Qubit dephasing affected by preparation-induced initial correlation with stochastic environment

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The reduced time evolution of a single qubit under the influence of a classical environment is studied by means of the stochastic Liouville equation. The qubit initially correlates with the environment, which is created by a preparation process of an initial state. The phase fluctuation caused by the environment is characterized by means of the Gauss-Markov process and the two-state-jump Markov process. It is shown that depending on values of the parameters, the initial correlation enhances or reduces the decoherence of the qubit. In the case of the slow modulation, the effect of the initial preparation of the qubit becomes significant and it cannot be ignored in the reduced time evolution of the qubit.

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I. INTRODUCTION

A quantum system in a real world is unavoidably influenced by a surrounding environment. As a result of an interaction with an environment, a quantum system evolves irreversibly from a prepared initial state into a stationary state [1,2], during which quantum-mechanical properties, such as coherence and entanglement [3,4], of a relevant system are eventually destructed. Such irreversible time evolution of a quantum system can be investigated by means of the quantum master equation method [5–10] and the path integral method [11–13]. In a study of irreversible time evolution, a noncorrelated initial state of a quantum system and an environment is usually assumed when the reduced dynamics of a quantum system is derived. In this case, the reduced dynamics of a relevant system is described by a completely positive map [2,4,14]. However, since an environment influences the time evolution of a quantum system, it is reasonable to consider that a preparation of an initial state of the system is also influenced by the environment. This implies that an initial correlation between a quantum system and an environment should be taken into account when the reduced dynamics is derived. Therefore it is important to investigate how an initial correlation between a quantum system and an environment influences the reduced time evolution of a quantum system.

Recently, in the many works [15–34], the effects of the initial correlation on the relevant system have been investigated in detail. One of the most important results is that the reduced time evolution in the presence of an initial correlation cannot be described by a completely positive map [15–17]. It has also been pointed out that the reduced time evolution may be quite different if the initial correlation is ignored [18]. The noncomplete positivity of the reduced time evolution has been studied in terms of a trace distance between quantum states [19–24], where the pure dephasing model, the Jaynes-Cummings model, and the exactly solvable single excitation model have been used. The quantum master equation for a relevant system which has an initial correlation has also been investigated by several methods [25–28]. Furthermore the linear response of a two-level system initially correlated with an environment to an external field has been also considered [29,30]. In these works, the initial states are classified into two types. One is given by a *ad hoc* manner without referring to a preparation process. The other is given by a thermal equilibrium state of a total system. All the initial states cannot be factorized into the system and environment states due to an initial correlation between them.

However, it is possible to consider a different type of initial correlation when a preparation of an initial state is performed by means of von Neumann measurement or filtering operation on a relevant system. In this case, although an initial state of a total system becomes a product state, the environmental part depends on the system part. Such a dependence creates an initial correlation between the system and the environment. It has been examined in detail how the initial correlation created by a preparation of a system initial state affects the decay of coherence and entanglement in the reduced time evolution of the quantum system [31–34]. In this paper, different from the previous works on the initial correlation, we will consider the case that a quantum system interacts with a classical environment [35–42], the influence of which is described by a stochastic process. Even in such a quantum-classical system, it will be found that the initial correlation yields a significant effect on the reduced time evolution of the quantum system. Here it should be noted that although the time evolution of a total system is given by a unitary map in the previous works, it is described by a nonunitary map in the present work.

In our previous paper [43], we have investigated the decoherence of a qubit interacting with a classical environment, where the time evolution of the total system is governed by the stochastic Liouville equation [38,44,45]. We have shown that the observation of the environmental variable (the classical stochastic variable [46]) can suppress the decay of distinguishability and entanglement of qubits. The present paper, on the other hand, investigates how the initial correlation between the qubit and the classical environment affects the reduced time evolution of the qubit, where the time evolution of the total system is determined by the stochastic Liouville equation which is the same as that used in [43]. We will find whether the initial correlation can reduce the decoherence of the qubit or not. Furthermore we will show that the effect of the initial correlation becomes significant and so it cannot be ignored in the reduced time evolution of the qubit, even if an environment is classical.

In this paper, we suppose that a classical environment induces a phase fluctuation of a quantum system, which is described by means of a classical stochastic process [46].

In the rest of this paper, we refer to such a classical environment as a stochastic environment. The time evolution of a total system which consists of a quantum system and a stochastic environment is governed by the stochastic Liouville equation [38,44,45]. Usually, to investigate the reduced time evolution of the relevant quantum system, the quantum master equation is derived by eliminating variables of the stochastic environment from the stochastic Liouville equation. To take the effect of the initial correlation into account, however, we need to solve the stochastic Liouville equation without the elimination. In Sec. II, we briefly summarize the stochastic Liouville equation [38,44,45] and the exact solutions when the Gauss-Markov process and the two-state-jump Markov process are considered [43]. We also explain a correlated initial state of a quantum system and a stochastic environment, which is prepared by means of the von Neumann measurement or the filtering operation on the relevant quantum system [31–34]. In Sec. III, we investigate how the initial correlation influences the reduced time evolution of a single qubit. It will be found that depending on the values of the parameters, the initial correlation enhances or reduces the decoherence of the qubit and the effect becomes significant in the case of the slow modulation of the dephasing. In Sec. IV, we provide concluding remarks.

II. TIME EVOLUTION AND INITIAL STATE

A. Stochastic Liouville equation

We suppose that a quantum system is placed under the influence of a stochastic environment. As a result, a random phase fluctuation described by means of a stochastic process [46] occurs in the quantum system. In this paper, the Gauss-Markov process and the two-state-jump Markov process are considered. The total system consisting of the quantum system and the stochastic environment can be described by a joint state $\hat{W}(t;\Omega)$ [44,45] with Ω being a value of the stochastic variable, the marginals of which are the reduced density matrix $\hat{\rho}(t)=\sum_{\Omega}\hat{W}(t;\Omega)$ of the relevant quantum system and the probability $P(t;\Omega)=\mathrm{Tr}\,\hat{W}(t;\Omega)$ of the stochastic variable. Here, the summation is taken over all possible values of the stochastic variable and Tr stands for the trace operation over the Hilbert space of the quantum system. When the stochastic variable takes continuous values, the summation is replaced with an appropriate integration. The time evolution of the joint state $\hat{W}(t;\Omega)$ is governed by the stochastic Liouville equation [44,45]

$$\frac{\partial}{\partial t}\hat{W}(t;\Omega)=-\frac{i}{\hbar}[\hat{H}+\hbar\hat{S}\Omega,\hat{W}(t;\Omega)]+\mathcal{L}_{\Omega}\hat{W}(t;\Omega),\quad (1)$$

where $\hat{H}$ is a Hamiltonian of the quantum system, and $\hat{S}$ is a system observable coupled with the stochastic variable. In this equation, $\mathcal{L}_{\Omega}$ stands for a map which characterizes the time evolution of the stochastic process and it satisfies the equality $\sum_{\Omega}\mathcal{L}_{\Omega}=0$ due to a conservation law of probability. Since we have assumed the time homogeneity of the stochastic process, the map $\mathcal{L}_{\Omega}$ does not depend on time t. The time evolution of the probability $P(t;\Omega)$ of the stochastic variable is determined by $\partial P(t;\Omega)/\partial t=\mathcal{L}_{\Omega}P(t;\Omega)$ [10,46]. In the dephasing, the equality $[\hat{H},\hat{S}]=0$ holds, and we ignore the unitary time evolution induced by the Hamiltonian $\hat{H}$ since it does not affect our results.

