^2). \quad (28)$$

In the first term, the summation runs over nucleon resonances, with $BW$ standing for a Breit-Wigner propagator $BW_a(W^2)$,
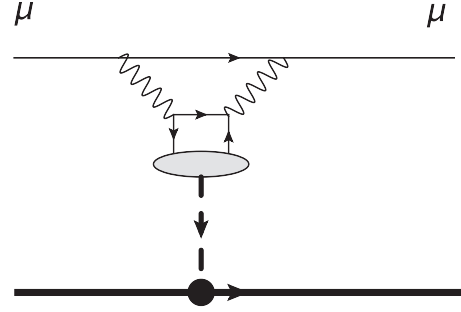


FIG. 2. Regge exchanges in the $t$ channel dominate the high-energy part of the Compton amplitude.

and electromagnetic transition form factors given by $F_a(Q^2)$. The second term represents a smooth background. Expressing $T_1(0, Q^2)$ in terms of the $J = 0$ pole contribution, $C_\infty(Q^2)$ yields

$$T_1(0, Q^2) = C_\infty(Q^2) - \frac{\nu_0}{2\pi^2} \sum_i \frac{c_i(Q^2)}{\alpha_i} \left[ \frac{N(Q^2)}{\nu_0} \right]^{\alpha_i}$$
$$+ \frac{1}{2\pi^2} \int_{\nu_\pi(Q^2)}^{N(Q^2)} d\nu' \left( 1 - \frac{Q^2}{2M\nu} \right) \sigma_T(\nu', Q^2), \quad (29)$$

which is the main result of this paper. It expresses the low-energy function $T_1(0, Q^2)$ that enters the Lamb shift through $E^{\mathrm{subt}}$ in Eq. (9) in terms of three distinct contributions with clear physical interpretation, which are diagrammatically shown in Figs. 1–3. The last two are the $t$-channel Regge exchanges and $s$-channel resonance contributions; the split between the two is determined by $N(Q^2)$. The first term is the $J = 0$ fixed-pole contribution to virtual Compton scattering $C_\infty(Q^2)$ [27], to which we now turn our attention.

### B. Analysis of the fixed pole

The $J = 0$ fixed pole in Compton scattering was introduced in Ref. [28] and studied in phenomenological models, e.g., in Refs. [26,27,29,30].

Such an $s$- and $t$-independent contribution has been analyzed in the kinematic region where both $-t$ and $s$ are large, $s, -t \gg M_N^2$, and the existing data in this region [31,32] support the existence of the fixed pole.
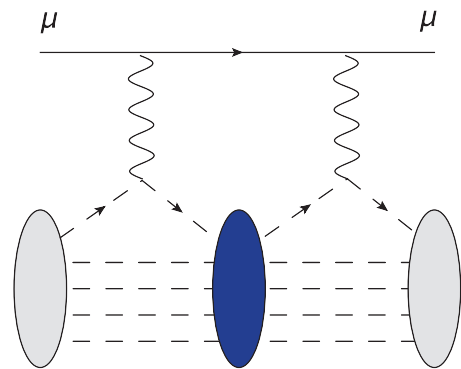


FIG. 3. (Color online) The low- and intermediate-energy region is described by a sum over $s$-channel resonances that are photoexcitations of the nucleon.

For real Compton scattering, $C_\infty(0)$ was determined in Ref. [24]; however, in Eq. (29), $C_\infty$ is evaluated at finite $Q^2$. Theory suggests that at asymptotic $Q^2$, $C_\infty(Q^2)$ is constant [26], but this has not been experimentally established; it might be so in the future with the help of the Deeply Virtual Compton Scattering program at Jefferson Laboratory. To allow for the possibility of a $Q^2$ dependence, we subtract Eq. (19) (real FESR) from Eq. (29) (virtual FESR), and change the integration variable from $\nu$ to $\omega = \nu - Q^2/2M$, to obtain

$$T_1(0,Q^2) = -\frac{\alpha}{M} + [C_\infty(Q^2) - C_\infty(0)] \qquad (30)$$

$$+ \frac{1}{2\pi^2} \int_{\nu_\pi(0)}^{N(0)} d\omega \left[ \frac{\omega}{\omega + \frac{Q^2}{2M}} \sigma_T(\omega, Q^2) - \sigma_T(\omega, 0) \right]$$

$$+ \frac{\nu_0}{2\pi^2} \sum_{i=P,R} \left\{ \frac{c_i(0)}{\alpha_i} \left[ \frac{N(0)}{\nu_0} \right]^{\alpha_i} \right.$$

$$\left. - \frac{c_i(Q^2)}{\alpha_i} \left[ \frac{N(Q^2)}{\nu_0} \right]^{\alpha_i} F_B(Q^2) \right\}. \qquad (31)$$

This is a rigorous representation of the subtraction term in the virtual Compton amplitude. If the fixed pole were $Q^2$ independent, as suggested by Ref. [26], $C_\infty$ would drop out of this equation. Since this is not established experimentally, we also provide an order of magnitude estimate under the assumption that $C_\infty(Q^2)$ falls with $Q^2$.

