Structure of generalized Heisenberg algebras and quantum decoherence analysis

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We study how catlike superpositions of generalized Heisenberg algebras (GHA) nonlinear coherent states behave under dissipative decoherence. Two cases are presented: the infinite square well potential and systems whose spectra, given by infinite strictly increasing sequences of nonnegative real numbers, can be considered perturbations of the harmonic oscillator. The decoherence effect caused by the interaction of these systems with a thermal bath is analyzed: from their fidelity behavior we see that a region always exists in the parameter space where the quantum coherence is better preserved as compared to the harmonic oscillator. Moreover, we show that the qualitative behavior of GHA systems under the studied mechanism of decoherence can be inferred from the algebraic structure via the analysis of their GHA characteristic functions.

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I. INTRODUCTION

Algebraic methods are important for understanding a large number of quantum physical systems [1–3]. The idea of using creation and annihilation operators, forming an algebra with the number operator, was applied for the first time to the harmonic oscillator [4]. Since then, many physical systems have been described using this algebraic method. Later, it was found that this algebra, called Heisenberg algebra, could be generalized, leading to the concept of deformed Heisenberg algebras [5,6] that have been used in many areas, such as nuclear physics [7–9], condensed matter [10], quantum field theory [11,12], and vibrational analysis [13].

A few years ago, a family of Heisenberg-type algebras that depend on a characteristic function of the Hamiltonian H, $f(H)$, called generalized Heisenberg algebra (GHA), was constructed [14,15]. The generators of the GHA are H and the ladder operators, A and $A^\dagger$, corresponding to a given physical system. The GHA describes general physical systems with one quantum number and contains [16,17] as particular cases many interesting physical systems, such as the harmonic oscillator, q-oscillator algebras, the infinite square well potential, and the Poschl-Teller potential. It is worthwhile to note that it also includes systems whose spectra are given by infinite strictly increasing sequences of nonnegative real numbers, which are a kind of small perturbations of the harmonic oscillator.

Coherent states for the harmonic oscillator were introduced mathematically a very long time ago [18–21] with the aim of finding the quantum counterparts of classical points in phase space. Later it was realized that they could be generated in the laboratory by a laser and prepared by photomultipliers [22]. One important feature of the GHA is that nonlinear coherent states satisfying Klauder's conditions [23] can be constructed [24] as eigenstates of its annihilation operator. Recently, it was shown that a single-atom laser is realized by a nonlinear coherent state which is an eigenstate of a deformed annihilation operator [25].

In quantum physics a quantum state can be prepared as a coherent superposition of other quantum states, while in classical physics the states are always a noncoherent mixture which does not present interference features. The passage from quantum to classical is therefore a process where coherence is lost and the mechanism by which the classical frame appears is the quantum decoherence [26]. Decoherence inevitably occurs when a quantum system interacts with the environment in a thermodynamically irreversible way. However, quantum superpositions of states are more or less resistant to decoherence for different quantum systems.

Several papers have studied the susceptibility of harmonic-oscillator coherent states to the loss of quantum coherence when in interaction with a "heat bath," both theoretically [27–29] and experimentally [30,31]. In particular, a series of works has studied experimentally the decoherence in mesoscopic systems created as a superposition of coherent harmonic oscillator states ("catlike states") of a trapped ion (see [30,31] and references therein).

We have at our disposal systems with an underlying GHA structure [24], whose nonlinear coherent states [25] can be superposed as catlike states. Our aim in this paper is to study how these quantum systems resist the decoherence effect caused by a dissipative interaction with an environment, as they evolve in time, as compared to the harmonic oscillator and among themselves. The answer to this question can be undoubtedly obtained through the behavior of the fidelity associated with the systems studied. The fidelity gives us the exact quantitative behavior, telling us how long the system will keep some amount of coherence. But here we show that the algebraic structure itself gives us the relevant qualitative information on this issue, through the characteristic function associated with each GHA. This is discussed in more detail in the conclusion.

The systems chosen to be studied here are (1) a particle in an infinite square well and (2) slight perturbations of the harmonic oscillator. In the second case, the reasons to consider perturbations of the harmonic oscillator are sustained by two main arguments: (i) the spectra measured in the laboratory are rather perturbations of it than of the harmonic oscillator itself, and (ii) in analyzing the possible perturbations we can learn which of them are able to improve the fidelity. These theoretical perturbations could, in principle, be realized in the laboratory. The slightly perturbed harmonic oscillators we

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analyze are particular cases of an interesting four-parameter deformed harmonic oscillator, here introduced by us.

This paper is organized as follows: in Sec. II we give a review of GHA and construct the nonlinear coherent states; in Sec. III we define nonlinear catlike states and present two examples of GHA spectra, a free particle in an infinite square well and a perturbation of the harmonic oscillator given by an infinite strictly increasing sequence of nonnegative real numbers; in Sec. IV using the Lindblad master equation we describe a process of dissipative interaction of a GHA quantum system; and, finally, in Sec. V we show the behavior of both the fidelity and the GHA characteristic function for the chosen systems and the harmonic oscillator, compare them, and present our final conclusions.