B. Gauss-Markov process

We first consider the case that the phase fluctuation caused by the stochastic environment is characterized by means of the Gauss-Markov process [10,46]. In this case, the stochastic Liouville equation for the joint state $\hat{W}(t;\Omega)$ is given by [44,45]

$$\frac{\partial}{\partial t}\hat{W}(t;\Omega)=-i\Omega\hat{S}^{\times}\hat{W}(t;\Omega)+\gamma\frac{\partial}{\partial\Omega}\left(\Omega+\Delta^2\frac{\partial}{\partial\Omega}\right)\hat{W}(t;\Omega),\quad (2)$$

where we set $\hat{A}^{\times}\hat{B}=[\hat{A},\hat{B}]$. In this equation, the non-negative parameters γ and Δ are related to the correlation function of the stochastic variable $\Omega(t)$ in the stationary state. The Doob's theorem provides $\langle\Omega(t)\Omega(t')\rangle_{s}=\Delta^2e^{-\gamma|t-t'|}$ [10,46], where $\langle\ldots\rangle_{s}$ represents the average over the stationary Gauss-Markov process. We can write the initial joint state at $t=t_0$ in the form of $\hat{W}(t_0;\Omega)=\hat{\rho}(t_0|\Omega)P(t_0;\Omega)$, where $P(t_0;\Omega)$ is an initial probability of the stochastic variable and $\hat{\rho}(t_0|\Omega)$ is a conditional density matrix of the quantum system for a given value Ω of the stochastic variable. When the initial probability of the stochastic variable is Gaussian with the equilibrium width Δ, that is,

$$P(t_0;\Omega)=\frac{1}{\sqrt{2\pi\Delta^2}}\exp\left[-\frac{(\Omega-\Omega_0)^2}{2\Delta^2}\right],\quad (3)$$

the joint state $\hat{W}(t;\Omega)$ of the total system becomes

$$\begin{aligned}\hat{W}(t;\Omega)=&\frac{1}{\sqrt{2\pi\Delta^2}}\exp[-f(t-t_0)(\hat{S}^{\times})^2\\&-i(\Omega_0/\Delta)\theta(t-t_0)\hat{S}^{\times}]\\&\times\exp\bigg(-\frac{1}{2\Delta^2}\{\Omega-[\Omega_0e^{-\gamma(t-t_0)}\\&-i\Delta\theta(t-t_0)\hat{S}^{\times}]\}^2\bigg)\hat{\rho}(t_0),\end{aligned}\quad (4)$$

where we set $f(t)=(\Delta/\gamma)^2(\gamma t-1+e^{-\gamma t})$ and $\theta(t)=(\Delta/\gamma)(1-e^{-\gamma t})$. The reduced density matrix of the quantum system is given by

$$\hat{\rho}(t)=\exp[-f(t-t_0)(\hat{S}^{\times})^2-i(\Omega_0/\Delta)\theta(t-t_0)\hat{S}^{\times}]\hat{\rho}(t_0),\quad (5)$$

and the probability of the stochastic variable is

$$P(t;\Omega)=\frac{1}{\sqrt{2\pi\Delta^2}}\exp\left[-\frac{(\Omega-\Omega_0e^{-\gamma(t-t_0)})^2}{2\Delta^2}\right].\quad (6)$$

The conditional density matrix of the quantum system for a given value Ω is provided by $\hat{\rho}(t|\Omega)=\hat{W}(t;\Omega)/P(t;\Omega)$. The more general case has been considered in Ref. [43].

C. Two-state-jump Markov process

Next we consider the case that the phase fluctuation of the quantum system is characterized by the two-state-jump Markov process [46,47]. In this case, the stochastic variable takes only two values which are denoted as $\pm\frac{1}{2}\Delta$. The joint state of the total system is given by the two-dimensional vector, the components of which are operators of the quantum system, where we denote the system operators as $\hat{W}(t;\frac{1}{2}\Delta) \equiv \hat{W}_+(t)$ and $\hat{W}(t;-\frac{1}{2}\Delta) \equiv \hat{W}_-(t)$. Then the joint state is represented by

$$\hat{W}(t) = \begin{pmatrix} \hat{W}_+(t) \\ \hat{W}_-(t) \end{pmatrix}. \tag{7}$$

The reduced density matrix of the relevant quantum system is $\hat{\rho}(t) = \hat{W}_+(t) + \hat{W}_-(t)$ and the probability that the stochastic variable takes the value $\pm\frac{1}{2}\Delta$ is given by $P_\pm(t) = \mathrm{Tr}\,\hat{W}_\pm(t)$. When we assume that $\langle\Omega(t)\rangle_s = 0$ and $\langle\Omega(t)\Omega(t')\rangle_s = (\Delta/2)^2 e^{-\gamma|t-t'|}$ in the stationary state, the stochastic Liouville equation for the joint state $\hat{W}(t)$ is provided by [44,45]

$$\frac{\partial}{\partial t}\hat{W}(t) = -i\mathbf{\Omega}\hat{S}^\times\hat{W}(t) + \mathbf{\Gamma}\hat{W}(t), \tag{8}$$

with

$$\mathbf{\Omega} = \begin{pmatrix} \frac{1}{2}\Delta & 0 \\ 0 & -\frac{1}{2}\Delta \end{pmatrix}, \quad \mathbf{\Gamma} = \begin{pmatrix} -\frac{1}{2}\gamma & \frac{1}{2}\gamma \\ \frac{1}{2}\gamma & -\frac{1}{2}\gamma \end{pmatrix}. \tag{9}$$