For the estimates of the uncertainty associated with $C_\infty(Q^2) - C_\infty(0)$, we use a parametrization

$$C_\infty(Q^2) - C_\infty(0) = \frac{Q^2}{\Lambda^2 + Q^2} [C_\infty(\infty) - C_\infty(0)], \qquad (32)$$

with a typical scale $\Lambda = 1$ GeV and $C_\infty(\infty) = 0$.

## IV. NUMERICAL ANALYSIS

If we substitute Eq. (31) in the expression for $E^{\text{subt}}$ in Eq. (10), we see that the result is infrared (IR) divergent. This is due to the Thomson term, $T_1(0,0) = -\frac{\alpha}{M}$. Physically, it corresponds to exchange of soft Coulomb photons that is already taken into account at the level of atomic wave functions and has to be subtracted in order to avoid double counting. We are left with the following convergent integral to be evaluated:

$$\Delta E^{\text{subt}} = 4\alpha \phi_n^2(0) \int_0^\infty dQ \gamma_1(\tau_l) \frac{T_1(0,Q^2) + \frac{\alpha}{M}}{Q^2}. \qquad (33)$$

The contribution from $T_1(0,Q^2)$ to the Lamb shift can be written as a sum of several terms,

$$\Delta E^{\text{subt}} = \sum_i \Delta E_i^{\text{res}} + \Delta E^{\text{Back}} + \Delta E^{\text{Regge}}. \qquad (34)$$

We evaluated the respective integrals in Eq. (33) numerically. Below, we quote the individual contributions from each of the well-established resonances, the nonresonant background, and the Regge part, respectively,

$$\Delta E_{\Delta(1232)} = (0.95 \pm 0.09) \, \mu\text{eV},$$
$$\Delta E_{S_{11}(1535)} = (-4.02 \pm 3.14) \, \mu\text{eV},$$
$$\Delta E_{D_{13}(1520)} = (0.41 \pm 0.09) \, \mu\text{eV},$$
$$\Delta E_{S_{11}(1665)} = (-0.23 \pm 0.16) \, \mu\text{eV},$$
$$\Delta E_{F_{15}(1680)} = (-0.32 \pm 0.06) \, \mu\text{eV},$$

$$\Delta E_{P_{11}(1440)} = (0.10 \pm 0.02) \, \mu\text{eV},$$
$$\Delta E_{F_{37}(1950)} = (-0.76 \pm 0.26) \, \mu\text{eV},$$
$$\Delta E^{\text{Back}} = (-29.34 \pm 2.93) \, \mu\text{eV},$$
$$\Delta E^{\text{Regge}} = (36.55 \pm 1.6) \, \mu\text{eV}. \qquad (35)$$

Adding the above contributions to the subtraction term,

$$\Delta E^{\text{subt}} = (3.3 \pm 4.6) \, \mu\text{eV}. \qquad (36)$$

It can be noted that there are strong cancellations between various terms. The size of the correction is almost entirely given by the sum of three contributions, $\Delta E^{\text{Regge}}, \Delta E^{\text{Back}}$, and $\Delta E_{S_{11}(1535)}$. To discuss the uncertainty, it thus suffices to constrain the uncertainty in these three contributions. Regge and background contributions are large, opposite in size, and cancel to about 80%. The background contribution is obtained from a fit to excellent experimental data over a wide range of $W^2, Q^2$ (see Ref. [17] for a full list of references) and a relative uncertainty of 10% is reasonable. The Regge contribution is related to the background since they are constructed to coincide at high energies, and assigning an extra uncertainty here would lead to double counting. We assign a 50% uncertainty on the $Q^2$ slope of the Reggeon strength $c_R(Q^2)$. For the resonances, we assign the uncertainties listed by the Particle Data Group (PDG) [33] for the $R \to N\gamma$ transition helicity amplitudes. The main uncertainty is due to $S_{11}(1535)$, and we believe that this estimate of uncertainties is very conservative. The actual fit describes the data in the second resonance region certainly better than $\pm 70\%$. We believe that this uncertainty can be further reduced.