II. GENERALIZED HEISENBERG ALGEBRA AND COHERENT STATES

Let us begin by reviewing the GHA as it is presented in [15]. Our GHA is described by the generators $\mathcal{H}, A, A^\dagger$ satisfying

$$\mathcal{H}A^\dagger = A^\dagger f(\mathcal{H}), \tag{1}$$

$$A\mathcal{H} = f(\mathcal{H})A, \tag{2}$$

$$[A^\dagger, A] = \mathcal{H} - f(\mathcal{H}), \tag{3}$$

where A and $A^\dagger$, $A = (A^\dagger)^\dagger$, are, respectively, the annihilation and creation operators and $\mathcal{H} = \mathcal{H}^\dagger$ the Hamiltonian of the physical system under consideration; $f(\mathcal{H})$ is an analytic function of $\mathcal{H}$, called the characteristic function of the algebra. A large class of type Heisenberg algebras[1] can be obtained just by appropriately choosing the function $f(\mathcal{H})$. The Casimir operator of this generalized algebra has the expression

$$C = A^\dagger A - \mathcal{H} = AA^\dagger - f(\mathcal{H}). \tag{4}$$

Let us also give a summary of the GHA representation theory: the n-dimensional irreducible representations of algebras (1)–(3) are given through the lowest eigenvalue of $\mathcal{H}$ with respect to the vacuum state $|0\rangle$:

$$\mathcal{H}|0\rangle = \epsilon_0|0\rangle. \tag{5}$$

It is clear that for each value of ϵ_0 and for a set of parameters of the algebra (related to the function f) we have a different vacuum, all of them denoted here for simplicity by $|0\rangle$. The solution of the representation theory problem is given in [15] for the linear and quadratic polynomials. The n-dimensional representation theory is given through a general vector $|m\rangle$ that is required to be an eigenvector of $\mathcal{H}$,

$$\mathcal{H}|m\rangle = \epsilon_m|m\rangle, \tag{6}$$

where $\epsilon_m = f^{(m)}(\epsilon_0)$, the mth iterate of ϵ_0 under f, and under the action of A and $A^\dagger$ we have

$$A^\dagger|m\rangle = N_m|m+1\rangle, \tag{7}$$

$$A|m\rangle = N_{m-1}|m-1\rangle, \tag{8}$$

where $N_m^2 = \epsilon_{m+1} - \epsilon_0$.

[1]A type Heisenberg algebra is an algebra having annihilation and creation operators among its generators.

In Ref. [15] it was shown that choosing for the characteristic function of the GHA the linear function $f(x) = x + 1$ the algebra in Eqs. (1)–(3) becomes the harmonic-oscillator algebra and for $f(x) = qx + 1$ we obtain in Eqs. (1)–(3) the deformed Heisenberg algebra. Moreover, it was shown in Ref. [32] that there is a class of quantum systems described by these generalized Heisenberg algebras. This class of quantum systems includes both linear and nonlinear systems and is characterized by energy eigenvalues written as

$$\epsilon_{n+1} = f(\epsilon_n), \tag{9}$$

where ϵ_{n+1} and ϵ_n are successive energy levels and $f(x)$ is a different function for each physical system. This function $f(x)$ is exactly the same function that appears in the construction of the algebra in Eqs. (1)–(3), which was called the characteristic function of the algebra. Therefore, the GHA characteristic function provides a strong connection between the algebraic structure and the spectrum of a given system.

Some important and well-known examples are the following [6,15–17,33]:

(1) $f(x) = x + 1$ is the characteristic function for the harmonic oscillator (Heisenberg algebra);

(2) $f(x) = qx + 1$, for the q-deformed algebra, q being the deformation parameter [6];

(3) $f(x) = (\sqrt{x} + \sqrt{1/2})^2$, for a free-particle in an infinite square well [16,33]; and

(4) $f(x) = x + 1 + \delta(x)$, for the Delone sequence $x_n = n + \alpha(n)$, where $\alpha(n)$ is a bounded function satisfying certain conditions [17].

The characteristic function for the different cases can be in general easily obtained [24]. Just to make it clear let us take a simple example, namely the spectrum given by

$$\epsilon_n = \frac{n}{n+1}, \quad n \geqslant 0. \tag{10}$$

In order to obtain a recurrence relation we write

$$\epsilon_{n+1} = \frac{n+1}{n+2} = \frac{1}{2 - \frac{n}{n+1}}, \tag{11}$$

which directly leads to

$$\epsilon_{n+1} = \frac{1}{2 - \epsilon_n}. \tag{12}$$

As

$$\epsilon_{n+1} = f(\epsilon_n), \tag{13}$$

we have the characteristic function $f(x) = \frac{1}{2-x}$ for spectrum (10).

Let us now construct nonlinear coherent states corresponding to some particular form of the characteristic function associated to some GHA [24]. We take a state $|z\rangle$ that we can expand in terms of the eigenvectors $|n\rangle$ according to

$$|z\rangle = \sum_{n=0}^{\infty} c_n|n\rangle, \tag{14}$$

and which is, by hypothesis, an eigenstate of the annihilation operator (8),

$$A|z\rangle = z|z\rangle, \tag{15}$$

where z is a complex number. Acting with A on $|z\rangle$ and equating the right-hand side of Eqs. (14) and (15), we obtain the coefficients $c_n = (c_0 z^n)/N_{n-1}!$, where $N_n! = N_0 N_1 N_2 \cdots N_n$ and $N_{-1}! = 1$. Denoting $c_0 = \mathcal{N}(z)$, the state (14) is then

$$|z\rangle = \mathcal{N}(z) \sum_{n=0}^{\infty} \frac{z^n}{N_{n-1}!} |n\rangle. \tag{16}$$

The states above satisfy the conditions of normalizability, continuity in the label, and completeness (see, for example, [21]) for a number of quantum systems which are described by the GHA [32], that is, systems having energy eigenvalues written as in Eq. (9). Therefore, they are nonlinear coherent states for the systems in question. From normalizability, that is, $\langle z|z\rangle = 1$, we find the normalization factor $\mathcal{N}(z)$ to be

$$\mathcal{N}^2(|z|) = \left[\sum_{n=0}^{\infty} \frac{|z|^{2n}}{N_{n-1}^2!} \right]^{-1}. \tag{17}$$

These three above-mentioned conditions are satisfied, in particular, for the systems that we examine here: a free particle in an infinite square well [16] and an example of small perturbation of the harmonic oscillator.

