Oscillator-field model of moving mirrors in quantum optomechanics

Chad R. Galley,[1,2] Ryan O. Behunin,[3] and B. L. Hu[4,5]

[1]*Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California 91125, USA*

[2]*Theoretical Astrophysics, California Institute of Technology, Pasadena, California 91106, USA*

[3]*Center for Nonlinear Studies and Los Alamos National Laboratory, Theoretical Division, Los Alamos, New Mexico 87545, USA*

[4]*Joint Quantum Institute and Maryland Center for Fundamental Physics, University of Maryland, College Park, Maryland 20742, USA*

[5]*Institute for Advanced Study and Department of Physics, Hong Kong University of Science and Technology,*

Clear Water Bay, Kowloon, Hong Kong, China

We present a microphysics model for the kinematics and dynamics of optomechanics describing the coupling between an optical field, modeled here by a massless scalar field, and the internal and mechanical degrees of freedom of a movable mirror. Instead of implementing boundary conditions on the field, we introduce an internal degree of freedom and its dynamics to describe the mirror's reflectivity. Depending on parameter values, the internal degrees of freedom of the mirror in this model capture a range of its optical activities, from those exhibiting broadband reflective properties to those reflecting only in a narrow band. After establishing the model we show how appropriate parameter choices lead to other well-known optomechanical models, including those of Barton and Calogeracos [Ann. Phys. (NY) **238**, 227 (1995)], Calogeracos and Barton, Ann. Phys. (NY) **238**, 268 (1995), Law [Phys. Rev. A **51**, 2537 (1995)], and Golestanian and Kardar [Phys. Rev. Lett. **78**, 3421 (1997); Phys. Rev. A **58**, 1713 (1998)]. As a simple illustrative application we derive classical radiation pressure cooling from this model. We then connect our microphysics model to the common descriptions of a moving mirror coupled to radiation pressure (e.g., with Nx coupling, where N is the photon number and x is the mirror displacement), making explicit the underlying assumptions made in these phenomenological models. Our model is also applicable to the lesser explored case of small N, which existing models based on sideband approximations [Kimble *et al.*, Phys. Rev. D **65**, 022002 (2001)] have not addressed. Interestingly, we also find that slow-moving mirrors in our model can be described by the ubiquitous Brownian motion model of quantum open systems. The scope of applications of this model ranges from a full quantum-mechanical treatment of radiation pressure cooling and quantum entanglement between macroscopic mirrors to the back reaction of Hawking radiation on black-hole evaporation in a moving mirror analog.

PACS number(s): 42.50.Pq, 42.50.Ct

I. INTRODUCTION

Optomechanics deals with the interaction of light with mechanical systems. (For an introductory review see, e.g., [1] and references therein.) Though old in name it is relatively new in content: optomechanics has a history at least as old as radiation pressure [2]. At the quantum level, optomechanics can be traced at least as far back as Casimir [3], who showed that there is an attractive force between two conducting plates from the change of ground-state energy in the presence of boundary conditions, and to Casimir and Polder [4], who calculated the force on an atom near an ideal mirror. The last decade has seen intense interest in several areas that are all under the umbrella of optomechanics. One such area is the dynamical Casimir effect [5], where a moving object, be it a moving mirror in the vacuum, a contracting gas bubble in a fluid (e.g., sonoluminescence as advocated in Ref. [6]), a time-varying magnetic flux bias threading a SQUID terminating a coplanar transmission line [7], or even a dynamical space-time (see below), can excite the vacuum, leading to the emission of radiation.

Optomechanics is of renewed current interest because of at least three new developments. The first relates to nanotechnology [8], where miniature mechanical motion can be transduced or manipulated with high precision by capacitive coupling or optical control or in nanoscale wave guides, where radiation pressure effects become important, e.g., leading to

large tailorable photon-phonon couplings, which give rise to a large enhancement of stimulated Brillouin scattering [9]. The second pertains to quantum information, where information stored in atoms and photons can interface with mechanical devices [10,11]. The third pertains to the use of atoms as optical elements [12].

Historically, the gravitation physics community also has explored mirror-field interactions in several ways. For example, cosmological particle creation in the early universe (studied by Parker and Zel'dovich in the 1960s–1970s [13,14]) is a form of the dynamical Casimir effect since it arises from the parametric amplification of vacuum fluctuations by the expansion of the universe. Another example was the use of a uniformly accelerated mirror as an analog model of Hawking-Unruh effects [15,16] developed by Davies and Fulling [17,18]. It should be noted that these effects are different from cosmological particle creation as both the black hole and the uniformly accelerated detector or mirror have event horizons, while the former, in general, does not, except for the de Sitter and anti–de Sitter universes. Another example of optomechanics realized in gravitational physics is the Laser Interferometer Gravitational-Wave Observatory (LIGO) [19–21]. Following the suggestion by Joseph Weber, Forward [22] proposed using laser interferometers for the detection of gravitational waves, which has since ushered in today's large-scale and international ground-based gravitational wave detection effort.

These gravitational wave detectors are probably the best illustration of the reverse function of optomechanics since in this case an impinging gravitational wave displaces the mirrors in the interferometer and the laser beam in the optical arms picks up the corresponding signal.

In terms of practical applications, an optomechanical process that is actively pursued now is mirror cooling by radiation pressure (see, e.g., [23]). Optomechanics also provides an excellent means for probing foundational issues in quantum physics. Sample studies include (1) reaching beyond the standard quantum limit using superconducting [24] and nanoelectromechanical [25] devices, (2) schemes for the improvement of the signal-to-noise ratio in gravitational wave interferometer detectors [26–29] based on earlier theoretical work of Braginsky and Khalili [30], Caves [2,31], Unruh [32], and Kimble *et al.* [33], (3) quantum superposition and entanglement of macroscopic objects such as between a mirror and the field [34] and between two mirrors [35], and (4) gravitational decoherence, both in its possible limitation to the precision of atom interferometry [36] and as a justification for a modified quantum theory [37].

Theoretical development for optomechanics also began quite early, most notably in the classic papers of Moore [38], Fulling and Davies [17], Jaekel and Reynaud [39], Barton and Calogeracos [40], Law [41], Dodonov and Klimov [42], and Schutzhold *et al.* [43], who took a canonical Hamiltonian approach, and Hu and Matacz [44], Golestanian and Kardar [45,46], Wu and Lee [47], and Fosco *et al.* [48], who took a path-integral approach. There is also a lineage of work on relativistically moving mirrors as analog models of the Hawking effect (see, e.g., [49–51]). However, many theoretical aspects remain untouched or were treated loosely (some even erroneously). In view of the momentous recent advances in optomechanics, we find it timely and necessary to construct a more solid and complete theoretical framework of moving mirrors interacting with a quantum field.

Our goal is to come up with models and theories capable of treating all of the problems listed above yet conceptually simple and theoretically systematic enough to be viable and useful. Admittedly not a simple task [52], we will delineate different aspects as we progress. Suffice it to mention that this first series of papers present the basic models and theories of optomechanics for both a closed (this paper) and an open (subsequent work) [53] system dynamics of moving mirrors in a quantum field. These models can be used to treat the broad class of problems related to the dynamical Casimir effect, among other things. The second series includes the back action of quantum fields on the mirror, which is needed for treating mirror cooling (for earlier work see references in Ref. [54]); the results therein could be applied to the related topics of quantum friction [55–58] and vacuum viscosity [14,46,59]. A third series will focus on basic issues in quantum information, making use of the stochastic equations derived in a subsequent work for moving mirrors interacting with a quantum field, specifically on quantum decoherence, superposition, and entanglement of mirrors and field. A different vein using similar techniques but staged in curved space-times is the moving mirror analog of the back reaction of Hawking radiation on the evolution of a black hole (for background, see references in, e.g., [60]).

In this paper we tend to the first order of business, namely, that of developing a useful microscopic model for any number of mirrors interacting with a field. We consider a massless scalar field in one spatial dimension for simplicity. The more realistic electromagnetic field in three spatial dimensions can be treated with a slight modification in the form of mirror-field coupling, known as the minimal coupling (see the appendix of [61]). In most prior considerations for the primary functions of a mirror its *reflective properties* (say, by the atom, molecular, and optical (AMO) community) and the *boundary conditions* it imposes on the ambient field (say, by the field theory community), namely, its amplitude has to vanish at the location of the mirror, are considered in a disjoint manner. The advantage of the present model is that it avoids the necessity for considering boundary conditions (e.g., à la Fulling and Davies). Only upon elimination of the explicit dependence of the internal degrees of freedom of the mirror would the field equations require careful attention to boundary conditions. We then consider the kinematics of *mirror motion*, which also has an effect on the field. For example, the motion can parametrically amplify the field modes, including its vacuum fluctuations, which results in particle creation (in the field theory language) or "motion-induced" or "acceleration" radiation (in the atom-optics language).

From practical experience physical mirrors have surfaces possessing "light" (as opposed to "heavy") degrees of freedom that interact with externally incident radiation in such a way as to maintain the appropriate boundary conditions that depend on the material composition of the mirror. Physical mirrors are transparent to sufficiently high frequency components of the field because the mirror's internal degrees of freedom are not energetic enough to (strongly) couple to field modes with arbitrarily high frequencies. For field modes with frequencies far below this cutoff frequency, known as the plasma frequency, the mirror becomes nearly perfectly reflecting.

In this paper, we treat the mirror's center-of-mass motion as that of a particle with mass M. To account for the mirror's reflectivity, we model the mirror's "light" internal degree of freedom $q(t)$ as a simple harmonic oscillator (with mass m and natural frequency Ω), which is coupled bilinearly to the massless scalar field $\Phi(t,x)$ at the mirror's location x. Because this model involves the mutual interaction of the internal oscillator, the field, and the center-of-mass motion of the mirror we shall call this model a *mirror-oscillator-field* (MOF) model for optomechanical applications. Further details and properties of the MOF model are given in Sec. II.