Using the Lie algebra method [48–51], we can solve the stochastic Liouville equation (8) to obtain the joint state [43],

$$\hat{W}(t) = \begin{pmatrix} \hat{\mathcal{G}}_{++}(t-t_0) & \hat{\mathcal{G}}_{+-}(t-t_0) \\ \hat{\mathcal{G}}_{-+}(t-t_0) & \hat{\mathcal{G}}_{--}(t-t_0) \end{pmatrix}\hat{W}(t_0), \tag{10}$$

where $\hat{\mathcal{G}}_{jk}(t)$'s are given by

$$\hat{\mathcal{G}}_{++}(t) = e^{-(1/2)\gamma t}\left[\cosh\left(\frac{\gamma t}{2\hat{A}}\right) - i\left(\frac{\Delta\hat{S}^\times}{\gamma}\right)\hat{A}\,\sinh\left(\frac{\gamma t}{2\hat{A}}\right)\right], \tag{11}$$

$$\hat{\mathcal{G}}_{--}(t) = e^{-(1/2)\gamma t}\left[\cosh\left(\frac{\gamma t}{2\hat{A}}\right) + i\left(\frac{\Delta\hat{S}^\times}{\gamma}\right)\hat{A}\,\sinh\left(\frac{\gamma t}{2\hat{A}}\right)\right], \tag{12}$$

$$\hat{\mathcal{G}}_{+-}(t) = \hat{\mathcal{G}}_{-+}(t) = e^{-(1/2)\gamma t}\hat{A}\,\sinh\left(\frac{\gamma t}{2\hat{A}}\right), \tag{13}$$

with $\hat{A} = 1/\sqrt{1-(\Delta\hat{S}^\times/\gamma)^2}$. The conditional density matrix of the quantum system for given value $\pm\frac{1}{2}\Delta$ of the stochastic variable is $\hat{\rho}(t|\pm) = \hat{W}_\pm(t)/P_\pm(t)$. The details are given in Ref. [43].

D. Initial state with correlation

Finally we explain a correlated initial state which is used to investigate the decoherence of a quantum system. The initial correlation is created in a preparation process which uses the von Neumann measurement. Such a process is called filtering or preselection. Suppose that to prepare an initial state at time $t = 0$, we first put a quantum system in a stochastic environment at time $t = -t_0$, where the quantum state is given by the density matrix $\hat{\rho}_i = |\psi_i\rangle\langle\psi_i|$ and the environment is in the stationary state before interacting with the quantum system. The joint state of the total system at this time is a product $\hat{\rho}_i P(\Omega)$ with $P(\Omega)$ being the stationary probability. Then the total system consisting of the quantum system and the stochastic environment evolves into the joint state $\hat{W}(t_0;\Omega)$ at $t = 0$, according to the stochastic Liouville equation, which can be written in the form of $\hat{W}(t_0;\Omega) = G_i(t_0;\Omega,\hat{S}^\times)\hat{\rho}_i$ [for instance, see Eq. (4)]. Here $G_i(t_0;\Omega,\hat{S}^\times)$ represents the quantum channel derived by the stochastic Liouville equation. We prepare a state of the quantum system at time $t = 0$ by means of the state reduction caused by the von Neumann measurement or by the filtering operation on the quantum system [31–34]. We denote the projectors of the von Neumann measurement as $\hat{M}_k = |\psi_k\rangle\langle\psi_k|$. When we obtain the kth measurement outcome, the joint state of the total system just after the measurement becomes

$$\hat{W}_k(0,\Omega) = |\psi_k\rangle\langle\psi_k|P(\Omega|k), \tag{14}$$

with the conditional probability

$$P(\Omega|k) = \frac{\langle\psi_k|\hat{W}(t_0,\Omega)|\psi_k\rangle}{\sum_\Omega\langle\psi_k|\hat{W}(t_0,\Omega)|\psi_k\rangle}. \tag{15}$$

The kth outcome is obtained with probability $\mathcal{P}(k) = \sum_\Omega\langle\psi_k|\hat{W}(t_0,\Omega)|\psi_k\rangle$. The joint state (14) is the initial state that we consider. If there is no correlation between the quantum system and the stochastic environment just before the preparation, the probability $P(\Omega|k)$ does not depend on the measurement outcome k and the joint state becomes $\hat{W}_k(0,\Omega) = |\psi_k\rangle\langle\psi_k|P(\Omega)$. Although the initial joint state $\hat{W}_k(0;\Omega)$ is factorized into the qubit part and the environmental part, it is really correlated unless $P(\Omega|k) = P(\Omega)$ [32–34]. After the preparation, the total system evolves, according to the stochastic Liouville equation with the initial joint state (14). The preparation and time evolution of the system is depicted in Fig. 1.

For a single qubit, the von Neumann measurement is described in terms of two orthonormal vectors $|\psi_+\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$ and $|\psi_-\rangle = e^{-i\phi}\sin(\theta/2)|0\rangle - \cos(\theta/2)|1\rangle$ with $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi \leqslant 2\pi$. Here $|0\rangle$ and $|1\rangle$ are eigenstates of the Pauli matrix $\hat{\sigma}_z$ such that $\hat{\sigma}_z|0\rangle = |0\rangle$ and $\hat{\sigma}_z|1\rangle = -|1\rangle$. In the case of the single qubit, the system operator $\hat{S}$ is given by $\hat{S} = \frac{1}{2}\hat{\sigma}_z$. Then we can obtain

$$\begin{aligned}\langle\psi_\pm|\hat{W}_i(t_0,\Omega)|\psi_\pm\rangle &= \tfrac{1}{2}G_i(t_0;\Omega,0)[1 \pm \langle\hat{\sigma}_z\rangle_i\cos\theta] \\ &\pm\tfrac{1}{2}G_i(t_0;\Omega,-1)\langle\hat{\sigma}_+\rangle_i e^{-i\phi}\sin\theta, \\ &\pm\tfrac{1}{2}G_i(t_0;\Omega,1)\langle\hat{\sigma}_-\rangle_i e^{i\phi}\sin\theta,\end{aligned} \tag{16}$$

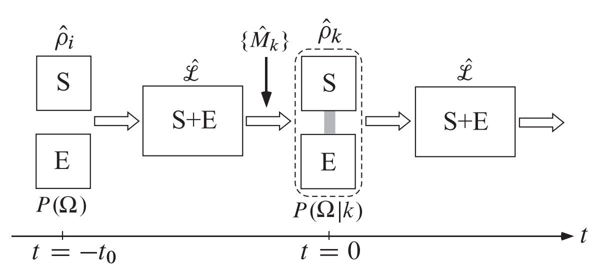

FIG. 1. A schematic representation of the state preparation and the time evolution of the quantum system (S) and the stochastic environment (E), where $\hat{\mathcal{L}}$ stands for the time evolution generated by the stochastic Liouville equation. The von Neumann measurement $\{\hat{M}_k\}$ makes the correlated initial state $\hat{W}_k(0,\Omega) = \hat{\rho}_k P(\Omega|k)$ at time $t = 0$, and after that the system evolves according to the stochastic Liouville equation.