III. GENERALIZED CATLIKE STATES

We now consider normalized superpositions of nonlinear coherent states (16), known as catlike states [34], or even and odd nonlinear coherent states [35,36], namely,

$$|\psi_\pm\rangle = \mathcal{N}_\pm(|z\rangle \pm |-z\rangle), \tag{18}$$

where $|\psi_+\rangle$ and $|\psi_-\rangle$ are mutually orthogonal, that is, $\langle \psi_\pm|\psi_\mp\rangle = 0$. From $\langle \psi_+|\psi_+\rangle = \langle \psi_-|\psi_-\rangle = 1$, we have

$$\mathcal{N}_\pm^2 = \left[2 \pm 2\mathcal{N}^2 \sum_{n=0}^{\infty} \frac{(-1)^n |z|^{2n}}{N_{n-1}^2!} \right]^{-1}, \tag{19}$$

where the nonlinear coherent state normalization factor $\mathcal{N}$ is given in Eq. (17). Therefore, Eq. (19) can also be written as

$$\mathcal{N}_\pm^2 = \left[2 \pm 2 \frac{\sum_{n=0}^{\infty} \frac{(-1)^n |z|^{2n}}{N_{n-1}^2!}}{\sum_{m=0}^{\infty} \frac{|z|^{2m}}{N_{m-1}^2!}} \right]^{-1}. \tag{20}$$

Here, we study two quantum cases described by a GHA. The first system is a free particle in an infinite square well, already known from the literature [16,33], for which we construct generalized catlike states. The second is the case of slightly perturbed harmonic oscillators, which we now present and show to be a GHA quantum system.

A. A free particle in an infinite square well

Let us consider the free particle in the square well potential $V(x) = 0$ for $0 < x < L$ and $V(x) = \infty$ elsewhere. The energy spectrum is then given by $\epsilon_n = b(n+1)^2$, where $b = \hbar^2\pi^2/2mL^2$, $n = 0,1,2,3,\ldots$. We can easily see that $\epsilon_{n+1} = b(n+2)^2 = \epsilon_n + 2\sqrt{b\,\epsilon_n} + \sqrt{b}$; therefore, from Eq. (9) we have that the characteristic function for this physical system is $f(x) = (\sqrt{x} + \sqrt{b})^2$ and $N_{n-1}^2 = b\,n(n+2)$. Here, we take $b = 1/2$ in order to have the same energy value at the fundamental state as the harmonic oscillator, for which we take $\hbar\omega = 1$. For that system the coherent states (16) are

$$|z\rangle = \left(\frac{|z|^2}{I_2(2\sqrt{2}|z|)} \right)^{1/2} \sum_{n=0}^{\infty} \frac{(\sqrt{2}\,z)^n}{\sqrt{n!(n+2)!}} |n\rangle, \tag{21}$$

as the respective coherent state norm is

$$\mathcal{N}^2(|z|) = \left[\sum_{n=0}^{\infty} \frac{(2|z|^2)^n}{n!(n+2)!} \right]^{-1} = \frac{|z|^2}{I_2(2\sqrt{2}|z|)}. \tag{22}$$

$I_n(x)$ above is the modified Bessel function of the first kind of order n.

The catlike state normalization factor (20) is

$$\mathcal{N}_\pm^2 = \left[2 \pm 2 \frac{J_2(2\sqrt{2}|z|)}{I_2(2\sqrt{2}|z|)} \right]^{-1} \tag{23}$$

where $J_m(x)$ is the Bessel function of the first kind of order n. The free particle in an infinite square well catlike state is then

$$|\psi_\pm\rangle = \left[2 \pm 2 \frac{J_2(2\sqrt{2}|z|)}{I_2(2\sqrt{2}|z|)} \right]^{-1/2} (|z\rangle \pm |-z\rangle), \tag{24}$$

with the coherent states $|z\rangle$ given by Eq. (21).

B. A perturbed spectrum of the harmonic oscillator

Let us consider a perturbed energy spectrum of the harmonic oscillator,[2] given by

$$\epsilon_n = \hbar\bar{\omega} x_n, \tag{25}$$

where

$$x_n = n + \alpha(n), \tag{26}$$

and $\mathbb{N} \ni n \mapsto \alpha(n)$ is a bounded function with values in the interval $(-1,1)$, $\alpha(0) \neq 0$, and its successive jumps $\alpha(n+1) - \alpha(n)$ have lower bound $r - 1$ with $r \in (0,1)$. A GHA is associated to this sequence, which is invertible according to

$$n = x_n + \gamma(x_n). \tag{27}$$

It is easy to see that $\gamma(x_n)$, defined by Eq. (27), shares the same properties as $\alpha(n)$.