In Sec. II we demonstrate the mirror's ability to reflect and transmit incident radiation and to *perfectly* reflect or transmit radiation upon judicious choices for the parameter values of the internal oscillator. We then compare our model with three commonly used models or descriptions for mirrors: (1) The first is the model of Barton and Calogeracos (BC) [40] for partially transmitting mirrors. We show that our model extends the BC model to nonadiabatic regimes of the internal oscillator dynamics. (2) The next is the auxiliary field approach of Golestanian and Kardar [45,46]. We also show how these two models are related to each other. (3) Finally, we discuss Law's model for moving mirrors forming a perfect cavity, obtained here as a subcase of a self-consistent solution of the dynamics for (a) the center of mass of the mirror,

(b) the mirror's internal degrees of freedom, and (c) the quantum field. In Sec. III we turn our attention to a moving mirror by extending our model to allow for arbitrary motion, relativistic or nonrelativistic. In Sec. IV we describe the MOF model for multiple moving mirrors and focus our attention on how our model appropriately describes multiple reflections and transmissions of radiation incident on a cavity. Hence, the MOF model is also applicable to multiple-mirror systems in general and to a cavity in particular, which should be useful for laboratory-related studies. In Sec. V we apply the MOF model to describe (classical) mirror cooling by radiation pressure and indicate the role of the mirror's internal oscillator. In Sec. VI we show how the bilinear coupling in the MOF model relates to the phenomenological model of moving mirrors wherein the radiation pressure acts on the mirror through the number of incident photons N times the position x of the mirror (which we refer to as an Nx-type coupling). In Sec. VII we show how the MOF model of N moving mirrors is related to models of quantum Brownian motion (QBM) involving N harmonic oscillators coupled to a bath of oscillators. The available and exact master equations for the latter model will facilitate, among other things, our later studies of entanglement between two mirrors, a prototype problem in macroscopic quantum phenomena as described above. Finally, in Sec. VIII we summarize our findings and mention further work in progress toward the construction of a more complete theory of optomechanics.

II. A MIRROR AT REST MODELED BY A BILINEAR OSCILLATOR-FIELD COUPLING

In this section we introduce a model for a mirror at rest interacting with a scalar field. Our system consists of a mirror with mass M that we treat as being pointlike, so that, when it is allowed to move, its trajectory is described by coordinates $Z(t)$. The light degrees of freedom, which are responsible for the reflective function of the mirror, are modeled as an internal oscillator $q(t)$ with mass $m \ll M$ and natural frequency Ω. For brevity, we will refer to this internal mirror oscillator as a *mirosc*. Modeling the light degrees of freedom by a simple harmonic oscillator is functionally similar to the idealization of the internal degrees of freedom of an atom to that of a "two-level" system when considering the atom's optical activities (such as spontaneous and stimulated emission) interacting with a field [62]. Last, we take the mirosc to couple to the external (possibly quantum) scalar field $\Phi(t,x)$ in a manner that is linear in both quantities (i.e., bilinearly coupled). Taken together, we will refer to this model categorically as a mirror-oscillator-field model for optomechanical applications. Different oscillator-field couplings in this model give rise to different models familiar in optics. However, we will always be considering a bilinear coupling in this paper.

We shall show below how this model can describe, with appropriate choices of parameters, a range of perfectly and imperfectly reflecting mirrors. We also show how it relates to the model of Barton and Calogeracos (BC) [40] used in the quantum optics community, to the auxiliary field model used more in the field-theory community [45,46], and to models of cavities formed by perfectly reflecting mirrors where the center-of-mass motion of one mirror is a dynamical variable [41].

A. Reflectivity of a mirror modeled by a bilinear oscillator-field coupling

To demonstrate that the MOF model described above actually possesses the ability to reflect incoming modes it is sufficient to put the mirror at rest at the origin so that the action is given by

$$S[\Phi,q] = \frac{1}{2} \int d^2x \, \partial_\alpha \Phi \partial^\alpha \Phi + \frac{m}{2} \int dt (\dot{q}^2 - \Omega^2 q^2)$$
$$+ \lambda \int dt \, q(t)\Phi(t,0), \quad (2.1)$$

where $\eta_{\alpha\beta} = \text{diag}(1,-1)$ is the metric of $(1+1)$-dimensional Minkowski space-time. In units where $\hbar = c = 1$ the coupling constant λ has dimensions of $(\text{mass})^{-2} = (\text{length})^2$. The equations of motion are obtained by varying (2.1) in the usual way,

$$\partial_\alpha \partial^\alpha \Phi = \partial_t^2 \Phi - \partial_x^2 \Phi = \lambda q(t)\delta(x), \quad (2.2)$$

$$m\ddot{q}(t) + m\Omega^2 q(t) = \lambda \Phi(t,0). \quad (2.3)$$

Let a plane wave with frequency ω be incident on the mirror from the left (L) so that the field is given by

$$\Phi_{\omega L} = e^{-i\omega t}\{\theta(-x)[e^{i\omega x} + R(\omega)e^{-i\omega x}] + \theta(x)T(\omega)e^{i\omega x}\},$$

where $R(\omega)$ and $T(\omega)$ are the frequency-dependent reflection and transmission coefficients, respectively. For the steady-state evolution of the mirosc-field system we can take q to oscillate with the same frequency as the incident radiation so that

$$q(t) = Ae^{-i\omega t}, \quad (2.4)$$

where the amplitude A is determined from (2.3) to be

$$A = \frac{\lambda}{\Omega^2 - \omega^2} \frac{T(\omega)}{m}. \quad (2.5)$$

The field is continuous at the location of the mirror, $\Phi_{\omega L}(t,0^+) = \Phi_{\omega L}(t,0^-)$, and the discontinuity of the spatial derivative is found by integrating (2.2) over a vanishingly small interval encompassing the mirror's position,

$$-\partial_x \Phi_{\omega L}(t,0^+) + \partial_x \Phi_{\omega L}(t,0^-) = \lambda q(t). \quad (2.6)$$

Using these conditions and the equations of motion in Eqs. (2.2) and (2.3), the reflection and transmission coefficients are given in terms of the three mirosc parameters (m,Ω,λ) by

$$R(\omega) = -\frac{i\lambda^2}{2m\omega(\Omega^2 - \omega^2) + i\lambda^2}, \quad (2.7)$$

$$T(\omega) = \frac{2m\omega(\Omega^2 - \omega^2)}{2m\omega(\Omega^2 - \omega^2) + i\lambda^2}. \quad (2.8)$$

The particle reflects most of the incident radiation whenever $\lambda^2 \gg |2m\omega(\Omega^2 - \omega^2)|$. Based on this, perfect reflection can be achieved in three cases: (1) in the infinitely strong oscillator-field coupling limit, $\lambda \to \infty$, (2) when the oscillator is resonantly excited by the monochromatic radiation, $\Omega = \omega$, and (3) in the limit that the mirosc mass vanishes, $m \to 0$. In all three cases, the reflection and transmission coefficients are $R(\omega) = -1$ and $T(\omega) = 0$, respectively.

Conversely, most of the incident radiation is transmitted when $\lambda^2 \ll |2m\omega(\Omega^2 - \omega^2)|$. Based on this, perfect transmission can be attained in three ways: (1) in the limit of

vanishingly small oscillator-field coupling, $\lambda \to 0$, (2) when the mirosc frequency is arbitrarily large, $\Omega \to \infty$, and (3) when the mirosc mass is arbitrarily large, $m \to \infty$. In all three cases, $R(\omega) = 0$ and $T(\omega) = 1$.

1. Reflective properties

The reflection and transmission coefficients possess interesting features that depend on the parameters of the theory (m,Ω,λ). The squared complex norm of the reflection coefficient from (2.7) is

$$|R(y)|^2 = \frac{1}{1 + \left[\frac{2m\Omega^3}{\lambda^2} y(1 - y^2)\right]^2}, \qquad (2.9)$$

where we have introduced the dimensionless quantity $y = \omega/\Omega$. To characterize the dependence of the reflection coefficient on the frequency ω of the incident-field mode we observe that the local minima and maxima occur for $y_{\min} = 1/\sqrt{3}$ and $y_{\max} = \{0,1\}$, respectively. The maximum at $\omega = 0$ is an artifact of the monopole coupling between the field and the mirosc. For a dipole coupling the reflection coefficient vanishes at $\omega = 0$. The reflection coefficient equals 1 at both maxima, and the value of $|R|^2$ at $y_{\min}$ is

$$|R(y_{\min})|^2 = \frac{1}{1 + r_p^2}, \qquad (2.10)$$

where we define the *plasma frequency* Ω_p for the partially transmitting mirror to be

$$\Omega_p \equiv \frac{3^{3/2}\lambda^2}{4m\Omega^2} \qquad (2.11)$$

and the index $r_p \equiv \Omega/\Omega_p = 4m\Omega^3/(3^{3/2}\lambda^2)$.

We can use this minimum in the reflected radiation to indicate when the two maxima of $|R|^2$ are sufficiently separated and distinguishable. While this is subjective, we take $|R(y_{\min})|^2 = 1/2$ to be our defining requirement, which fixes $r_p = 1$. The implication is that if $r_p \gg 1$, then the reflection coefficient is sharply peaked about $\omega = \Omega$. Under this condition, the parameters of the mirosc can be tuned to selectively reflect incident radiation in a narrow bandwidth centered on $\omega = \Omega$, which occurs when the mirosc is resonantly excited (or very nearly so) by the incident field. Whenever the mirosc parameters are such that $r_p \ll 1$, then the local minimum at $y_{\min} = 1/\sqrt{3}$ is close to 1, and the mirosc reflects modes over a broader frequency bandwidth. Furthermore, the mirror will reflect more than half of the incident radiation so long as the frequency of the field is less than $\sim \Omega$ to a good approximation. Hence, if the mirosc mass is made smaller or the oscillator-field coupling constant λ is made larger, then more modes will be reflected more strongly by the mirror. Figure 1 shows the basic features of the mirror's scattering properties studied in this section and contains plots of the reflection coefficient $|R(y)|^2$ for r_p equal to, much larger than, and much smaller than 1.

2. Model justification

Although simplified, our description of the mirror's internal dynamics by a harmonic oscillator captures the salient features of a mirror exhibiting a dispersive interaction with an electromagnetic field, which can be described by the undamped Lorentz oscillator model (or the plasma model). However, by
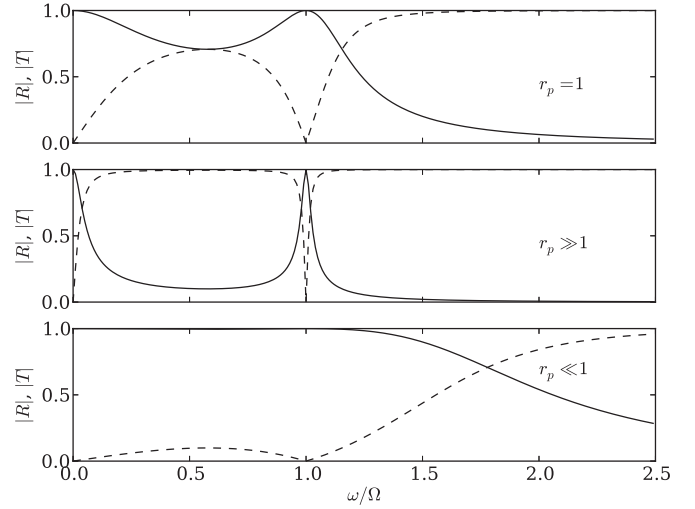

FIG. 1. Complex norm of the reflectivity (solid lines) and transmissivity (dashed lines) for the case when the minimum in $|R(\omega)|^2$ is 50% (top) and when the oscillator reflects incident modes in a narrow (middle) and a broad bandwidth (bottom).

a straightforward generalization, one can obtain the Huttner-Barnett model [63], which describes a lossy dielectric and generalizes the optical response of the mirror to include Ohmic dissipation.