where we have used the relations

$$\mathrm{Tr}[|0\rangle\langle 1|f(\hat{S}_z^\times)\hat{A}] = f(-1)\langle 1|\hat{A}|0\rangle, \tag{17}$$

$$\mathrm{Tr}[|1\rangle\langle 0|f(\hat{S}_z^\times)\hat{A}] = f(1)\langle 1|\hat{A}|0\rangle, \tag{18}$$

$$\mathrm{Tr}[|k\rangle\langle k|f(\hat{S}_z^\times)\hat{A}] = f(0)\langle k|\hat{A}|k\rangle \ \ (k=0,1), \tag{19}$$

and we set $\langle\hat{A}\rangle_i = \mathrm{Tr}(\hat{A}\hat{\rho}_i)$ and $\hat{\sigma}_\pm = (\hat{\sigma}_x \pm i\hat{\sigma}_y)/2$. The correlated initial state of the single qubit and the stochastic environment is provided by Eqs. (14)–(16). The initial correlation is really created unless $\langle\psi_\pm|\hat{W}_i(t_0,\Omega)|\psi_\pm\rangle$ is factorized into the qubit part and the environmental part.

The initial state that we consider in this paper has the characteristic features. First, it is obvious that there is no initial correlation between the qubit and the stochastic environment if $t_0 = 0$ since the qubit does not interact with the stochastic environment before the preparation of the initial state. Furthermore if the time t_0 is greater than the phase relaxation time τ_{ph}, the coherence of the qubit is destroyed by the stochastic environment. In that case, the initial correlation cannot be created by the preparation procedure. In fact, if $t_0 > \tau_{\mathrm{ph}}$, we can approximate as

$$\langle\psi_\pm|\hat{W}_i(t_0,\Omega)|\psi_\pm\rangle \approx \frac{1}{2}G_i(t_0;\Omega,0)[1 \pm \langle\hat{\sigma}_z\rangle_i \cos\theta], \tag{20}$$

$$\sum_\Omega \langle\psi_\pm|\hat{W}_i(t_0,\Omega)|\psi_\pm\rangle \approx \frac{1}{2}[1 \pm \langle\hat{\sigma}_z\rangle_i \cos\theta], \tag{21}$$

where we have used $G(t_0;\Omega,\pm 1) \approx 0$ for $t_0 > \tau_{\mathrm{ph}}$. In this case, from Eq. (15), we derive $P(\Omega|k) \approx G_i(t_0;\Omega,0)$, which does not depend on the result k of the state preparation. In the intermediate time region ($0 < t_0 < \tau_{\mathrm{ph}}$), we can obtain the initial state correlated with the stochastic environment. The formal solution of the stochastic Liouville equation (1) with $\hat{H} = 0$ is written as $\hat{W}(t) = e^{-it\Omega\hat{S}^\times + t\mathcal{L}_\Omega}\hat{\rho}_k P(\Omega|k)$, where we set $\hat{\rho}_k = |\psi_k\rangle\langle\psi_k|$. Then the reduced density matrix of the qubit is given by $\hat{\rho}(t) = \hat{\mathcal{G}}(t|k)\hat{\rho}_k$, where $\hat{\mathcal{G}}(t|k) = \langle e^{-it\Omega\hat{S}^\times + t\mathcal{L}_\Omega}\rangle_k$ and $\langle\ldots\rangle_k = \sum_\Omega \cdots P(\Omega|k)$ means the conditional average of the stochastic variable. The dependence of the time evolution operator $\hat{\mathcal{G}}(t|k)$ on the initial state of the qubit is clear evidence of the initial correlation [14].

III. DECOHERENCE OF QUBIT

In this section, we investigate how the initial correlation between the qubit and the stochastic environment affects the decoherence of the qubit, where the Gauss-Markov process and the two-state-jump Markov process are considered. We will find that depending on the values of the parameters, the initial correlation enhances or suppresses the decoherence and the effect becomes significant in the case of the slow modulation.

A. Gauss-Markov process

For the Gauss-Markov process, the quantum channel $G_i(t_0;\Omega,\hat{S}_z^\times)$ that generates the joint state $\hat{W}(t_0;\Omega)$ just before the preparation of the initial state is obtained by putting $\Omega_0 = 0$ in Eq. (4),

$$G_i(t_0;\Omega,\hat{S}_z^\times) = \frac{1}{\sqrt{2\pi\Delta^2}}\exp\bigg[-f(t_0)(\hat{S}_z^\times)^2 - \frac{1}{2\Delta^2}(\Omega + i\Delta\theta(t_0)\hat{S}_z^\times)^2\bigg]. \tag{22}$$

Substituting this equation into Eq. (16), we find out the initial joint states $\hat{W}_\pm(0;\Omega)$ of the qubit and the stochastic environment just after the preparation with the projector $\hat{M}_\pm = |\psi_\pm\rangle\langle\psi_\pm|$,

$$\hat{W}_\pm(0;\Omega) = |\psi_\pm\rangle\langle\psi_\pm|[Q_\pm(t_0;\Omega) \pm R_\pm(t_0;\Omega) \pm R_\pm^*(t_0;\Omega)], \tag{23}$$

where $Q_\pm(t_0;\Omega)$ and $R_\pm(t_0;\Omega)$ are given, respectively, by

$$Q_\pm(t_0;\Omega) = \frac{P(\Omega)(1 \pm \langle\hat{\sigma}_z\rangle_i\cos\theta)}{1 \pm \langle\hat{\sigma}_z\rangle_i\cos\theta \pm \Phi(t_0)\langle\hat{\sigma}_-\rangle_i e^{i\phi}\sin\theta \pm \Phi(t_0)\langle\hat{\sigma}_+\rangle_i e^{-i\phi}\sin\theta}, \tag{24}$$

$$R_\pm(t_0;\Omega) = \frac{P(t_0;\Omega)\Phi(t_0)\langle\hat{\sigma}_+\rangle_i e^{-i\phi}\sin\theta}{1 \pm \langle\hat{\sigma}_z\rangle_i\cos\theta \pm \Phi(t_0)\langle\hat{\sigma}_-\rangle_i e^{i\phi}\sin\theta \pm \Phi(t_0)\langle\hat{\sigma}_+\rangle_i e^{-i\phi}\sin\theta}. \tag{25}$$