In order to find the GHA for these sequences, we write the $(n+1)$th term of the sequence in terms of x_n:

$$x_{n+1} = x_n + 1 + \delta(x_n), \tag{28}$$

where

$$\delta(x_n) = \gamma(x_n) + \alpha(x_n + 1 + \gamma(x_n)). \tag{29}$$

The characteristic function is trivially obtained from Eq. (29):

$$f(x) = x + 1 + \delta(x), \tag{30}$$

and the corresponding GHA is then

$$[\mathcal{H}, A^\dagger] = A^\dagger(1 + \delta(\mathcal{H})) \tag{31}$$

$$[A, A^\dagger] = 1 + \delta(\mathcal{H}). \tag{32}$$

[2]For these perturbations, $x_0 \neq 0$; this is their only difference from a Delone sequence, where x_0 is zero, by definition.

We see from Eq. (29) that δ is the difference of two perturbation terms and is hence itself a perturbation. Equations (31) and (32) show that the invertible sequences associated with GHA are perturbations of the quantum harmonic oscillator.

One- and two-parameter deformed oscillators have been studied in the literature (see, for example, [37] and references therein). Here we introduce a four-parameter oscillator, choosing for the invertible sequence (26)

$$\alpha(n) = \frac{an+e}{cn+d}, \quad n \geqslant 0, \tag{33}$$

where the real constants a, c, d, and e, all of them different from zero, have to satisfy the following conditions:

(a) $|\frac{a}{c}| < 1$,
(b) $-\frac{4ad-4ce}{c^2} \geqslant r-1$,
(c) $\frac{d}{c} > 0$.

Condition (a) is necessary to guarantee that Eq. (33) is a small perturbation even when $n \to \infty$; conditions (b) and (c) guarantee that the sequence $x_n = n + \frac{an+e}{cn+d}$ is strictly increasing and that Eq. (33) is real, for any n. We also choose $\frac{e}{d} > 0$ in order to have the zero-point energy positive.

The conditions above restrict the range of possible values for the constants a, c, d, and e, but in spite of them, the deformed oscillator

$$\epsilon_n = \hbar\bar{\omega}x_n = \hbar\bar{\omega}\left(n + \frac{an+e}{cn+d}\right) \tag{34}$$

still has four parameters, which gives us many possibilities of finding cases resistant to decoherence as the system evolves in time. Here we choose a to be a small real parameter and $c = 1$. In this example,

$$\epsilon_n = \hbar\bar{\omega}\left(n + \frac{an+e}{n+d}\right), \tag{35}$$

where the harmonic oscillator limit is recovered by $a = 1/2$ and $e = d/2$. Also, the energy spectrum (25) with x_n given by Eq. (35) has the harmonic oscillator as the $n \to \infty$ asymptotic limit when $d = 2e$.

The characteristic function in this case is

$$f(x) = x + 1 + \gamma(x) + \frac{a[x+1+\gamma(x)]+e}{x+1+\gamma(x)+d}, \tag{36}$$

with

$$\gamma(x_n) = \frac{\sqrt{x_n^2 + 2(d-a)x_n + (a+d)^2 - 4e} - x_n - a - d}{2}. \tag{37}$$

Then, the coherent states (16) for the particular system given by Eqs. (35) is

$$|z\rangle = \mathcal{N}(z)\sum_{n=0}^{\infty}\frac{z^n}{N_{n-1}!}|n\rangle, \tag{38}$$

where

$$N_{n-1}^2 = \hbar\bar{\omega}\left(\frac{n^2+(a+d)n+e}{n+d} - \frac{e}{d}\right), \tag{39}$$

and from $\langle z|z\rangle = 1$, we have

$$\mathcal{N}^2(|z|) = \left[\sum_{n=0}^{\infty}\frac{(n+d)!}{(n+d+a-e/d)!}\frac{(|z|^2/\hbar\bar{\omega})^n}{n!}\right]^{-1}. \tag{40}$$

From the definition of the confluent hypergeometric function,

$${}_1F_1(a,b;t) = \sum_{n=0}^{\infty}\frac{a(a+1)\cdots(a+n-1)t^n}{b(b+1)\cdots(b+n-1)n!}. \tag{41}$$

We can rewrite Eq. (40) as

$$\mathcal{N}^2(|z|) = [{}_1F_1(d+1,d+a-e/d+1;|z|^2/\hbar\bar{\omega})]^{-1}. \tag{42}$$

The catlike state normalization factor (19) for the spectrum given by Eq. (35), where we have taken $\hbar\bar{\omega} = d/2e$ in order to have the same energy value at the fundamental state as the harmonic oscillator, is

$$\mathcal{N}_\pm^2 = \left[2 \pm 2\mathcal{N}^2\sum_{n=0}^{\infty}\frac{(n+d)!}{(n+d+a-e/d)!}\frac{(-2e/d\,|z|^2)^n}{n!}\right]^{-1}. \tag{43}$$

Noting that the sum in Eq. (43) is the same as the one in Eq. (40), replacing $|z|^2$ by $-|z|^2$, and using Eq. (42), we see that

$$\mathcal{N}_\pm^2 = \left[2 \pm 2\frac{{}_1F_1(d+1,a+d-e/d+1;-2e/d\,|z|^2)}{{}_1F_1(d+1,d+a-e/d+1;2e/d\,|z|^2)}\right]^{-1}. \tag{44}$$

Finally, the catlike state (18) for the system described by the sequence given by Eq. (35) is

$$|\psi_\pm\rangle = \left[2 \pm 2\frac{{}_1F_1(d+1,a+d-e/d+1;-2e/d\,|z|^2)}{{}_1F_1(d+1,d+a-e/d+1;2e/d\,|z|^2)}\right]^{-1/2} \times (|z\rangle \pm |-z\rangle), \tag{45}$$

where $|z\rangle$ is the coherent state given by Eq. (38).