As a final note, the MOF model relates to the familiar description of light-matter interactions in terms of a frequency-dependent dielectric permittivity. This can be seen by eliminating the appearance of the mirror's internal degrees of freedom in the equations of motion. We start from the equations of motion for the field (2.2) and the oscillator (2.3). Solving for the oscillator, we find

$$q(t) = q_0(t) + \lambda \int dt' \, \chi(t - t')\Phi(t',0), \qquad (2.12)$$

where $q_0(t)$ is the homogeneous solution for the oscillator carrying information about its initial conditions. The second term on the right-hand side is the particular solution which quantifies the response of the oscillator to the field. The function $\chi(t - t')$ is the oscillator's susceptibility. Next, we substitute (2.12) into the equation for the field, which gives

$$\partial_\alpha \partial^\alpha \Phi(t,x) - \lambda^2 \int dt' \, \chi(t - t')\delta(x)\Phi(t',x) = \lambda q_0(t)\delta(x).$$

Notice that the particular solution for the oscillator modifies the free evolution of the field. After Fourier transforming to the frequency domain the interpretation becomes clearer,

$$\{\partial_x^2 + \underbrace{[\omega^2 + \lambda^2 \chi(\omega)\delta(x)]}_{\omega^2 \varepsilon(\omega,x)}\}\Phi(\omega,x) = -\lambda q_0(\omega)\delta(x).$$

The key point is that the oscillator induces a space- and frequency-dependent permittivity given by $\varepsilon(\omega,x) = 1 + \lambda^2 \chi(\omega)\delta(x)/\omega^2$, as claimed.

B. Relation to other mirror models and approaches

The bilinearly coupled oscillator-field model introduced above possesses interesting physical limits that relate to two well-known and commonly used mirror models. The first

model arises when the mirosc evolves adiabatically with the field and gives rise to the model of Barton and Calogeracos for a partially transmitting mirror. The second model arises when the mass of the mirosc becomes arbitrarily small, in which case the mirosc serves the role of an auxiliary field that relates to the path-integral approaches of [45,46,48], which describe a quantum field interacting with a perfectly reflecting mirror(s). Last, we connect our model to the dynamical description of a moving mirror [41]. In Sec. VI we also relate the MOF formulation of optomechanics to a commonly used model that describes the effects of radiation pressure by invoking a phenomenological coupling between the number of photons impinging the mirror and the mirror's position.

1. Barton-Calogeracos model

The BC model has been used quite often in quantum optics, and it is worth summarizing its primary properties before showing how it can be derived from our MOF model. Much of Barton and Calogeracos's attention focused on quantizing the nonrelativistic limit of the theory where the mirror velocity is much smaller than c. We do not present their results here but refer the reader to the original papers of [40] for further details.

The action for a mirror at rest in the BC model is, in $1+1$ dimensions of space-time,

$$S_{BC}[\Phi] = \frac{1}{2} \int d^2x \, \partial_\alpha \Phi \partial^\alpha \Phi - \gamma \int dt \, \Phi^2(t, x=0),$$

where γ is related to the plasma frequency of the mirror [40]. Extremizing the action gives the equations of motion

$$\partial_\alpha \partial^\alpha \Phi = 2\gamma \delta(x) \Phi(t,0). \quad (2.13)$$

The reflection and transmission of a normal mode of the field incident on the mirror from the left ($x < 0$) is

$$\Phi_{\omega L} = e^{-i\omega t} \{\theta(-x)[e^{i\omega x} + R(\omega)e^{-i\omega x}] + \theta(x)T(\omega)e^{i\omega x}\}, \quad (2.14)$$

where $R(\omega)$ and $T(\omega)$ are the frequency-dependent reflection and transmission coefficients, reflectively, with the property that $|R|^2 + |T|^2 = 1$.

We demand that the field be continuous across the mirror $\Phi_{\omega L}(t, 0^+) = \Phi_{\omega L}(t, 0^-)$ and that its derivative satisfy

$$-\partial_x \Phi_{\omega L}(t, 0^+) + \partial_x \Phi_{\omega L}(t, 0^-) = 2\gamma \Phi_{\omega L}(t,0). \quad (2.15)$$

This jump condition follows from integrating the field equations across the mirror's position at $x = 0$. Together with the field equation these conditions imply that

$$R(\omega) = -\frac{i\gamma}{\omega + i\gamma}, \quad (2.16)$$

$$T(\omega) = \frac{\omega}{\omega + i\gamma}. \quad (2.17)$$

As the parameter γ becomes arbitrarily large, we see that the reflection becomes perfect, and the incoming phase of the field changes by π radians,

$$\lim_{\gamma \to \infty} R(\omega) = -1. \quad (2.18)$$

The ability of the BC model to reproduce perfect and imperfect reflections comes from using the quadratic interaction $\Phi^2(t,0)$.

With this specific coupling to the mirror the jump condition across the origin (2.15) is linear in Φ at the mirror, which is vital for obtaining the normal mode in Eq. (2.14).

The MOF model in Eq. (2.1) can be related, under appropriate conditions, to the BC model. Observe from (2.3) that if $q(t)$ evolves adiabatically with time,

$$\left|\frac{\ddot{q}}{\Omega^2 q}\right| \ll 1, \quad (2.19)$$

then the mirosc follows the time dependence of the field at the mirror's position,

$$q(t) \approx \frac{\lambda}{m\Omega^2} \Phi(t,0). \quad (2.20)$$

Substituting this approximation for the oscillator variable into the scalar field equation (2.2) gives

$$\partial_\alpha \partial^\alpha \Phi \approx \left(\frac{\lambda^2}{m\Omega^2}\right) \delta(x) \Phi(t,0). \quad (2.21)$$

Comparing with (2.13), we recover the model of BC by identifying γ with the parameters of the mirosc and hence to the plasma frequency of the MOF model,

$$\gamma = \frac{\lambda^2}{2m\Omega^2} = \frac{2}{3^{3/2}} \Omega_p. \quad (2.22)$$

Therefore, in the limit that the mirosc changes adiabatically, the MOF model yields the BC model.

An equivalent way of connecting to the BC model is to take the mass of the mirosc to zero, $m \to 0$, but keep the quantity $m\Omega^2 \equiv \kappa$ constant in this limit, which requires the mirosc natural frequency to simultaneously approach infinity, $\Omega \to \infty$. In this limit, the mirosc also follows the time dependence of the field,

$$q(t) \to \frac{\lambda}{\kappa} \Phi(t,0). \quad (2.23)$$

The identification with the BC model then follows the same steps as in the previous paragraph, and in particular, one finds that $\gamma = \lambda^2/(2\kappa)$. It is worth pointing out that the massless limit $m \to 0$ here does not imply that the mirror is perfectly reflecting as in the previous section. This is because of the additional requirement that $m\Omega^2 = \kappa$ remain constant. In fact, the reflection coefficient (2.7) in this limit becomes

$$R(\omega) \to -\frac{i\lambda^2}{2\omega\kappa + i\lambda^2}, \quad (2.24)$$

and the mirror becomes perfectly reflecting when $\lambda \to \infty$.

Through the identification in Eq. (2.22) we may attach heuristic physical interpretations to m, Ω (or κ), and λ. In Ref. [40], Barton and Calogeracos observe that their model is equivalent to a jellium sheet of zero width, i.e., a surface of vanishing thickness having a surface current density generated by the motion of small charge elements with charge density n_s. If these elements have charge $n_s e$ per unit area and mass $n_s m_e$ per unit area, then BC find

$$\gamma = \frac{2\pi n_s e^2}{m_e}. \quad (2.25)$$

Identifying these microscopic variables to those in our MOF model via (2.22) gives the following relationship:

$$\frac{4\pi n_s e^2}{m_e} = \frac{\lambda^2}{\kappa}. \qquad (2.26)$$

This suggests identifying the mirosc field coupling as a charge per unit area, $\lambda \to n_s e$, and κ as a mass per unit area, $\kappa \to n_s m_e/(4\pi)$. That is, λ can be viewed as a surface charge density and $\kappa = m\Omega^2$ as a surface mass density. This interpretation may be useful for developing a similar MOF model for a mirror in $3 + 1$ dimensions.

2. Models using auxiliary fields

The MOF model reduces to another well-known description of mirrors if we take the limit $m \to 0$. In this limit our model describes a perfectly reflecting mirror, as discussed earlier, and the action (2.1) becomes

$$\lim_{m \to 0} S[\Phi, q] = \frac{1}{2} \int d^2x \, \partial_\alpha \Phi \partial^\alpha \Phi + \lambda \int dt \, q(t) \Phi(t, 0). \qquad (2.27)$$

The key point is that the mirosc possesses no dynamics in this limit. Thus, the quantity $\psi(t) \equiv \lambda q(t)$ possesses no dynamics of its own and can be regarded as an *auxiliary field*.

In the path-integral formulation of the quantum theory, the massless mirosc limit gives rise to the following generating functional [64]:

$$\lim_{m \to 0} Z[J] = \int \mathcal{D}\Phi \int \mathcal{D}\psi \, \exp \left\{ \frac{i}{2} \int d^2x \, \partial_\alpha \Phi \partial^\alpha \Phi \right.$$
$$\left. + i \int dt \, \psi(t) \Phi(t, 0) + i \int d^2x \, J\Phi \right\}. \quad (2.28)$$

Then, noting that the path integral over $\psi(t)$ is just the Fourier representation of the Dirac δ functional,

$$\int \mathcal{D}\psi \, \exp \left\{ i \int dt \, \psi(t) \Phi(t, 0) \right\} = \delta[\Phi(t, 0)], \quad (2.29)$$

it follows that the generating functional

$$\lim_{m \to 0} Z[J] = \int \mathcal{D}\Phi \, \exp \left\{ \frac{i}{2} \int d^2x \, \partial_\alpha \Phi \partial^\alpha \Phi + i \int d^2x \, J\Phi \right\}$$
$$\times \delta[\Phi(t, 0)] \qquad (2.30)$$

describes a quantum scalar field constrained to vanish at the location of the mirror (only those field configurations that vanish at $x = 0$ will contribute to the path integral). The vanishing of the field at the location of the mirror is equivalent to the perfect reflection of an incident field [45,46]. The generating functional above was first introduced by Golestanian and Kardar in Refs. [45,46].