In these equations, we set $P(\Omega) = (2\pi\Delta^2)^{-1/2}\exp(-\Omega^2/2\Delta^2)$ and

$$P(t_0;\Omega) = \frac{1}{\sqrt{2\pi\Delta^2}}\exp\bigg\{-\frac{1}{2\Delta^2}[\Omega - i\Delta\theta(t_0)]^2\bigg\}, \tag{26}$$

with $\Phi(t_0) = \exp[-f(t_0)]$. The correlated initial state $\hat{W}_\pm(0;\Omega)$ is obtained with probability

$$P_i(t_0;\pm) = \tfrac{1}{2}[1 \pm \langle\hat{\sigma}_z\rangle_i\cos\theta \pm \Phi(t_0)\langle\hat{\sigma}_-\rangle_i e^{i\phi}\sin\theta \pm \Phi(t_0)\langle\hat{\sigma}_+\rangle_i e^{-i\phi}\sin\theta]. \tag{27}$$

When we choose appropriate measurement vectors $|\psi_\pm\rangle$ for the qubit, we can obtain the initial qubit state that we would like to prepare with the finite success probability. On the other hand, the probability of the stochastic variable just after the preparation is given by $P_\pm(0;\Omega) = Q_\pm(t_0;\Omega) \pm R_\pm(t_0;\Omega) \pm R_\pm^*(t_0;\Omega)$, which is apparently not Gaussian. This means that the stochastic process deviates from a Gaussian process. Thus the preparation of the initial state changes the property of the phase fluctuation of the qubit.

Since we have obtained the initial joint state $\hat{W}_\pm(0;\Omega)$, we can derive the time evolution by solving the stochastic Liouville equation. It is a straightforward task to find the time evolution of each term of the right-hand side of Eq. (23),

$$|\psi_\pm\rangle\langle\psi_\pm|P(\Omega) \to \frac{1}{\sqrt{2\pi\Delta^2}}e^{-f(t)(\hat{S}_z^\times)^2} \times e^{-[\Omega + i\Delta\theta(t)\hat{S}_z^\times]^2/2\Delta^2}|\psi_\pm\rangle\langle\psi_\pm|, \tag{28}$$

$$|\psi_\pm\rangle\langle\psi_\pm|P(t_0;\Omega) \to \frac{1}{\sqrt{2\pi\Delta^2}}e^{-f(t)(\hat{S}_z^\times)^2+\Theta(t,t_0)\hat{S}_z^\times} \times e^{-[\Omega-i\Delta\theta(t_0)e^{-\gamma t}+i\Delta\theta(t)\hat{S}_z^\times]^2/2\Delta^2}|\psi_\pm\rangle\langle\psi_\pm|, \quad (29)$$

$$|\psi_\pm\rangle\langle\psi_\pm|P^*(t_0;\Omega) \to \frac{1}{\sqrt{2\pi\Delta^2}}e^{-f(t)(\hat{S}_z^\times)^2-\Theta(t,t_0)\hat{S}_z^\times} \times e^{-[\Omega+i\Delta\theta(t_0)e^{-\gamma t}+i\Delta\theta(t)\hat{S}_z^\times]^2/2\Delta^2}|\psi_\pm\rangle\langle\psi_\pm|, \quad (30)$$

where we set $\Theta(t,t_0) = (\Delta/\gamma)^2(1-e^{-\gamma t_0})(1-e^{-\gamma t})$. Then the reduced density matrix of the qubit at time t is obtained:

$$\hat{\rho}_\pm(t) = \frac{1\pm\cos\theta}{2}|0\rangle\langle 0| + \frac{1\mp\cos\theta}{2}|1\rangle\langle 1| \pm \frac{1}{2}\Psi_\pm(t,t_0)e^{-i\phi}\sin\theta|0\rangle\langle 1| \pm \frac{1}{2}\Psi_\pm^*(t,t_0)e^{i\phi}\sin\theta|1\rangle\langle 0|, \quad (31)$$

where the function $\Psi_\pm(t,t_0)$ that characterizes the decoherence of the qubit is given by

$$\Psi_\pm(t,t_0) = \Phi(t)\frac{1\pm\langle\hat{\sigma}_z\rangle_i\cos\theta \pm \Upsilon_-(t,t_0)\langle\hat{\sigma}_-\rangle_i e^{i\phi}\sin\theta \pm \Upsilon_+(t,t_0)\langle\hat{\sigma}_+\rangle_i e^{-i\phi}\sin\theta}{1\pm\langle\hat{\sigma}_z\rangle_i\cos\theta \pm \Phi(t_0)\langle\hat{\sigma}_-\rangle_i e^{i\phi}\sin\theta \pm \Phi(t_0)\langle\hat{\sigma}_+\rangle_i e^{-i\phi}\sin\theta}, \quad (32)$$

with $\Upsilon_\pm(t,t_0) = \Phi(t_0)e^{\pm\Theta(t,t_0)}$. The second factor on the right-hand side of this equation represents the effect of the initial correlation on the decoherence of the qubit. It is obvious from Eq. (32) that the decoherence function $\Psi_\pm(t,t_0)$ depends on the initial state of the qubit. Note that the decoherence function is equivalent to the characteristic function of the stochastic process. If the process is Gaussian, it is given by a simple exponential function since all the cumulants greater than the second order vanish. The decoherence function given by Eq. (32) clearly shows that the process is non-Gaussian due to the effect of the initial correlation. By the way, if we do not refer to the outcome of the measurement for the state preparation, we obtain

$$\hat{\rho}(t) = P_i(t_0;+)\hat{\rho}_+(t) + P_i(t_0;-)\hat{\rho}_-(t) = \frac{1\pm\cos\theta}{2}|0\rangle\langle 0| + \frac{1\mp\cos\theta}{2}|1\rangle\langle 1| \pm \frac{1}{2}\Phi(t)e^{-i\phi}\sin\theta|0\rangle\langle 1| \pm \frac{1}{2}\Phi(t)e^{i\phi}\sin\theta|1\rangle\langle 0|, \quad (33)$$

which does not depend on the measurement and is equal to that obtained in the absence of the initial correlation.