IV. DISSIPATIVE DECOHERENCE OF NONLINEAR CATLIKE STATES

We are interested in studying the effect on quantum systems caused by a dissipative interaction with an environment, which is a "heat bath" described by an assembly of harmonic oscillators. This process can be described in the interaction picture by the master equation in the Lindblad form [26]:

$$\dot{\rho}(t) = -[\rho,H_S] + \gamma a\rho(t)a^\dagger - \frac{\gamma}{2}\{a^\dagger a,\rho(t)\}, \tag{46}$$

where $\rho(t)$ is the reduced density matrix for the quantum system in question at time t, γ is a parameter describing the damping rate, and a and $a^\dagger$ are respectively the annihilation and creation operators of the harmonic oscillator, which satisfy the Heisenberg algebra. H_S is the Hamiltonian of the quantum system. We are assuming that at zero temperature the density operator commutes with H_S and, therefore,

$$\dot{\rho}(t) = \gamma a\rho(t)a^\dagger - \frac{\gamma}{2}\{a^\dagger a,\rho(t)\}. \tag{47}$$

The quantum system starts at $t = 0$ as a coherent superposition of the type from Eqs. (18)–(20); that is,

$$\rho(0) = |\psi_\pm\rangle\langle\psi_\pm|. \tag{48}$$

The time evolution of $\rho(t)$ is given by

$$\rho(t) = \sum_{j=0}^{\infty} \mathcal{S}_j(t)\rho(0)\mathcal{S}_j(t). \tag{49}$$

The $S_j(t)$ are evolution operators that can be written as

$$S_j(t) = \sum_{n,m=j}^{\infty} s_{n,m}(t)|m\rangle\langle n|. \tag{50}$$

Therefore, solving Eq. (47) for $s_{n,m}(t)$ gives us [38]

$$s_{n,m}(t) = \sqrt{\frac{n!}{m!(n-m)!}}e^{-m\gamma t/2}[1-e^{-\gamma t}]^{(n-m)/2} \tag{51}$$

and Eq. (49) is a solution of Eq. (47) with

$$\mathcal{S}_j(t) = \sum_{n=j}^{\infty}\sqrt{\frac{n!}{(n-,j)!j!}}[e^{-(n-j)\gamma t/2}[1-e^{-\gamma t}]^{j/2}|n-j\rangle\langle n|]. \tag{52}$$

In order to know to what extent the evolution in time of the catlike states preserves coherence, we appeal to the concept of fidelity $\mathcal{F}$; by definition,

$$\mathcal{F}(t) = \mathrm{Tr}\{\rho(t)\rho(0)\}, \tag{53}$$

and it says how much the evolved state, described in t by $\rho(t)$, is faithful to the initial one.

For a coherent superposition as in Eqs. (18)–(20) constructed with the nonlinear coherent state (16) the fidelity will be

$$\mathcal{F}_{\pm}(t) = \mathcal{N}^4\mathcal{N}_{\pm}^4\sum_{j=0}^{\infty}\sum_{n,m=j}^{\infty}\sqrt{\frac{n!m!}{(j!)^2(n-j)!(m-j)!}}e^{-\gamma t[(n+m)/2-j]}(1-e^{-\gamma t})^j \times \frac{[z^n\pm(-z)^n]}{N_{n-1}!}\frac{[z^m\pm(-z)^m]}{N_{m-1}!}\frac{[z^{n-j}\pm(-z)^{n-j}]}{N_{n-j-1}!}\frac{[z^{m-j}\pm(-z)^{m-j}]}{N_{m-j-1}!}. \tag{54}$$

Then, given a certain quantum system, once we know its spectrum and the norms $\mathcal{N}$ and $\mathcal{N}_{\pm}$, we can calculate the fidelity.

In the next section we calculate the fidelity of the quantum systems for which we have constructed nonlinear catlike states in Sec. III, that is, a free particle in an infinite square well and a perturbed spectrum of the harmonic oscillator. We analyze the robustness of those catlike states and of the usual nondeformed catlike state [34] when interacting with a dissipative environment. Our aim is to compare the behavior of those systems concerning their loss of coherence in the physical situation described by Eq. (47).

V. CATLIKE STATES EVOLUTION UNDER DISSIPATION: FIDELITY AND GHA CHARACTERISTIC FUNCTION

The quantum harmonic oscillator is a special case of GHA for which the characteristic function is $f(x) = x + 1$, and so its coherent state is

$$|z\rangle = \left(e^{z^2}\right)^{-1/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle. \tag{55}$$

The normalization factor of the catlike state constructed with Eq. (55) is

$$\mathcal{N}(z)_{\pm} = \left(2\pm 2e^{-2z^2}\right)^{-1/2}, \tag{56}$$

and the fidelity reduces to

$$\mathcal{F}_{\pm}(t) = \frac{e^{-2z^2}}{\left(2+2e^{-2z^2}\right)^2}\sum_{j=0}^{\infty}\sum_{n,m=j}^{\infty}\sqrt{\frac{n!m!}{(j!)^2(n-j)!(m-j)!}}e^{-\gamma t[(n+m)/2-j]}(1-e^{-\gamma t})^j \times \frac{[z^n\pm(-z)^n]}{\sqrt{n!}}\frac{[z^m\pm(-z)^m]}{\sqrt{m!}}\frac{[z^{n-j}\pm(-z)^{n-j}]}{\sqrt{(n-j)!}}\frac{[z^{m-j}\pm(-z)^{m-j}]}{\sqrt{(m-j)!}}. \tag{57}$$

Here, we compare the behavior of the harmonic oscillator, the free particle in an infinite square well, and two possible perturbed oscillators by calculating the time evolution of the fidelity for those systems and analyzing their GHA characteristic function. In all cases, we will take 1 as the value of the the damping rate parameter γ in the fidelity expression.