Our bilinearly coupled MOF model (2.1) has successfully reproduced two models describing the interactions of a field with a mirror at rest: (1) the partially transmitting BC mirror model when $q(t)$ evolves adiabatically and (2) an auxiliary field approach that enforces the field to vanish at the mirror when the mass of the mirosc is vanishingly small. In turn, these two models can be related to each other. Specifically, noting that the δ functional above can be approximated by a narrow
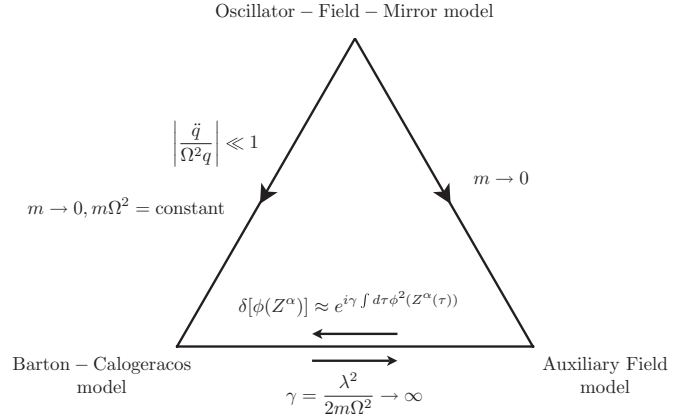

FIG. 2. Relationships among our mirror-oscillator-field model of a partially transmitting mirror, the Barton-Calogeracos model of a partially transmitting mirror, and the auxiliary field approach for a perfectly reflecting mirror.

Gaussian, it follows that (2.30) becomes

$$\lim_{m \to 0} Z[J] \approx \int \mathcal{D}\Phi \, \exp \left\{ \frac{i}{2} \int d^2x \, \partial_\alpha \Phi \partial^\alpha \Phi \right.$$
$$\left. - i\gamma \int d^2x \, \Phi^2(t, 0) + i \int d^2x \, J\Phi \right\}, \quad (2.31)$$

which is increasingly more accurate for larger values of γ. Hence, BC falls out from the generating functional approach if we smear the δ functional constraint that enforces the field to vanish on the surface of the mirror. Likewise, using the action for the BC model in the generating functional formalism gives the perfect reflection limit when $\gamma \to \infty$. See Fig. 2 for the relationships among these theories.

III. A MOVING MIRROR IN THE MOF MODEL

As pointed out in the Introduction, the physics is quite different in the two cases when the mirror is moving relativistically compared to the case when it is moving slowly. The former relates to cosmological particle creation and radiation emitted from black holes or in uniformly accelerated detectors in the Hawking-Unruh effects, while the latter is closer to accessible laboratory situations such as mirror movements caused by the passing of gravitational waves in interferometer detectors and mirror cooling from the field's back action in the form of radiative pressure and quantum friction.

The MOF model presented here provides a unified framework for treating both, albeit very different, situations. For cases when the mirror motion is prescribed, such as coplanar waveguides terminated by a superconducting quantum interference device (SQUID) [7,65], or when the mirror possesses nontrivial reflective properties [12], our model can meet the needs of current experiments by providing a rich set of reflective properties and a tractable formalism capable of providing analytical insight. For systems where the mirror motion is dynamically determined by the mutual interaction of the mirror's center of mass, its internal motion, and the field, our model provides a computational ease. This simplification results from the fact that boundary conditions are not imposed on the field from the outset but are determined by a

self-consistent elimination of the mirror's internal motion. This facilitates the derivation of equations of motion for the mirror's mean position, which will be adopted in Sec. V to describe classical radiation pressure cooling and in later papers in this series to provide a fully quantum-mechanical treatment of mirror cooling and the mirror analog of the black-hole back reaction.

A. Action formulation

Allowing the mirror to move requires the addition of an extra term in the action describing its motion along the world line $Z^\mu(\lambda)$, where λ is an affine parameter and $\mu = 0,1$ or (t,x). The physics must remain invariant under any reparametrization of the mirror's world line $\lambda \to \lambda(\bar\lambda)$, which requires modifying the action (2.1) for a static mirror in the following way:

$$S[\Phi,q,Z^\mu] = \frac{1}{2}\int d^2x\, \partial_\alpha\Phi\partial^\alpha\Phi - M\int d\sigma\sqrt{U^\alpha U_\alpha}$$
$$+ \frac{m}{2}\int d\sigma\left(\frac{\dot q^2}{\sqrt{U^\alpha U_\alpha}} - \Omega^2 q^2\sqrt{U^\alpha U_\alpha}\right)$$
$$+ \lambda\int d\sigma\sqrt{U^\alpha U_\alpha}\, q(\sigma)\Phi(Z^\mu(\sigma)), \quad (3.1)$$

where an overdot denotes differentiation with respect to the world line parameter, $U^\mu(\sigma) = \dot Z^\mu(\sigma)$ is the two-velocity of the mirror, $d\sigma\sqrt{U^\alpha U_\alpha} = d\tau$ is the invariant proper time element as measured by an observer on the world line, and indices with Greek letters are raised and lowered with the Minkowski metric $\eta_{\alpha\beta} = \mathrm{diag}(1,-1)$. The field still couples bilinearly to the mirosc via the last term of the action, so that the reflective properties studied in the previous section are retained by the model. The corresponding Euler-Lagrange equations of motion are easily found to be

$$\partial_t^2\Phi - \partial_x^2\Phi = \lambda q(\tau)\delta^2(x^\mu - Z^\mu(\tau)), \quad (3.2)$$

$$m\ddot q + m\Omega^2 q = \lambda\Phi(Z^\mu(\tau)), \quad (3.3)$$

$$M_{\mathrm{eff}}(\tau)\dot U^\mu = -\lambda q(\tau)(\eta^{\mu\nu} - U^\mu U^\nu)\partial_\nu\Phi(Z^\mu(\tau)), \quad (3.4)$$

where we conveniently have chosen to parametrize the world line by the proper time τ at this point since then $U^\alpha U_\alpha = 1$ and $U^\alpha\dot U_\alpha = 0$, which help simplify the expressions. The quantity $M_{\mathrm{eff}}(\tau)$ in Eq. (3.4) is an effective mass for the mirror and is given by

$$M_{\mathrm{eff}}(\tau) \equiv M + \tfrac{1}{2}m\dot q^2(\tau) + \tfrac{1}{2}m\Omega^2 q^2(\tau) - \lambda q(\tau)\Phi(Z^\mu(\tau)). \quad (3.5)$$

Notice that the effective mass has contributions from the rest mass of the mirror (M), the energy of the oscillator ($m\dot q^2/2 + m\Omega^2 q^2/2$), and the interaction energy of the mirror-oscillator-field system [$-\lambda q\Phi(Z)$]. In other words, the effective mass is the rest mass of the mirror plus the total internal energy of the mirosc.

B. Hamiltonian formulation

Here, we provide a Hamiltonian formulation of the MOF model. To do this, we find it convenient to parametrize the world line by the coordinate time t wherein the action (3.1)

becomes

$$S[\Phi,q,Z] = \frac{1}{2}\int d^2x\, \partial_\alpha\Phi\partial^\alpha\Phi - M\int dt\sqrt{1 - U^2}$$
$$+ \frac{m}{2}\int dt\left(\frac{\dot q^2}{\sqrt{1 - U^2}} - \Omega^2 q^2\sqrt{1 - U^2}\right)$$
$$+ \lambda\int dt\sqrt{1 - U^2}\, q(t)\Phi(t,Z(t)), \quad (3.6)$$

where $U(t) = dZ(t)/dt$ and from which the Lagrangian is

$$L = \frac{1}{2}\int dx\, \partial_\alpha\Phi\partial^\alpha\Phi + \frac{m}{2}\left(\frac{\dot q^2}{\sqrt{1 - U^2}} - \Omega^2 q^2\sqrt{1 - U^2}\right)$$
$$- M\sqrt{1 - U^2} + \lambda\sqrt{1 - U^2}\, q(t)\Phi(t,Z(t)). \quad (3.7)$$

To derive the Hamiltonian H we first identify the conjugate momenta,

$$\Pi(t,x) = \frac{\partial L}{\partial\dot\Phi(t,x)} = \dot\Phi(t,x), \quad (3.8)$$

$$p(t) = \frac{\partial L}{\partial\dot q(t)} = \frac{m\dot q(t)}{\sqrt{1 - U^2(t)}}, \quad (3.9)$$

$$P(t) = \frac{\partial L}{\partial\dot Z(t)} = \frac{M_{\mathrm{eff}}(t)U(t)}{\sqrt{1 - U^2(t)}}, \quad (3.10)$$

where the effective mass in terms of the conjugate momenta is

$$M_{\mathrm{eff}}(t) = M + \frac{p^2(t)}{2m} + \frac{1}{2}m\Omega^2 q^2(t) - \lambda q(t)\Phi(t,Z(t)). \quad (3.11)$$

The Legendre transformation of (3.7) yields the Hamiltonian after some algebra,

$$H = \frac{1}{2}\int dx\,[\Pi^2 + (\partial_x\Phi)^2] + \sqrt{P^2 + M_{\mathrm{eff}}^2(t)}. \quad (3.12)$$

For completeness, we give Hamilton's equations of motion:

$$\dot\Phi = \Pi, \quad (3.13)$$

$$\dot\Pi = \Phi'' + \frac{\lambda q(t)M_{\mathrm{eff}}(t)}{\sqrt{P^2 + M_{\mathrm{eff}}^2(t)}}\delta(x - Z(t)), \quad (3.14)$$

$$\dot q = \frac{p}{m}, \quad (3.15)$$

$$\dot p = -\frac{M_{\mathrm{eff}}(t)}{\sqrt{P^2 + M_{\mathrm{eff}}^2(t)}}[m\Omega^2 q - \lambda\Phi(t,Z(t)], \quad (3.16)$$

$$\dot Z = \frac{P}{\sqrt{P^2 + M_{\mathrm{eff}}^2(t)}}, \quad (3.17)$$

$$\dot P = \frac{\lambda q(t)M_{\mathrm{eff}}(t)}{\sqrt{P^2 + M_{\mathrm{eff}}^2(t)}}\partial_x\Phi(t,Z(t)), \quad (3.18)$$

which can be shown to be equivalent to the Euler-Lagrange equations in Eqs. (3.2)–(3.4). As discussed at the end of Appendix A, an external source (as from a laser) must be introduced to generate nontrivial forces on the mirror.