To examine the effect of the initial correlation on the decoherence of the qubit, we suppose that the qubit state $\hat{\rho}_i$ at time $t=-t_0$ is equal to one of the measurement vectors, that is, $\hat{\rho}_i = |\psi_+\rangle\langle\psi_+|$ (or $\hat{\rho}_i = |\psi_-\rangle\langle\psi_-|$). Furthermore we assume that the qubit is prepared in $\hat{\rho}_+(0) = |\psi_+\rangle\langle\psi_+|$ (or $\hat{\rho}_-(0) = |\psi_-\rangle\langle\psi_-|$) by performing the von Neumann measurement. In this case, the decoherence function $\Psi(t,t_0)$ of the qubit becomes

$$\Psi(t,t_0) = \Phi(t)\frac{1+\cos^2\theta+\Phi(t_0)\cosh\Theta(t,t_0)\sin^2\theta}{1+\cos^2\theta+\Phi(t_0)\sin^2\theta}. \quad (34)$$

It is obvious that the inequality $\Psi(t,t_0)\geqslant\Phi(t)$ is always satisfied. Hence we find that the initial correlation between the qubit and the stochastic environment reduces the decoherence of the qubit. This result suggests as follows: When we need the qubit in the state $|\psi_\pm\rangle$, we can obtain the qubit that is more robust against the phase fluctuation if we first put the qubit of the state $|\psi_\pm\rangle$ in the environment at $t=-t_0$ and then we perform the von Neumann measurement (the filtering operation or the preselection) at $t=0$ to generate the qubit state $|\psi_\pm\rangle$, where the success probability of the generation is given by

$$P_i(t_0;\pm) = \tfrac{1}{2}[1+\cos^2\theta+\Phi(t_0)\sin^2\theta]. \quad (35)$$

On the other hand, if we obtain the initial qubit state $\hat{\rho}_-(0) = |\psi_-\rangle\langle\psi_-|$ (or $\hat{\rho}_+(0) = |\psi_+\rangle\langle\psi_+|$) with the probability $P_i(t_0;\pm) = \frac{1}{2}[1-\Phi(t_0)]\sin^2\theta$, the decoherence function becomes

$$\Psi(t,t_0) = \Phi(t)\frac{1-\Phi(t_0)\cosh\Theta(t,t_0)}{1-\Phi(t_0)}, \quad (36)$$

which is independent of the parameter θ. If $\Phi(t_0)[1+\cosh\Theta(t,t_0)] > 2$, the inequality $|\Psi(t,t_0)| > \Phi(t)$ is fulfilled. Otherwise, we obtain $|\Psi(t,t_0)| \leqslant \Phi(t)$. Hence in this case, depending on the values of the parameters, the decoherence is suppressed or enhanced by the initial correlation.

Next we suppose that the qubit state at $t=-t_0$ is $|\tilde{\psi}_i\rangle = \cos(\theta/2)|0\rangle \pm ie^{i\phi}\sin(\theta/2)|1\rangle$. The value of the relative phase ϕ is different from that of the measurement vector by $\pm\pi/2$. In this case, we find the decoherence function from Eq. (32),

$$\Psi(t,t_0) = \Phi(t)\left[1\pm i\frac{\sin^2\theta}{1+\cos^2\theta}\Phi(t_0)\sinh\Theta(t,t_0)\right], \quad (37)$$

for the initial state $\hat{\rho}(0) = |\psi_+\rangle\langle\psi_+|$, which is obtained with the probability $P_i(t_0;+) = \frac{1}{2}(1+\cos^2\theta)$, and

$$\Psi(t,t_0) = \Phi(t)[1\mp i\Phi(t_0)\sinh\Theta(t,t_0)], \quad (38)$$

for the initial state $\hat{\rho}(0) = |\psi_-\rangle\langle\psi_-|$ with probability $P_i(t_0;-) = \frac{1}{2}\sin^2\theta$, both of which always satisfy the inequality $|\Psi(t,t_0)| \geqslant \Phi(t_0)$. Hence the initial correlation reduces the decoherence of the qubit. The time dependence of the decoherence function is depicted in Fig. 2. It is found from the figure that the initial correlation can enhance or reduce the decoherence of the qubit, depending on the values of the parameters. In the case of $\Delta/\gamma = 4.0$ and $t_0/T_2 = 6.0$ [Fig. 2(a)], the decoherence function (36) [long

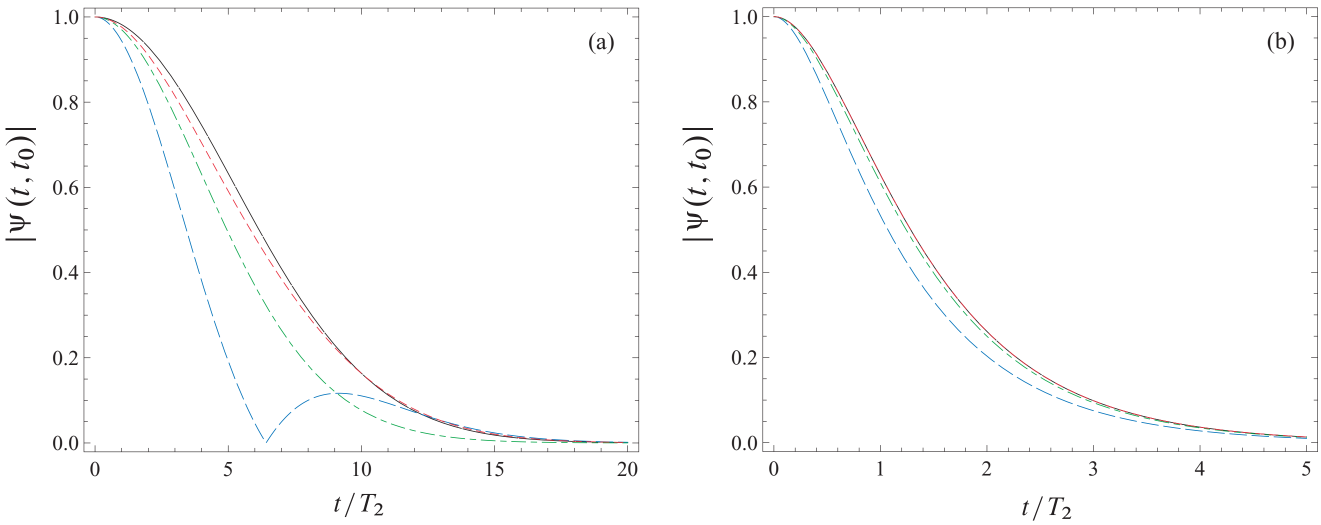

FIG. 2. (Color online) The time dependence of the absolute value of the decoherence function $|\Psi(t,t_0)|$, where the solid line (black) stands for Eq. (34) with $\theta = \pi/2$, the long dashed line (blue) for Eq. (36), the short dashed line (red) for Eq. (37) with $\theta = \pi/2$ [or Eq. (38)], and the dash-dotted line (green) for $\Phi(t) = \exp[-f(t)]$. We set $\Delta/\gamma = 4.0$ and $t_0/T_2 = 6.0$ in panel (a) and $\Delta/\gamma = 0.8$ and $t_0/T_2 = 1.0$ in panel (b). In the figure, time is scaled by the dephasing time $T_2 = \gamma/\Delta^2$, which is derived in the narrowing limit of the Gauss-Markov process.