In Fig. 1, the time evolution of the fidelity [Eq. (54)] of the free particle in an infinite square well and the harmonic oscillator is compared for $z = 1.7$. We can see that the Schrödinger catlike states constructed with those nonlinear coherent states are more resistant to decoherence than the harmonic oscillator. The figure shows a case where $z > 1$, but the same holds true also for values of z smaller than 1.

In Fig. 2 we show the time evolution of the fidelity of the perturbed harmonic oscillator with energy spectrum (35), $\epsilon_n = \frac{d}{2e}[n(n+a+d)+e]/(n+d)$, with the parameter values $d = 2e$, $d = 0.2$, and $e = 0.1$, for $z = 1.7$. We compare three cases, $a = 0.5$, $a = 0.9$, and $a = 0.1$. We must remember that the harmonic oscillator is exactly the particular case of Eq. (35) with $\hbar\bar{\omega} = \frac{d}{2e} = 1$ and $a = 0.5$. We see from this

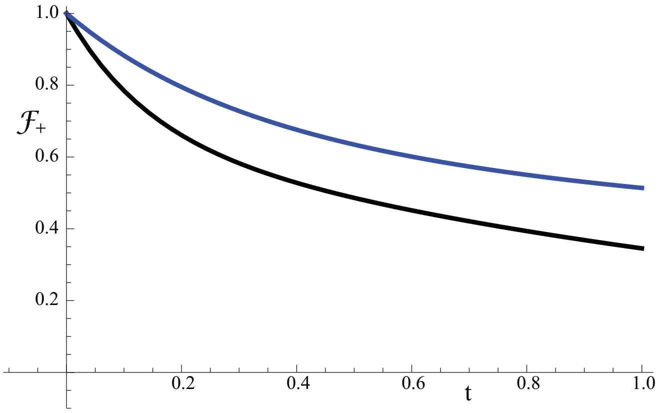

FIG. 1. (Color online) Comparison of the fidelity behavior of the harmonic oscillator (lower curve) and the free particle in an infinite square well (upper curve) for $z = 1.7$, as a function of t, in seconds.

figure that the case where the coherent catlike state is more resistant to decoherence is for the value of the parameter a larger than the harmonic oscillator value $a = 0.5$. This happens because the larger the value of a, the larger the difference between the first energy levels, as can be seen in Fig. 3; therefore, it is more difficult for the thermal bath to change the coherent catlike state configuration for a larger energy difference between levels. For these systems the same results appear when $z < 1$ also. We have seen thus that our freedom in choosing the four parameters in the energy spectrum of Eq. (35) allows us to find examples of perturbed oscillators which are more resistant to decoherence.

In Figs. 4 and 5 we see how the free particle in an infinite square well, the harmonic oscillator, and the perturbed spectra of the harmonic oscillator with $a = 0.9$ and $a = 0.1$ vary with the coherent state parameter z—which measures how closely spaced are the states in the superposition—for $t = 1$. We see that all those systems are more robust under decoherence for smaller values of z, that is, for states more closely spaced in the coherent state superposition. We note also that, as expected, the free particle in an infinite square well generates more

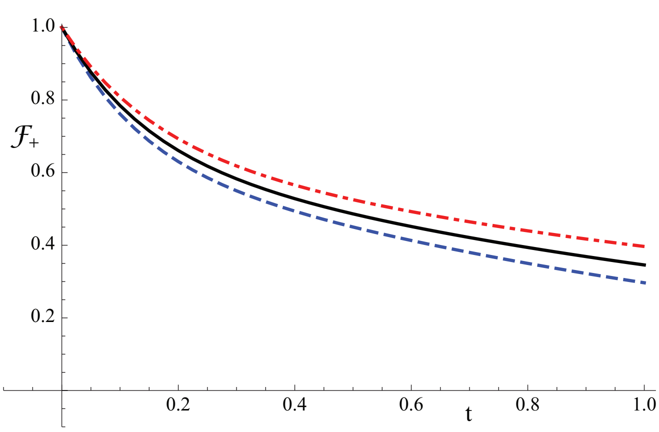

FIG. 2. (Color online) Comparison of the fidelity behavior (as a function of t, in seconds) of the harmonic oscillator (middle curve) with the deformed harmonic oscillator $N_{n-1}^2 = n(n + a + 0.2 - 1/2)/(n + 0.2)$ for $a = 0.9$ (upper curve) and $a = 0.1$ (lower curve), for $z = 1.7$.