Depending on the application, it may be more convenient to work in a reference frame wherein the field and the mirror's world line decouple from each other so that the mirror always remains at rest at the origin. A transformation to such a noninertial frame is advocated in Ref. [40] and may be useful

for canonically quantizing the MOF model. However, we will not pursue this representation here.

C. A slowly moving mirror in the MOF model

Under all laboratory conditions to date the speed of the mirror is small compared to c and justifies developing the nonrelativistic limit of the mirror-oscillator-field model. For example, it was recently demonstrated that film bulk acoustic resonators (FBARs) [66] as large as ≈ 0.5 mm can be mechanically oscillated up to 3 GHz. The corresponding speed of the FBAR (having a modulation depth of 10^{-8}) is only $v \approx 4.4$ m/s, which is much smaller than c. Thus, for laboratory applications, the nonrelativistic limit of the MOF action in Eq. (3.1) is entirely appropriate.

The relativistic Lagrangian (3.7) expanded in powers of $\dot{Z} \ll 1$ and retaining the lowest-order contributions in the velocity yields

$$L = \frac{1}{2} \int dx\, \partial_\alpha \Phi \partial^\alpha \Phi + \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \Omega^2 q^2$$
$$+ \frac{1}{2} M \dot{Z}^2 - V(Z) + \lambda q(t) \Phi(t, Z(t)), \quad (3.19)$$

where we have dropped the term depending solely on the constant mass of the mirror M and $V(Z)$ accounts for an external (e.g., trapping) potential. The related Hamiltonian follows from a Legendre transform of (3.19) and is found to be

$$H = \frac{1}{2} \int dx (\Pi^2 + \Phi'^2) + \frac{p^2}{2m} + \frac{1}{2} m \Omega^2 q^2$$
$$+ \frac{P^2}{2M} + V(Z) - \lambda q(t) \Phi(t, Z(t)). \quad (3.20)$$

The equations of motion are easily derived from (3.19) or (3.20), so we do not give them here.

D. Relation to other moving mirror models

Our MOF model can also be connected to previous work on moving mirrors forming a perfect cavity [41]. For a slowly moving mirror, the equations of motion in the MOF model follow from Lagrangian (3.19) or Hamiltonian (3.20),

$$\partial_\alpha \partial^\alpha \Phi(x^\alpha) = \lambda q(t) \delta(x - Z(t)), \quad (3.21)$$

$$\ddot{q}(t) + \Omega^2 q(t) = \frac{\lambda}{m} \Phi(t, Z(t)), \quad (3.22)$$

$$M \ddot{Z}(t) = \lambda q(t) \overline{\partial_x \Phi(t, Z(t))}, \quad (3.23)$$

where in this section only we average the derivative of the field over both sides of the mirror, which we denote with an overline, as a form of regularization. Next, we solve for the internal motion of the mirror and eliminate its appearance in the remaining equations of motion, resulting in

$$\partial_\alpha \partial^\alpha \Phi(x^\alpha) = \frac{\lambda^2}{m} \delta(x - Z(t)) \int dt' \chi(t - t') \Phi(t', Z(t')),$$
$$(3.24)$$

$$M \ddot{Z}(t) = \frac{\lambda^2}{m} \overline{\partial_x \Phi(t, Z(t))} \int dt' \chi(t - t') \Phi(t', Z(t')), \quad (3.25)$$

where we have dropped the homogeneous solution for $q(t)$, which is irrelevant in the perfectly reflecting limit.

By integrating the equation of motion for the field in space from $Z(t) - 0^+$ to $Z(t) + 0^+$ we find the following boundary condition for the field:

$$-\partial_x \Phi(t, x)\big|_{x=Z(t)-0^+}^{x=Z(t)+0^+} = \frac{\lambda^2}{m} \int dt' \chi(t - t') \Phi(t', Z(t')), \quad (3.26)$$

which, when plugged into the equation of motion for the mirror's center-of-mass motion, gives

$$M \ddot{Z}(t) = -\frac{1}{2} \partial_x \Phi(t, Z(t) + 0^+) \partial_x \Phi(t, Z(t) + 0^+)$$
$$+ \frac{1}{2} \partial_x \Phi(t, Z(t) - 0^+) \partial_x \Phi(t, Z(t) - 0^+), \quad (3.27)$$

which is the equation of motion for the mirror being acted on by radiation pressure. This equation is identical to (2.3) in Ref. [41], which is the starting point of that work, with the exception that our equation takes into account the radiation pressure on both sides of the mirror and we are ignoring external potential forces. In the limit that the internal dynamics of the mirror becomes very light the susceptibility χ is proportional to a δ function. Then taking the strong-coupling limit (see Sec. II A) to achieve perfect reflection, we find that the right side of (3.24) becomes

$$\frac{\lambda^2}{m} \delta(x - Z(t)) \int dt' \chi(t - t') \Phi(t', Z(t'))$$
$$\rightarrow \lim_{\lambda \to \infty} \frac{\lambda^2}{m \Omega^2} \delta(x - Z(t)) \Phi(t, Z(t)). \quad (3.28)$$

All terms in the equation of motion must be finite in this limit, implying that the field vanishes along the trajectory of the mirror, $\Phi(t, Z(t)) = 0$, thereby producing the boundary conditions of a perfectly reflecting mirror.

IV. MULTIPLE MOVING MIRRORS IN THE MOF MODEL

In the previous sections we introduced a model for a mirror whose scattering and reflective properties are described by an oscillator, the mirosc, coupled bilinearly to the field. In this section we extend the MOF model to include multiple spatially separated partially transmitting mirrors that interact mutually via the field.

The Lagrangian for N moving mirrors (possibly relativistically) with masses M_a ($a = 1, \ldots, N$) can be written as

$$L = \frac{1}{2} \int dx\, \partial_\alpha \Phi \partial^\alpha \Phi - \sum_{a=1}^{N} M_a \sqrt{1 - U_a^2}$$
$$+ \sum_{a=1}^{N} \frac{m_a}{2} \left(\frac{\dot{q}_a^2}{\sqrt{1 - U_a^2}} - \Omega_a^2 q_a^2 \sqrt{1 - U_a^2} \right)$$
$$+ \sum_{a=1}^{N} \lambda_a \sqrt{1 - U_a^2}\, q_a(t) \Phi(t, Z_a(t)), \quad (4.1)$$

and the Euler-Lagrange equations of motion follow straightforwardly and are simply given by Eqs. (3.2)–(3.4) with all mirosc and world line parameters and variables receiving a subscript a to label the mirror. For completeness and for later

use, the corresponding Hamiltonian is

$$H = \frac{1}{2} \int dx [\Pi^2 + (\partial_x \Phi)^2] + \sum_{a=1}^{N} \sqrt{P_a^2 + M_{\text{eff},a}^2(t)}, \quad (4.2)$$

where the effective mass of the mirror has the same interpretation as before (i.e., mirror rest mass plus total internal energy), except now the total internal energy includes the energy of all N miroscs and their interaction energies with the field,

$$M_{\text{eff},a}(t) = M_a + \frac{p_a^2(t)}{2m_a} + \frac{1}{2} m_a \Omega_a^2 q_a^2(t) - \lambda_a q_a(t) \Phi(t, Z_a(t)). \tag{4.3}$$

In the nonrelativistic limit, the Lagrangian and the Hamiltonian are

$$L = \frac{1}{2} \int dx \, \partial_\alpha \Phi \partial^\alpha \Phi + \sum_{a=1}^{N} \left[\frac{1}{2} m_a \dot{q}_a^2 - \frac{1}{2} m_a \Omega_a^2 q_a^2 \right.$$
$$\left. + \frac{1}{2} M_a \dot{Z}_a^2 - V(Z_a) + \lambda_a q_a(t) \Phi(t, Z_a(t)) \right], \quad (4.4)$$

$$H = \frac{1}{2} \int dx [\Pi^2 + (\partial_x \Phi)^2] + \sum_{a=1}^{N} \left[\frac{p_a^2}{2m_a} + \frac{1}{2} m_a \Omega_a^2 q_a^2 \right.$$
$$\left. + \frac{P_a^2}{2M_a} + V(Z_a) - \lambda_a q_a(t) \Phi(t, Z_a(t)) \right]. \quad (4.5)$$

In Appendix B we study the scattering of incident radiation by two mirrors in this model and demonstrate that they possess the expected reflective properties.

V. CLASSICAL MIRROR COOLING WITH THE MOF MODEL

In this section, we give an example showing how the MOF model can be used to describe mirror cooling within a completely classical context. In a subsequent work, we discuss quantum effects in mirror cooling using the MOF model [54].

The setup is as follows. Consider a cavity formed by two mirrors. We take one of the mirrors to be fixed at the origin and perfectly reflecting, so that the (classical scalar) field satisfies Dirichlet boundary conditions, $\Phi(t,0) = 0$. As this fixed and perfectly reflecting mirror will, by assumption, possess no dynamics, then we will model the second mirror only by the MOF model. This second mirror possesses a mirosc internal degree of freedom and will be free to move in response to the forces imparted by the field. The motion of this second mirror is assumed to be small relative to the size of the cavity, and to move on a time scale much longer than all other time scales in the problem. The partial reflectivity of the second mirror allows, for example, a laser field, generated by an external source $J_{\text{ext}}(x^\alpha)$, to couple to the cavity.