dashed line (blue)] becomes greater than $\Phi(t)$ in the time region $t/T_2 \gtrsim 9.1$. Thus the initial correlation always suppresses the decoherence of the qubit in the long-time region. On the other hand, when $\Delta/\gamma = 0.8$ and $t_0/T_2 = 1.0$ [Fig. 2(b)] the decoherence function (36) is always smaller than $\Phi(t)$. Furthermore it is obvious from the figure the effect of the initial correlation is significant in the case of the slow modulation ($\Delta/\gamma > 1$). In the case of the fast modulation ($\Delta/\gamma < 1$), the preparation effect is very small since the correlation just before the preparation cannot be large. In the latter case, since the correlation time of the stochastic environment is sufficiently short, the phase information of the qubit is lost before the preparation of the initial state is performed. Finally we summarize the effect of the initial preparation on the decoherence in Table I.

B. Two-state-jump Markov process

We consider the case that the phase fluctuation caused by means of the stochastic environment is described by the two-state-jump Markov process. In this case, the quantum channel

TABLE I. A summary of the effect of the initial preparation on the decoherence of the qubit under the influence of the phase fluctuation of the Gauss-Markov process.

Qubit state at $t = -t_0$	Initial state at $t = 0$	Decoherence function
$\lvert\psi_+\rangle$	$\lvert\psi_+\rangle$	$\Psi(t,t_0) \geqslant \Phi(t)$
	$\lvert\psi_-\rangle$	$\Psi(t,t_0) \geqslant \Phi(t)$ or $\Psi(t,t_0) \leqslant \Phi(t)$
$\lvert\psi_-\rangle$	$\lvert\psi_+\rangle$	$\Psi(t,t_0) \geqslant \Phi(t)$ or $\Psi(t,t_0) \leqslant \Phi(t)$
	$\lvert\psi_-\rangle$	$\Psi(t,t_0) \geqslant \Phi(t)$
$\lvert\tilde{\psi}_i\rangle$	$\lvert\psi_\pm\rangle$	$\Psi(t,t_0) \geqslant \Phi(t)$

$G(t_0;\pm\Delta/2,\hat{S}_z^\times)$ is obtained from Eq. (10):

$$G(t_0;\pm\Delta/2,\hat{S}_z^\times) = e^{-(1/2)\gamma t_0}\left[\cosh\left(\frac{\gamma t_0}{2\hat{A}}\right) + \left(1 \mp i\frac{\Delta\hat{S}_z}{\gamma}\right)\hat{A}\,\sinh\left(\frac{\gamma t_0}{2\hat{A}}\right)\right]. \tag{39}$$

Then we can derive the initial joint state prepared by the von Neumann measurement or the filtering operation on the qubit

$$\hat{W}_\pm(0) = |\psi_\pm\rangle\langle\psi_\pm| \begin{pmatrix} \frac{1+w_\pm(t_0)}{2} \\ \frac{1-w_\pm(t_0)}{2} \end{pmatrix}, \tag{40}$$

where the real parameter $w_\pm(t_0)$ is given by

$$w_\pm(t_0) = \pm i\frac{v(t_0)[\langle\hat{\sigma}_+\rangle_i e^{-i\phi} - \langle\hat{\sigma}_-\rangle_i e^{i\phi}]\sin\theta}{1 \pm \langle\hat{\sigma}_z\rangle_i \cos\theta \pm u(t_0)[\langle\hat{\sigma}_+\rangle_i e^{-i\phi} + \langle\hat{\sigma}_-\rangle_i e^{i\phi}]\sin\theta}. \tag{41}$$

In this equation, the functions $u(t)$ and $v(t)$ are

$$u(t) = e^{-(1/2)\gamma t}\left[\cosh\left(\frac{\gamma t}{2a}\right) + a\,\sinh\left(\frac{\gamma t}{2a}\right)\right], \tag{42}$$

$$v(t) = e^{-(1/2)\gamma t}\left(\frac{\Delta}{\gamma}\right)a\,\sinh\left(\frac{\gamma t}{2a}\right), \tag{43}$$

with $a = 1/\sqrt{1-(\Delta/\gamma)^2}$. The initial joint state $\hat{W}_\pm(0)$ is obtained with probability

$$P_\pm(t_0) = \tfrac{1}{2}[1 \pm \langle\hat{\sigma}_z\rangle_i \cos\theta \pm u(t_0)(\langle\hat{\sigma}_+\rangle_i e^{-i\phi} + \langle\hat{\sigma}_-\rangle_i e^{i\phi})\sin\theta]. \tag{44}$$

Here it should be noted that although the initial preparation of the qubit changes the probability of the stochastic variable from $\pm 1/2$ to $[1 \pm w_\pm(t_0)]/2$, the stochastic process is still a two-state-jump Markov process. Since the fluctuation of the