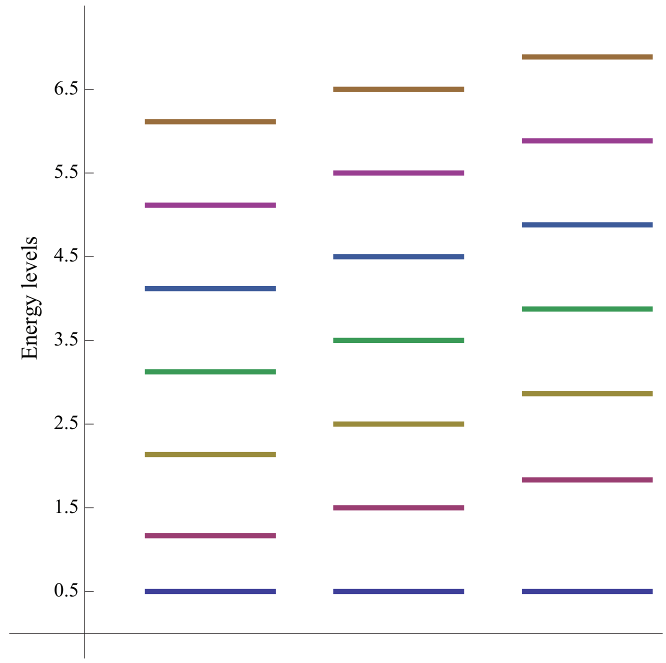

FIG. 3. (Color online) First dimensionless energy levels ($n = 0,1,\ldots,6$) for the harmonic oscillator (center) and perturbed oscillators with $d = 2e = 0.2$; $a = 0.9$ (right) and $a = 0.1$ (left).

robust catlike states for all the values of z considered. The perturbations of the harmonic oscillator in Eq. (35) present the same behavior as for $z = 1.7$, shown in Fig. 2, for all values of z.

As was also found in Refs. [27–29] for other systems, like a q oscillator [6] and an ion-trapped system [28], for the free particle in a square well potential and the perturbed oscillator with $a = 0.9$ the catlike states constructed with their respective coherent states were shown to be more robust under decoherence as time evolves than the harmonic oscillator coherent states.

The use of the fidelity as a means to measure the resistance of quantum states to processes of decoherence is quite well known. But we show here that the analysis of the GHA algebraic structure can give us an indication of the quantum

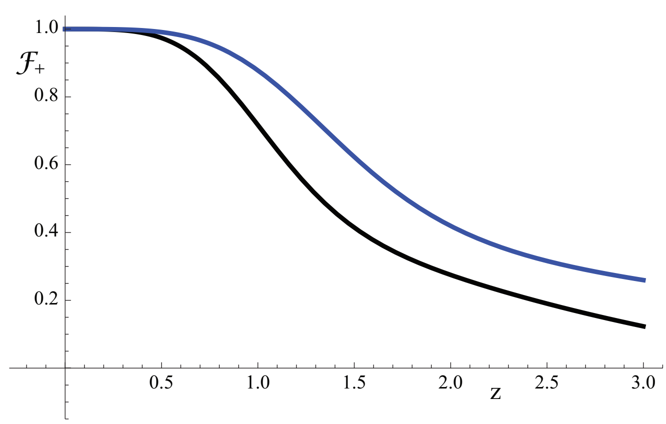

FIG. 4. (Color online) Comparison of the fidelity variation with respect to z of the harmonic oscillator (lower curve) and the free particle in an infinite square well (upper curve), for $t = 1$.

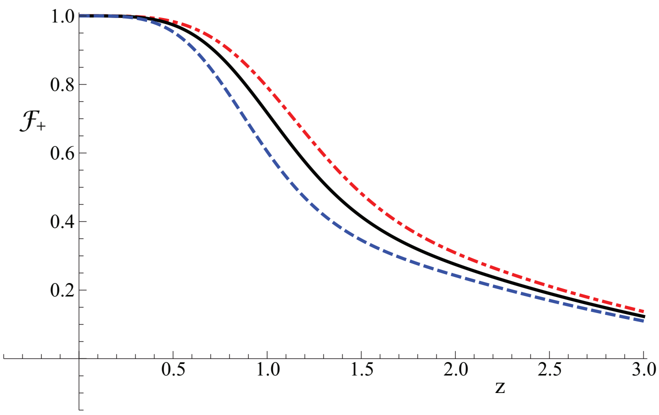

FIG. 5. (Color online) Comparison of the fidelity variation with respect to z of the harmonic oscillator (middle curve) with the perturbed harmonic oscillator with spectrum $N_{n-1}^2 = n(n + a + 0.2 - 1/2)/(n + 0.2)$, for $a = 0.9$ (upper curve) and $a = 0.1$ (lower curve), for $t = 1$.

system's resistance to the process of decoherence presented in Sec. IV. The construction of coherent catlike states for a given system is based on its spectrum, and the robustness of those states under decoherence depends on the distance between the energy levels. As presented in Ref. [15], given an initial state ϵ_0, the n iterations of the characteristic function give the whole spectrum through

$$f^{(n)}(\epsilon_0) = \epsilon_n. \tag{58}$$

Therefore, the larger the value of $f(\epsilon_i)$ for a given ϵ_i, the larger the difference between the two levels ϵ_{i+1} and ϵ_i. So, the higher is the absolute value of the characteristic function, more difficult it is for the thermal bath to cause decoherence.

This can be seen directly by observing the GHA characteristic functions of the systems. That is exactly what is inferred from Fig. 6, where we show the characteristic function, as a function of the energy levels, of the oscillators with respectively $a = 0.9$, $a = 0.5$, and $a = 0.1$, whose fidelities' time evolution is shown in Fig. 2. The absolute value of the characteristic function is higher for the case $a = 0.9$, which is the case where there is more resistance to decoherence.