A. Arbitrary bilinear coupling strength

The MOF Lagrangian for the system described in the previous paragraph is given by Eq. (3.7):

$$L = \frac{1}{2} \int dx [\dot{\Phi}^2(x^\alpha) - \Phi'^2(x^\alpha) + 2J_{\text{ext}}(x^\alpha)\Phi(x^\alpha)]$$

$$+ \frac{m}{2} [\dot{q}^2(t) - \Omega^2 q^2(t)] + \frac{M}{2} \left[\dot{Z}^2(t) - \Omega_0^2 Z^2(t) \right]$$
$$+ \lambda q(t) \Phi(t, L + Z(t)), \tag{5.1}$$

where we have included an external source $J_{\text{ext}}(x^\alpha)$ for the field and the second mirror (the dynamical one) has coordinates $x = L + Z(t)$ and moves within a harmonic potential with natural frequency Ω_0. The Euler-Lagrange equations for the field, the mirosc, and the coordinates of the movable mirror are

$$\partial_t^2 \Phi(x^\alpha) - \partial_x^2 \Phi(x^\alpha) = J_{\text{ext}}(x^\alpha) + \lambda q(t) \delta(x - L - Z(t)), \tag{5.2}$$

$$\ddot{q}(t) + \Omega^2 q(t) = \frac{\lambda}{m} \Phi(t, L + Z(t)), \tag{5.3}$$

$$\ddot{Z}(t) + \Omega_0^2 Z(t) = \frac{\lambda}{M} q(t) \, \partial_x \Phi(t, L + Z(t)). \tag{5.4}$$

Our procedure is to solve (5.2) for the field and eliminate its appearance in the remaining equations of motion. We then solve for the mirosc to find the equations of motion for the mirror's motion in the regime where its displacement is small compared to the cavity length L. The technical details of the calculation are given in Appendix C. In the end, we find the following equation of motion for the mirror's center-of-mass motion:

$$M\ddot{Z}(t) + M\Omega_0^2 Z(t) = \mathcal{F}[Z(t)], \tag{5.5}$$

where $\mathcal{F}[Z(t)]$ accounts for the external forces and back reaction from the cavity field and mirosc and is given by

$$\mathcal{F}[Z(t)] = \lambda q_0(t) \left(\partial_x F_{\text{ext}}(t,L) + \frac{\lambda}{2} q_0(t - 2L) \right)$$
$$+ \lambda q_0(t) \left(\partial_x^2 F_{\text{ext}}(t,L) - \frac{\lambda}{2} \dot{q}_0(t - 2L) \right) Z(t)$$
$$+ \lambda q_0(t) \left(-\frac{\lambda}{2} \dot{q}_0(t - 2L) Z(t) \right.$$
$$- \frac{\lambda}{2} q_0(t - 2L) \dot{Z}(t - 2L)$$
$$\left. - \frac{\lambda}{2} \dot{q}_0(t - 2L) Z(t - 2L) \right)$$
$$+ \lambda q_1(t) \left(\partial_x F_{\text{ext}}(t,L) + \frac{\lambda}{2} q_0(t - 2L) \right)$$
$$+ \frac{\lambda^2}{2} q_0(t) q_1(t - 2L). \tag{5.6}$$

Note that the general motion of the mirror as influenced by the cavity field is described by a delay integro-differential equation.

The back-reaction terms above will be shown to lead to several effects. First, the driven field will build up in amplitude inside the cavity formed by the perfect mirror and the mirror-oscillator. This will lead to a spatially varying radiation pressure and a shift in the frequency of the mirror's mechanical motion. Next, depending on the equilibrium position of the mirror the cavity field can either accept from or donate energy to the mirror's motion arising from retardation effects (see, e.g., [1] for a detailed explanation of cooling due to retardation).

Finally, non-Markovian effects will be present which show how the mirror's motion in the past influences its future movements; these effects are accounted for in time-delayed and integral terms.

B. The weak-coupling regime

We will specialize here to a specific external source J_{ext} and explore the mirror cooling in the weak-coupling regime, when $\lambda^2/(m\Omega^3) \ll 1$. For many systems of physical interest there exists a large separation between the values of the cavity frequency and the oscillation frequency Ω_0, which allows for a multiple-time-scale analysis. In the following we will assume that the cavity frequency, the mirosc's frequency Ω, and the pump frequency Ω_D are all much larger than the frequency of the mirror's mechanical motion Ω_0. Under these circumstances we may time average the mirror's equation of motion in Eq. (5.5) over the pump period $2\pi/\Omega_D$. Since the mirror's mechanical motion is very slow compared to this pumping time scale its trajectory can be safely factored out of any time-averaging integrals so that

$$\langle\langle Z(t)(\cdots)\rangle\rangle \equiv \frac{1}{T}\int_t^{T+t} dt'\, Z(t')(\cdots) \approx \frac{Z(t)}{T}\int_t^{T+t} dt'(\cdots),$$
(5.7)

where $\langle\langle\cdot\rangle\rangle$ denotes the time average.

The details of the calculation are given in Appendix D. In the end, we find that (5.5) becomes

$$M\ddot{Z}(t) + \Gamma(L)\dot{Z}(t) + M[\Omega_0^2 - \Delta\Omega^2(L)]Z(t) = F_{rad}(L),$$
(5.8)

which is simply the equation for a forced, damped harmonic oscillator with mass M, frequency $[\Omega_0^2 - \Delta\Omega^2(L)]^{1/2}$, and damping coefficient $\Gamma(L)$. Notice that the latter two quantities depend explicitly on the length of the cavity. After time averaging, the explicit form for the radiation pressure is given by

$$F_{rad}(L) = \lambda^2\alpha\tilde{D}(\Omega_D)\left(\alpha'^* + \frac{\lambda^2}{2}\alpha^*\tilde{D}^*(\Omega_D)e^{-i2\Omega_D L}\right),$$

the frequency shift is given by

$$\begin{aligned}
M\Delta\Omega^2(L) = &-i\Omega_D\lambda^2\alpha\tilde{D}(\Omega_D) \\
&\times\left(\alpha'^* + \frac{3\lambda^2}{2}\alpha^*\tilde{D}^*(\Omega_D)e^{-i2\Omega_D L}\right) \\
&+ \lambda^2\tilde{D}(\Omega_D)(\alpha' + \lambda^2\alpha\tilde{D}(\Omega_D)e^{i2\Omega_D L}) \\
&\times(\alpha'^* + \lambda^2\alpha^*\tilde{D}^*(\Omega_D)\cos 2\Omega_D L),
\end{aligned}$$

and the damping coefficient is given by

$$\Gamma(L) = \frac{\lambda^4}{2}|\alpha\tilde{D}(\Omega_D)|^2\cos 2\Omega_D L. \tag{5.9}$$

Figure 3 shows the force on the mirror due to the resulting radiation pressure $F_{rad}(L)$ (solid line) and the damping constant $\Gamma(L)$ (dashed line) as a function of the movable mirror's unperturbed position L from the static mirror at the origin. The parameter values chosen for these plots are given in the caption. We observe that when the cavity is pumped by an external source, the field energy inside builds up and
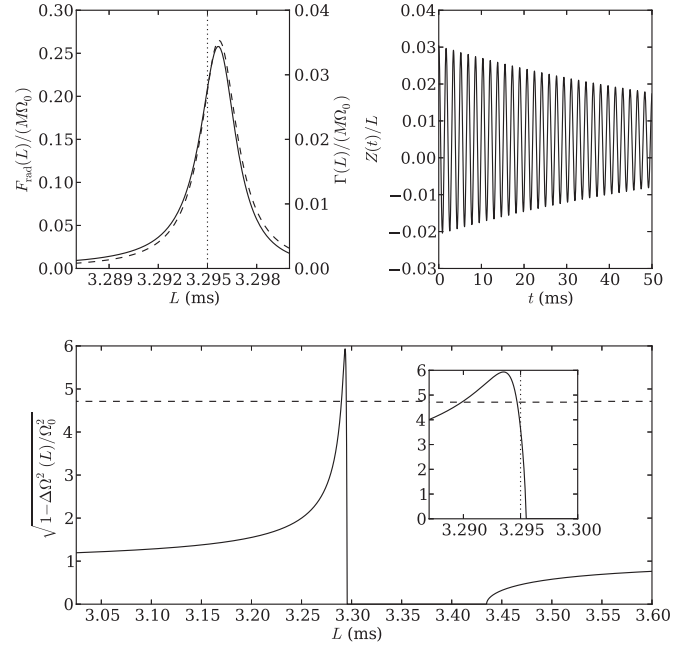

FIG. 3. Plots of the force on the mirror from radiation pressure (top left, solid line), the damping constant $\Gamma(L)$ (top left, dashed line), and the fractional change in the mirror's natural frequency (bottom) as a function of the unperturbed cavity length L. The dotted vertical line indicates the value of $L = 3.295$ ms. The dashed line in the bottom panel shows an estimate of where the weak-coupling approximation begins to break down. The inset shows the fractional change in frequency for $L \in [3.287, 3.300]$ ms. The top right panel shows the evolution of the perturbed trajectory as a function of time with $L = 3.295$ ms. The parameter values, in units where $c = 1$, used to make these plots are as follows: $(m,M) = (1, 2\,000)$ kg, $(\Omega_0, \Omega_D, \Omega) = (1, 300\pi \approx 942.5, 942)$ Hz, $\lambda = 5\,000$ s^2, and $A = 10$ s^{-2}.

results in a force from radiation pressure $F_{rad}(L)$ that varies depending on the size of the unperturbed cavity.

The gradient of the radiation pressure and the Markov approximation of the integral terms, i.e., those terms containing $q_1(t)$, leads to a shift in the oscillation frequency of the mirror's center-of-mass motion. These optical spring effects are quantified by the term $\Delta\Omega(L)$ and change depending on L. The bottom panel of Fig. 3 shows the fractional change in the mirror's natural frequency $[1 - \Delta\Omega^2(L)/\Omega_0^2]^{1/2}$, which can become imaginary precisely where the real part goes to zero in that plot. An important point to note here is that our weak-coupling approximation is valid when $|\Omega_0 - \Delta\Omega(L)| \ll \Omega_D$. For the values indicated in the caption and with $L = 3.295$ ms we see that the mirror's modified natural frequency satisfies $\sqrt{\Omega_0^2 - \Delta\Omega^2} \approx 3.64$ Hz $\ll \Omega_D$, which is consistent with the weak-coupling approximation. It is also important to mention that the mirror's motion can become unstable when the mirror is to the right of the resonance, namely, the mirror's spring constant, i.e., $K = M[\Omega^2 - \Delta\Omega^2(L)]$, becomes negative as shown in Fig. 3. For the parameter values given in the caption of Fig. 3, the moving mirror's motion is damped (the top right panel in Fig. 3) and exemplifies the "cooling" aspect of this classical system to dissipate its input energy into the cavity field.

VI. REDUCTION OF MOF MODEL TO MODELS WITH Nx COUPLING

In the previous section we used the MOF model in the classical regime to describe the damped motion of one mirror of a cavity forced by interactions with an external (laser) field. In the corresponding quantum theory of mirror cooling, one usually models the interaction between the mirror and the field by the radiation pressure $\sim \hat{N}\hat{x}$, where $\hat{N}$ is the number operator of quanta (photons) impinging on the mirror's surface and $\hat{x}$ is the position operator of the mirror [1,2,31]. We will refer to this interaction as "Nx coupling."