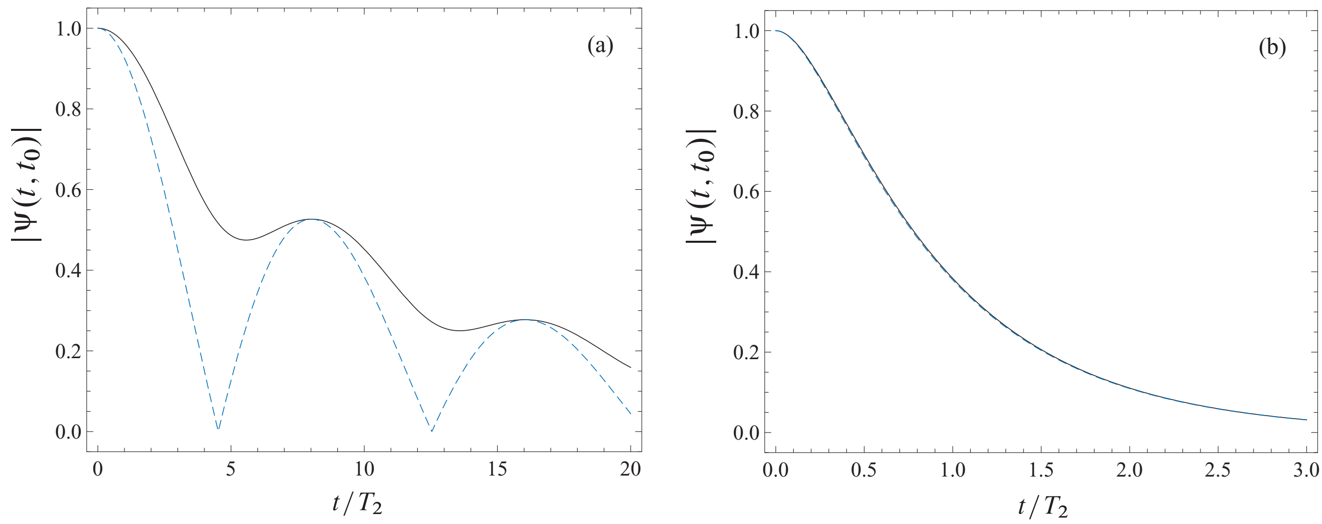

FIG. 3. (Color online) The time dependence of the absolute value of the decoherence function $|\Psi(t,t_0)|$, where the solid line (black) stands for Eq. (46) with $\theta=\pi/2$ [or Eq. (47)] and the long dashed line (blue) for $\Phi(t)=u(t)$. We set $\Delta/\gamma=5.0$ and $t_0/T_2=4.0$ in panel (a) and $\Delta/\gamma=0.8$ and $t_0/T_2=0.6$ in panel (b). In the figure, time is scaled by the phase relaxation time $T_2=4\gamma/\Delta^2$ provided in the narrowing limit of the two-state-jump Markov process.

stochastic variable at the initial time reduces from $(\Delta/2)^2$ to $(\Delta/2)^2[1-w_\pm^2(t_0)]$ due to the initial preparation, the decoherence of the qubit may be suppressed by the initial correlation. The reduced density matrix $\hat{\rho}(t)$ of the qubit at time t is derived by solving the stochastic Liouville equation (8) with the initial condition (40). The result is provided by replacing the decoherence function $\Psi_\pm(t,t_0)$ in Eq. (31) with

$$\Psi_\pm(t,t_0) = u(t) \pm \frac{v(t_0)v(t)[\langle\hat{\sigma}_+\rangle_i e^{-i\phi} - \langle\hat{\sigma}_-\rangle_i e^{i\phi}]\sin\theta}{1 \pm \langle\hat{\sigma}_z\rangle_i \cos\theta \pm u(t_0)[\langle\hat{\sigma}_+\rangle_i e^{-i\phi} + \langle\hat{\sigma}_-\rangle_i e^{i\phi}]\sin\theta}. \quad (45)$$

The second term on the right-hand side of this equation represents the effect of the initial correlation. As in the case of the Gauss-Markov process, the decoherence function depends on the initial qubit state. We can see from Eqs. (42) and (43) that the effect of the initial correlation disappears at time $t=(2\pi n/\gamma)\sqrt{(\Delta/\gamma)^2-1}$ $(n=1,2,\ldots)$ when the inequality $\Delta>\gamma$ is satisfied. Furthermore, if we do not refer to the outcome of the preparation measurement, the initial correlation disappears and the decoherence function becomes $\Psi_\pm(t,t_0)=u(t)$.

When the qubit state at $t=-t_0$ is $\hat{\rho}_i=|\psi_\pm\rangle\langle\psi_\pm|$, we find out $w_\pm(t_0)=0$ from Eq. (41). This means that the preparation of the initial qubit state at $t=0$ does not make any correlation between the qubit and the stochastic environment. Hence the decoherence function becomes $\Psi(t,t_0)=u(t)$. On the other hand, when $\hat{\rho}_i=|\tilde{\psi}_\pm\rangle\langle\tilde{\psi}_\pm|$ with $|\tilde{\psi}_\pm\rangle=\cos(\theta/2)|0\rangle \pm i e^{i\phi}\sin(\theta/2)|1\rangle$, we obtain the decoherence function

$$\Psi(t,t_0) = u(t) \pm i\,\frac{v(t_0)v(t)\sin^2\theta}{1+\cos^2\theta} \quad (46)$$

for the initial state $\hat{\rho}(0)=|\tilde{\psi}_+\rangle\langle\tilde{\psi}_+|$ with probability $P_i(t_0;+)=\frac{1}{2}(1+\cos^2\theta)$ and

$$\Psi(t,t_0) = u(t) \mp i\,v(t_0)v(t) \quad (47)$$

for the initial state $\hat{\rho}(0)=|\tilde{\psi}_-\rangle\langle\tilde{\psi}_-|$ with probability $P_i(t_0;-)=\frac{1}{2}\sin^2\theta$. It is obvious that the decoherence functions always satisfy the inequality $|\Psi_\pm(t,t_0)|\geqslant|u(t)|$. Hence the initial correlation reduces the decoherence of the qubit. This result means that when we need the qubit state $|\psi_\pm\rangle$, we first put the qubit of the state $|\tilde{\psi}_\pm\rangle$ in the environment and then we perform the von Neumann measurement with the projector $|\psi_\pm\rangle\langle\psi_\pm|$. The time dependence of the decoherence function is depicted in Fig. 3. The figure shows that the initial correlation significantly reduces the decoherence of the qubit in the case of the slow modulation. As in the Gauss-Markov process, the effect of the initial-state preparation becomes negligible in the case of the fast modulation. The effect of the initial preparation on the decay of coherence is summarized in Table II.

TABLE II. A summary of the effect of the initial preparation on the decoherence of the qubit the influence of the phase fluctuation of the two-state-jump Markov process.

Qubit state at $t=-t_0$	Initial state at $t=0$	Decoherence function
$\|\psi_\pm\rangle$	$\|\psi_\pm\rangle$	$\Psi(t,t_0)=\Phi(t)$
$\|\tilde{\psi}_\pm\rangle$	$\|\psi_\pm\rangle$	$\Psi(t,t_0)\geqslant\Phi(t)$

IV. CONCLUDING REMARKS

In this paper, we have studied how the initial correlation between the qubit and the stochastic environment influences the decoherence of the qubit during the reduced time evolution, where the Gauss-Markov process and the two-state-jump Markov process have been used to describe the phase fluctuation of the qubit. The time evolution of the qubit and the environment is determined by the stochastic Liouville equation. We have supposed that the initial correlation is created by the von Neumann measurement or by the filtering operation on the qubit, which is performed to prepare the initial qubit state. We have found that depending on the value of the parameters, the decoherence of the qubit is suppressed or enhanced by the initial correlation. In the case of the slow modulation, the effect of the initial correlation becomes significant and thus it cannot be ignored in the reduced time evolution. On the other hand, in the case of the fast modulation (or the narrowing limit), the preparation of the initial state slightly affects the reduced time evolution of the qubit since the phase information of the qubit is lost before the initial correlation is created. Although we have considered the specific type of the initial correlation in this paper, it may be sufficient to recognize the importance of the effect of the initial correlation between the quantum system and the environment.

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