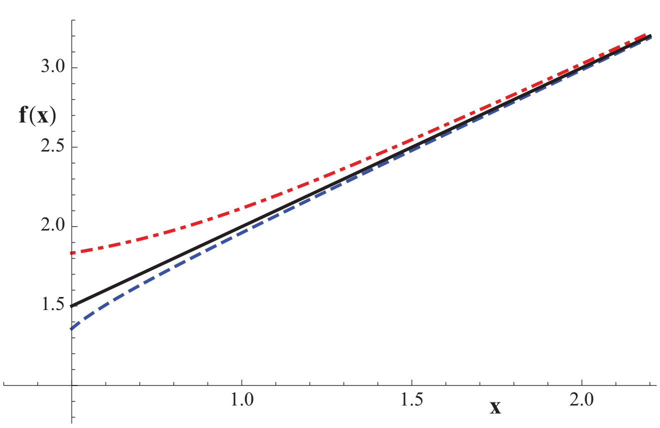

FIG. 6. (Color online) Comparison of the behavior of the characteristic function $f(x)$ for the harmonic oscillator (middle curve) with the perturbed harmonic oscillator $N_{n-1}^2 = n(n + a + 0.2 - 1/2)/(n + 0.2)$, for $a = 0.9$ (upper curve) and $a = 0.1$ (lower curve).

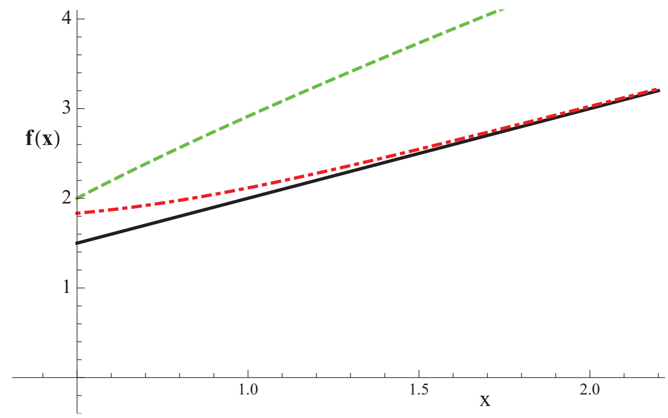

FIG. 7. (Color online) Comparison of the behavior of the characteristic function $f(x)$ for the harmonic oscillator (lower curve) with the perturbed harmonic oscillator $N_{n-1}^2 = n(n + a + 0.2 - 1/2)/(n + 0.2)$, for $a = 0.9$ (middle curve) and the free particle in an infinite square well (upper curve).

Note that, although we are presenting a particular case, this seems to be true in general. As another example, the characteristic function $f(\epsilon_i) = \epsilon_i + \sqrt{2\epsilon_i} + 1/2$ of the particle in the infinite square well potential is always larger than the harmonic oscillator characteristic function $f(\epsilon_i) = \epsilon_i + 1$ and larger than the deformed oscillator with $a = 0.9$, for any value of $i > 0$, as can be seen in Fig. 7. When we compare the fidelity of the free particle in the square well with those two systems, the harmonic oscillator and the deformed oscillator with $a = 0.9$, we find that the free particle is indeed the most resistant to the decoherence process here studied (Fig. 8). We can conclude then that the analysis of the characteristic function indicates the qualitative behavior of a given system under decoherence, meaning that this information is already contained in the algebraic structure itself. A possible next step is to find for some physical systems a phenomenological description of the fidelity in terms of the characteristic function.

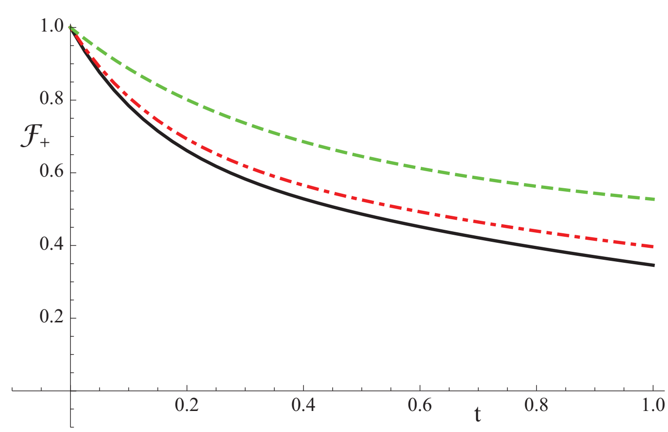

FIG. 8. (Color online) Comparison of the fidelity behavior (as a function of t, in seconds) of the harmonic oscillator (lower curve) with the deformed harmonic oscillator $N_{n-1}^2 = n(n + a + 0.2 - 1/2)/(n + 0.2)$ for $a = 0.9$ (middle curve) and the free particle in the infinite square well (upper curve), for $z = 1.7$.

VI. CONCLUSIONS

The four-parameter perturbed oscillator that we have introduced in this paper is advantageous because by varying the values of the parameters we can find many different systems, all of them being slightly different from the usual harmonic oscillator. This is particularly useful when we look for coherent Schrödinger catlike states that resist the effect of decoherence caused by a dissipative interaction with the environment.

Studying the behavior of the harmonic oscillator, the free particle in the infinite square well, and the perturbed oscillator for some particular values of its four parameters, we have been able to compare these systems and select those that have coherent catlike states that are more resistant under decoherence. First, these results are obtained by calculating the fidelity—a well-known measure of quantum states' resistance to decoherence. Second, we have been able to show a relation between the resistance to decoherence and the value of the GHA characteristic function, whose analysis indicates the qualitative behavior of the quantum systems in interaction with a heat bath. This implies that this information is somehow contained in the algebraic structure of the GHA systems and we can hope that further investigation of this relation will lead us to a phenomenological description of the fidelity in terms of the characteristic function.

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