The basic motivation for this type of interaction can be easily understood by considering the Hamiltonian for a single cavity mode of the form $H_{\text{cav}} \sim \omega_{\text{cav}}(L)a^{\dagger}a$, where a ($a^{\dagger}$) is the annihilation (creation) operator for field quanta and $\omega_{\text{cav}}(L)$ is the frequency of a cavity mode of size L. Since the frequency of the cavity modes scales as the inverse cavity length $\omega_{\text{cav}}(L) \sim 1/L$, when we allow the cavity length to vary by a small amount x, the frequency is perturbed to leading order as $\omega_{\text{cav}}(L + x) \approx \omega_{\text{cav}}(L)(1 - x/L + \cdots)$. For small cavity-length changes the Hamiltonian becomes $H_{\text{cav}} \approx \omega_{\text{cav}}(L)(1 - x/L)a^{\dagger}a$.

In this section, we show how the quantum MOF model relates to models with Nx coupling. In doing so, we highlight the assumptions that must be made to connect the two models. We thereby demonstrate that the MOF model should be an improvement of the oft-used background-field approximation for the cavity field [30,33]. In particular, the MOF model should be very useful for studying optomechanical systems having low numbers of cavity photons, where quantum effects can become quite interesting and important.

Consider a cavity formed by two mirrors. As in Sec. V, we take the first mirror (at $x = 0$) to be fixed for all time and perfectly reflecting so that the field satisfies Dirichlet boundary conditions at the origin. We assume the second mirror to be partially transmitting and dynamical with small perturbations to its equilibrium position at $x = L > 0$. The second mirror will be described by the MOF model. Recall Hamiltonian (3.20) for a slowly moving mirror:

$$H = \frac{1}{2} \int dx \left\{ \Pi^2(x^{\alpha}) + [\partial_x \Phi(x^{\alpha})]^2 - 2J_{\text{ext}}(x^{\alpha})\Phi(x^{\alpha}) \right\}$$
$$+ \frac{p^2(t)}{2m} + \frac{1}{2}m\Omega^2 q^2(t) + \frac{P^2(t)}{2M} + V(Z(t))$$
$$- \lambda q(t)\Phi(t, L + Z(t)), \tag{6.1}$$

where we have included an external source $J_{\text{ext}}(x^{\alpha})$ for the field. We shall show, by making a number of assumptions, that the interaction component of the above Hamiltonian

$$H_{\text{int}} = -\lambda \int dx \, q(t)\Phi(t, L + Z(t)) \tag{6.2}$$

can be reduced to the Nx coupling.

The internal physics of the mirror for many standard radiation pressure cooling calculations is accounted for phenomenologically through the introduction of a cavity quality factor which accounts for the dissipation of field energy from within the cavity. Specifically, the MOF model accounts for the detailed information of the mirror's internal dynamics. We first will solve for the mirosc to find its effect on the mirror's motion. In this way we trade the microscopic information

about the mirror for a macroscopic description in terms of the mirror's susceptibility, which will establish the link between Nx coupling and the MOF model.

The Heisenberg equations of motion for the field [Eq. (3.2)] and the mirosc [Eq. (3.3)] variables are

$$\partial_t^2 \Phi(x^{\alpha}) - \partial_x^2 \Phi(x^{\alpha}) = \lambda q(t)\delta(x - L - Z(t)) + J_{\text{ext}}(x^{\alpha}), \tag{6.3}$$

$$\ddot{q}(t) + \Omega^2 q(t) = \frac{\lambda}{m} \Phi(t, L + Z(t)). \tag{6.4}$$

We can eliminate the field's explicit dependence on the mirosc $q(t)$ by solving (6.4) and plugging the solution into the wave equation (6.3). In the regime where the mirosc evolves adiabatically so that $|\ddot{q}| \ll |\Omega^2 q|$ the approximate solution to (6.4) is given by

$$q(t) \approx \frac{\lambda}{\kappa} \Phi(t, L + Z(t)) - \frac{\lambda}{\kappa} \frac{\ddot{\Phi}(t, L + Z(t))}{\Omega^2} + \cdots, \tag{6.5}$$

where the second term on the right side is a correction to the leading-order, instantaneous solution and is due to the fact that the full mirosc dynamics is generally non-Markovian. This can be seen from the general solution of (6.4):

$$q(t) = q_h(t) + \frac{\lambda}{m} \int dt' \, \chi(t - t')\Phi(t', L + Z(t')), \tag{6.6}$$

where $q_h(t)$ is the homogeneous solution and $\chi(t - t')$ is the retarded Green's function (or susceptibility) for the mirosc

$$\chi(t - t') = \theta(t - t')\frac{\sin \Omega(t - t')}{\Omega}, \tag{6.7}$$

where $\theta(t - t')$ is the Heaviside step function. More specifically, the mirosc receives contributions from the past as implied by the integral in Eq. (6.6). However, the approximation (6.5) is valid if the mirosc degree of freedom is light, thus responding *nearly* instantaneously to external influences.

Substituting the approximate mirosc solution (6.5) into the wave equation (6.3) gives the effective dynamics for the field:

$$\partial_t^2 \Phi(x^{\alpha}) - \partial_x^2 \Phi(x^{\alpha})$$
$$= J_{\text{ext}}(x^{\alpha}) + \frac{\lambda^2}{\kappa} \Phi(t, L + Z(t))\delta(x - L - Z(t))$$
$$- \frac{\lambda^2}{\kappa\Omega^2} \ddot{\Phi}(t, L + Z(t))\delta(x - L - Z(t)) + \cdots. \tag{6.8}$$

Notice that (6.8) can be derived from the following effective Lagrangian:

$$L_{\text{eff}} = \frac{1}{2} \int dx (\partial_{\alpha}\Phi\partial^{\alpha}\Phi + 2J_{\text{ext}}\Phi) + \frac{1}{2}M\dot{Z}^2 - V(Z)$$
$$+ \frac{\lambda^2}{2\kappa} \int dt \, \Phi^2(t, L + Z(t))$$
$$+ \frac{\lambda^2}{2\kappa\Omega^2} \int dt \, \dot{\Phi}^2(t, L + Z(t)) + \cdots, \tag{6.9}$$

where $\cdots$ denotes the higher-order terms from (6.5). The interaction Hamiltonian corresponding to the above effective

Lagrangian is found to be

$$H_{\text{eff int}} = -\frac{\lambda^2}{2\kappa} \Phi^2(t, L + Z(t))$$
$$+ \frac{\lambda^2}{2\kappa\Omega^2} \Pi^2(t, L + Z(t)) + \cdots . \quad (6.10)$$

Assuming that $Z(t) \ll L$, we may expand the effective interaction Hamiltonian in Eq. (6.10) to find

$$H_{\text{eff int}} = H_{\text{eff int}}^{(0)} + H_{\text{eff int}}^{(1)} + O(Z^2), \quad (6.11)$$

where

$$H_{\text{eff int}}^{(0)} = -\frac{\lambda^2}{2\kappa} \Phi^2(t, L) + \frac{\lambda^2}{2\kappa\Omega^2} \Pi^2(t, L) + \cdots, \quad (6.12)$$

$$H_{\text{eff int}}^{(1)} = -\frac{\lambda^2}{\kappa} Z(t) \Phi(t, L) \partial_x \Phi(t, L)$$
$$+ \frac{\lambda^2}{\kappa\Omega^2} Z(t) \Pi(t, L) \partial_x \Pi(t, L) + \cdots . \quad (6.13)$$

Notice that the leading-order interaction Hamiltonian is independent of $Z(t)$, so that it exerts no force on the movable mirror. In fact, one can group $H_{\text{eff int}}^{(0)}$ with the free Hamiltonian for the field that, when taken together, describes the free evolution of the field in a cavity where one mirror is fixed at $x = 0$ and is perfectly reflecting and the other mirror is fixed at $x = L$ but is partially transmitting. The remaining terms in the effective interaction Hamiltonian describe the perturbative response of the second mirror to its coupling with the field and vice versa.

To leading order in $Z(t)$, we can express the field in terms of a homogeneous solution via the cavity's normal modes and in terms of the external source J_{ext},

$$\Phi(x^\alpha) \approx \sum_k N_k [a_k u_k(x) e^{-i\omega_k t} + \text{H.c.}]$$
$$+ \int d^2 x' \, G_{\text{ret}}^{\text{cav}}(x^\alpha, x'^\alpha) J_{\text{ext}}(x'^\alpha), \quad (6.14)$$

where H.c. is the Hermitian conjugate of the preceding terms and $u_k(x)$ are the normal modes of the cavity and satisfy

$$\left(\partial_x^2 + \omega_k^2 + \frac{\lambda^2}{\kappa} \delta(x - L) \right) u_k(x) = 0, \quad (6.15)$$

such that $u_k(0) = 0$ since the mirror at $x = 0$ is perfectly reflecting. The retarded Green's function here satisfies

$$\left(\partial_x^2 + \omega_k^2 + \frac{\lambda^2}{\kappa} \delta(x - L) \right) G_{\text{ret}}^{\text{cav}}(\omega_k; x, x') = -\delta(x - x'),$$

with Dirichlet boundary conditions at the origin, $G_{\text{ret}}^{\text{cav}}(\omega_k; x, 0) = 0$ and $G_{\text{ret}}^{\text{cav}}(\omega_k; 0, x') = 0$.

N_k is chosen so that $[\Phi(t, x), \Pi(t, x')] = i\hbar\delta(x - x')$ for x and x' greater than zero. These commutation relations require a_k and $a_k^\dagger$ to be annihilation and creation operators, respectively. For the following, we focus entirely on the component of the interaction Hamiltonian coming from the field inside the cavity. The field outside of the cavity gives rise to a constant and position-independent radiation pressure that only yields a shift in the equilibrium position of the mirror at $x = L$.