Finally, we obtain, for the hadronic $O(\alpha^5)$ contribution to the $2P - 2S$ Lamb shift in muonic hydrogen set forth in Eq. (5),

$$\Delta E = (-40 \pm 5) \, \mu\text{eV}. \qquad (37)$$

## V. DISCUSSION

We have split the contribution of the nucleon's Compton tensor to the Lamb shift of the muonic hydrogen atom into three parts, $E^{\text{el}}$, $E^{\text{inel}}$, and $E^{\text{subt}}$. The first two, corresponding to elastic scattering off the proton and photoexcitation of resonances, are in agreement with previous work by other authors. The last term contains the contribution of the real subtraction to the Compton tensor and is the only one where significant uncertainty has remained. Specifically, in the analyses of Ref. [2], the subtraction function was identified with

$$T_1(0,Q^2) = -\frac{\alpha}{M} F_D^2(Q^2) + Q^2 \beta(Q^2), \qquad (38)$$

where $F_D(Q^2)$ stands for the Dirac form factor, and $\beta(Q^2)$ stands for the generalized magnetic polarizability that for real photons reduces to the usual magnetic polarizability of Compton scattering, $\beta(0) = \beta_M$. Its $Q^2$ dependence was taken by analogy with elastic form factors. The term $-(\alpha/M)F_D^2(Q^2)$ in Eq. (38) was originally included in the elastic contribution in Ref. [2].

In Ref. [14], it was argued that

$$T_1(0,Q^2) = -\frac{\alpha}{M} + Q^2 \beta(Q^2), \qquad (39)$$

where we put together the two contributions identified in Ref. [14] as $T_1^{NB}(0,Q^2) = Q^2\beta(Q^2)$ and $T_1^{B,\text{no-pole}}(0,Q^2) = -\frac{\alpha}{M}$ for clarity. The common feature of the two approximations is that at $Q^2 = 0$, they reduce to the Thomson term. However, they differ already in the first derivative, and they effectively operate with two different values of $\beta$ that is a measured quantity. We define

$$\bar{T}_1(Q^2) \equiv \frac{T_1(0,Q^2) + \frac{\alpha}{M}}{Q^2}, \tag{40}$$

the function that enters the calculation of the Lamb shift, and evaluate this function at $Q^2 = 0$. With the model of Ref. [2], one obtains

$$\bar{T}_1(0) = -\frac{\alpha}{M}2F_D'(0) + \beta, \tag{41}$$

while the model of Ref. [14] gives

$$\bar{T}_1(0) = \beta. \tag{42}$$

The difference is not small and amounts to $3.4 \times 10^{-4}$ fm$^3$, which is of the same size as the polarizability itself.

Consequently, Birse and McGovern [34] argued that Pachucki's prescription of Eq. (38) should be used, rather than Carlson and Vanderhaeghen's version of Eq. (39), while claiming a theory uncertainty due to the subtraction constant at the level of 1 $\mu$eV. To our knowledge, no exhaustive theory evidence for such small uncertainty was given. Hill and Paz advocated for increasing the theory uncertainty by an order of magnitude [13]. Here we show [cf. Eq. (36)] this to be unnecessary.

What complicates the issue is the impossibility to measure $T_1(0,Q^2)$ directly since the kinematical arguments are in the unphysical region. The problem of a low-energy expansion of doubly virtual Compton scattering was approached by two of us in [35] in terms of a fully model-independent low-energy theorem. It was found that it is only possible to unambiguously identify $T_1(0,Q^2)$ with a combination of known or measurable quantities (form factors and polarizabilities) modulo a dispersion integral in the annihilation channel that is largely unknown. Rewriting the findings of Ref. [35] for $T_1(0,Q^2)$, we find

$$T_1(0,Q^2) = -\frac{\alpha}{M}\left[F_D^2(Q^2) - \tau F_P^2(Q^2)\right] + Q^2\beta(Q^2) + \cdots, \tag{43}$$

where we omitted terms coming from that dispersion integral in the annihilation channel.

The reason for such detailed discussion is to remind the reader that to relate the *unphysical* subtraction constant $T_1(0,Q^2)$ to *measurable* quantities such as the polarizability and elastic form factors, a good deal of caution should be exercised.

Following the analysis presented in this paper, the systematic uncertainty in the Lamb shift from this term has been significantly reduced. We have employed the method of the finite-energy sum rules to analyze this term, explicitly displaying the contributions it receives from the known $t$-channel Regge and $s$-channel resonances. There is no double counting of these resonances with respect to $E^{\text{inel}}$. The alternative analysis presented here provides information on the
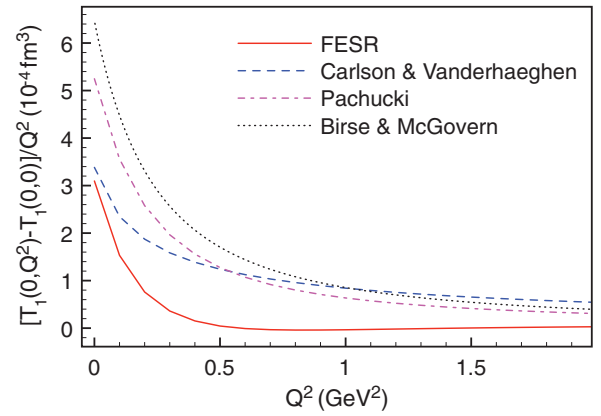


FIG. 4. (Color online) Subtraction function $[T_1(0,Q^2) - T_1(0,0)]/Q^2$ in units of $10^{-4}$ fm$^3$ as obtained from FESR (red solid line), and from the models of Ref. [14] (blue dashed line), Ref. [2] (magenta dash-dotted line), and Ref. [34] (black dotted line).

subtraction term from Regge theory and the resonance region, reducing the unknowns to the fixed pole of Compton scattering. Our finite-energy sum rule in Eq. (31) has made it possible to predict the $Q^2$ dependence of the subtraction function directly from existing experimental data. In Fig. 4, we compare the function $\bar{T}_1(Q^2)$ as obtained from FESR to phenomenological *Ansätze* of previous analyses. We observe that all approaches effectively have similar values of $\bar{T}_1(0)$, but in view of the complicated situation with the low-energy theorem discussed above, we stress that this is a coincidence. Neglecting the $t$-channel contributions in Eq. (43) and removing the contributions of the form factors ($3.4 \times 10^{-4}$ fm$^3$ and $0.5 \times 10^{-4}$ fm$^3$), we would arrive at $\beta = -0.9 \times 10^{-4}$ fm$^3$. Note that the most recent determination of the magnetic polarizability was given in the heavy baryon chiral perturbation theory (HBChPT) framework in Ref. [36],

$$\beta = [3.15 \mp 0.35 \pm 0.2 \mp 0.3] \times 10^{-4} \text{ fm}^3, \tag{44}$$

with the three uncertainties identified in Ref. [36] as "statistical," "Baldin," and "theory," respectively. It suggests that to connect the result of this work for the subtraction constant $T_1(0,Q^2)$ in terms of the FESR to the value of the magnetic polarizability, the aforementioned $t$-channel contributions should not be neglected.

We have shown that the contribution of the subtraction term $\Delta E^{\text{subt}}$ is small, $\approx 3$ $\mu$eV, and its large relative error on the order of 5 $\mu$eV does not alter the conclusion that the overall contribution of the nucleon photoexcitation processes to the Lamb shift in muonic hydrogen is about $-40 \pm 5$ $\mu$eV, in agreement with previous evaluations. A numerical comparison with existing calculations is shown in Table I.

Our overall estimated uncertainty has increased slightly with respect to that by Pachucki [2], Carlson and Vanderhaeghen [14], as well as chiral perturbation theory [34,37], while it is reduced compared to Hill and Paz [13]. The method of the finite-energy sum rule presented in this work allows for a reliable estimate of the subtraction constant contribution and the uncertainty thereof, based on virtual photoabsorption data and on the natural $Q^2$ dependence of the $J = 0$ pole. Recent model calculations by Miller *et al.*, designed to resolve

TABLE I. Numerical results for the $O(\alpha^5)$ proton structure corrections to the $2P - 2S$ Lamb shift in muonic hydrogen in $\mu$eV. The entry $\Delta E^{\text{sub}}$ from Ref. [34] is obtained by summing the Born nonpole and polarizability contributions; that work uses the values obtained for $\Delta E^{\text{el}}$ and $\Delta E^{\text{inel}}$ in Ref. [14].

|              | This work      | Ref. [2] | Ref. [14]       | Ref. [34]      |
| ------------ | -------------- | -------- | --------------- | -------------- |
| $\Delta E^{\text{subt}}$ | $3.3 \pm 4.6$  | 6.6      | $5.3 \pm 1.9$   | $9.0 \pm 1.0$  |
| $\Delta E^{\text{el}}$   | $-30.1 \pm 1.2$ | $-27.8$  | $-29.5 \pm 1.3$ | $-29.5 \pm 1.3$ |
| $\Delta E^{\text{inel}}$ | $-13.0 \pm 0.6$ | $-13.9$  | $-12.7 \pm 0.5$ | $-12.7 \pm 0.5$ |
| $\Delta E$   | $-39.8 \pm 4.8$ | $-35.1$  | $-36.9 \pm 2.4$ | $-33 \pm 2$    |

the proton radius puzzle in terms of the two-photon exchange contribution, are not supported by resonance region data at low $Q^2$ [38] and require an unnaturally large value of the $J = 0$ pole for hard virtual photons [39].

The 300 $\mu$eV discrepancy between the direct muonic-hydrogen Lamb shift measurement and estimates for it based on usual (electronic) hydrogen is unnaturally large for the hadronic structure-dependent corrections at the order of $O(\alpha^5)$ that have been proposed in the literature, basically Eq. (5), and the explanation must be looked for elsewhere.

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