If the cavity is pumped by a laser beam with a frequency slightly detuned from one of the cavity resonances and if the

cavity quality factor is large, then the cavity field, represented as a mode sum, can be well approximated by a single mode so long as the frequency of the mirror's center-of-mass motion is small enough that it cannot couple adjacent cavity modes. Expressing the field in terms of the fundamental cavity resonance, we find, at linear order $Z(t)$, that the interaction Hamiltonian (6.13) is given by

$$H_{\text{int}}^{(1)} \approx -\frac{\lambda^2}{\kappa} Z(t) \{ N_k [a_k u_k(L) e^{-i\omega_k t} + \text{H.c.}] + \tilde{F}_{\text{ext}}(t, L) \}$$
$$\times \{ N_k [a_k u_k'(L) e^{-i\omega_k t} + \text{H.c.}] + \partial_x \tilde{F}_{\text{ext}}(t, L) \} + \cdots,$$

where $\cdots$ refers to corrections arising from time derivatives of the field appearing in Eq. (6.13) and

$$\tilde{F}_{\text{ext}}(x^\alpha) \equiv \int d^2 x' \, G_{\text{ret}}^{\text{cav}}(x^\alpha; x'^\alpha) J_{\text{ext}}(x'^\alpha). \quad (6.16)$$

For many systems of interest the frequency of the fundamental cavity mode is much larger than the typical frequency of the mirror's motion (i.e., $\Omega_0/\omega_k \ll 1$). Under such conditions the mirror's position changes adiabatically over many oscillations of the cavity field, allowing a time average (denoted by double angle brackets) of the effective interaction Hamiltonian

$$\langle\langle H_{\text{int}}^{(1)} \rangle\rangle = \frac{1}{NT} \int_0^{NT} dt \, H_{\text{int}}^{(1)}. \quad (6.17)$$

Here, T is the period of the cavity's fundamental mode, and N is a large integer such that $(2\pi)/\Omega_0 \gg NT$. Since $Z(t)$ is approximately constant over the entire integration range, it can be taken outside of the time average, giving

$$\langle\langle H_{\text{int}}^{(1)} \rangle\rangle \approx -\frac{\lambda^2}{\kappa} Z(t) \tilde{F}_{\text{ext}}(t, L) \partial_x \tilde{F}_{\text{ext}}(t, L)$$
$$- \left(\frac{\lambda^2}{2\kappa} |N_k|^2 u_k'(L) u_k^*(L) \right) Z(t) a_k^\dagger a_k + \text{H.c.}$$

This step is equivalent to taking the rotating-wave approximation. The key point is that the first term on the right side is a classical radiation pressure originating solely from the external source, while the second term is a quantum-mechanical radiation pressure and is, in fact, the Nx coupling.

Before concluding this section, we collect the main assumptions used in relating the MOF model to the phenomenological radiation pressure interaction Hamiltonian. The assumptions are as follows: (1) The movable mirror is only ever slightly perturbed from its otherwise equilibrium position at $x = L$. (2) The cavity frequency is much less than the natural frequency of the mirosc. (3) The cavity has a high quality factor. (4) The cavity is pumped by a laser at a frequency slightly detuned from one of the cavity resonances. (5) The cavity frequency is much greater than the typical time scale associated with the mirror's motion (i.e., the natural period if in a harmonic trap). Under these assumptions we have shown that the effective interaction between the mirror and the cavity field is given by an Nx coupling. One can extend our derivation here by including the corrections from the generally non-Markovian mirosc dynamics, as discussed below (6.5).

It is possible that the Nx coupling can be obtained using a different setup and assumptions. However, our purpose here is not to elucidate all the ways that the Nx coupling can be

derived from the MOF model, but rather to show that it *can* be derived from a microphysics model of a moving mirror.

VII. MOF MODEL AND QUANTUM BROWNIAN MOTION

In this section we shall establish a connection between the MOF model for N moving mirrors and N harmonic oscillators interacting with a bath of harmonic oscillators that constitute an environment for the N oscillators. The latter system has a long and well-developed history for providing a simple model with which to study quantum Brownian motion. Hence, if a relationship between the MOF model and QBM exists, then one should be able to exploit the results of many previous studies [regarding decoherence, (dis)entanglement, fluctuation-dissipation relations, etc.] to apply towards moving mirror systems. We show here that such a relationship does indeed exist.

A. Static mirrors and QBM

Consider a mirror at rest that is fixed at $Z(t) = 0$ for all time. The MOF Hamiltonian for this configuration follows from (3.20):

$$H = \frac{1}{2} \int dx \{\Pi^2(x^\alpha) + [\partial_x \Phi(x^\alpha)]^2\}$$
$$+ \frac{p^2(t)}{2m} + \frac{1}{2}m\Omega^2 q^2(t) - \lambda q(t)\Phi(t,0). \quad (7.1)$$

It is well known that a field can be represented as a continuum of harmonic oscillators, some of which have arbitrarily large natural frequencies. However, such large frequencies are not usually physically relevant (and often lead to divergences that must be properly handled with well-established renormalization techniques) so that one can simply impose a cutoff frequency Λ, which has the effect of ensuring that all calculated quantities are finite [67].

The mode decomposition of the field is

$$\Phi(t,x) = \sum_k \sum_{\sigma=1}^{2} \varphi_k^\sigma(t) u_k^\sigma(x). \quad (7.2)$$

If we restrict the field to the interior of a one-dimensional (but large) volume V, then the normal modes of the field are simply

$$u_k^1(x) = (2V\omega_k)^{-1/2} \cos kx, \quad (7.3)$$

$$u_k^2(x) = (2V\omega_k)^{-1/2} \sin kx, \quad (7.4)$$

so that the time dependence of the kth mode has the following representation in terms of creation and annihilation operators:

$$\varphi_k^1(t) = a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}, \quad (7.5)$$

$$\varphi_k^2(t) = i(a_k e^{-i\omega_k t} - a_k^\dagger e^{i\omega_k t}). \quad (7.6)$$

In terms of this mode decomposition, the Hamiltonian is

$$H = \frac{1}{2} \sum_k \sum_{\sigma=1}^{2} \left[(\pi_k^\sigma)^2 + k^2(\varphi_k^\sigma)^2\right] + \frac{p^2}{2m} + \frac{1}{2}m\Omega^2 q^2$$
$$- \lambda \sum_k \sum_{\sigma=1}^{2} q(t) u_k^\sigma(0)\, \varphi_k^\sigma(t). \quad (7.7)$$

Notice that the coupling constant λ in the last term can be grouped with the mode function $u_k^\sigma(0)$ to give an effective coupling constant that depends on the particular mode $C_k^\sigma \equiv \lambda u_k^\sigma(0)$. Therefore, the Hamiltonian for this system is

$$H = \frac{1}{2} \sum_k \sum_{\sigma=1}^{2} \left[(\pi_k^\sigma)^2 + k^2(\varphi_k^\sigma)^2\right] + \frac{p^2}{2m} + \frac{1}{2}m\Omega^2 q^2$$
$$- \sum_k \sum_{\sigma=1}^{2} C_k^\sigma q(t)\varphi_k^\sigma(t)$$
$$=: H_{1-\text{HOQBM}}, \quad (7.8)$$

which is precisely the Hamiltonian for a harmonic oscillator $q(t)$ coupled to an environment composed of a bath of harmonic oscillators $\{\varphi_k^\sigma(t)\}$. In other words, the MOF model for a mirror at rest can be related to quantum Brownian motion where the field provides the environment with which the mirosc interacts. QBM has a long history and is well studied, so that results already found in that literature can be applied directly to the interaction of a field with a static mirror via the MOF model. For example, the master equation is exactly known for this system [68], so one can study its behavior near the perfectly reflecting limit where $\lambda \to \infty$ or, equivalently, $m \to 0$, as well as in a nonzero-temperature regime.

A similar result holds for N mirrors held at rest at positions $x = L_a$, with $a = 1, \ldots, N$. It is straightforward to see that the corresponding Hamiltonian, when decomposing the field into harmonic oscillators, is

$$H = \frac{1}{2} \sum_k \sum_{\sigma=1}^{2} \left[(\pi_k^\sigma)^2 + k^2(\varphi_k^\sigma)^2\right]$$
$$+ \sum_{a=1}^{N} \left(\frac{p_a^2}{2m_a} + \frac{1}{2}m_a\Omega_a^2 q_a^2 - \sum_k \sum_{\sigma=1}^{2} C_{ka}^\sigma q_a(t)\varphi_k^\sigma(t)\right)$$
$$=: H_{N-\text{HOQBM}}, \quad (7.9)$$

where the effective bilinear coupling constant is $C_{ka}^\sigma \equiv \lambda u_k^\sigma(L_a)$. Therefore, N static mirrors in the MOF model correspond to N harmonic oscillators (mirosc variables) coupled to a bath of oscillators (the field). For $N = 2$ oscillators coupled to a general environment, the exact master equation has been derived in Ref. [69] and thus can be used to provide a different perspective and new insights in the description of a field coupled to two partially transmitting mirrors via the MOF model.

B. Slowly moving mirrors and QBM

Next, we will find the relationship between slowly moving mirrors in the MOF model and quantum Brownian motion. Let us first consider one mirror since the result for N mirrors will generalize in an obvious way. Assume that the mirror is in an externally generated potential $V(x)$, such as a harmonic trap. Then the Hamiltonian in Eq. (7.7) is

$$H = \frac{1}{2} \sum_k \sum_{\sigma=1}^{2} \left[(\pi_k^\sigma)^2 + k^2(\varphi_k^\sigma)^2\right] + \frac{p^2}{2m} + \frac{1}{2}m\Omega^2 q^2$$
$$+ \frac{P^2}{2M} + V(Z) - \lambda \sum_k \sum_{\sigma=1}^{2} q(t) u_k^\sigma(Z(t))\, \varphi_k^\sigma(t), \quad (7.10)$$

where we have included the world-line variable in the Hamiltonian. Notice that from a QBM perspective, the effective coupling constant acquires a time dependence since the mode function is now time dependent, $u_k^\sigma(Z(t))$. However, if the potential $V(x)$ restricts the motion of the mirror to be only small perturbations from its equilibrium position at $x = 0$, then we may expand the mode function about the origin so that the interaction term above becomes

$$
-\lambda \sum_k \sum_{\sigma=1}^2 q(t) u_k^\sigma(Z(t)) \varphi_k^\sigma(t)
$$
$$
= -\sum_k \sum_{\sigma=1}^2 C_k^\sigma q(t) \varphi_k^\sigma(t)
$$
$$
- \lambda Z(t) \sum_k \sum_{\sigma=1}^2 \partial_x u_k^\sigma(0) q(t) \varphi_k^\sigma(t) + O(Z^2). \quad (7.11)
$$

Therefore, Hamiltonian (7.10) is equal to an unperturbed Hamiltonian, given by the one-harmonic oscillator QBM Hamiltonian in Eq. (7.8) plus an interaction Hamiltonian that describes perturbations due to the small displacement of the mirror that arise from interactions between the field oscillators and the mirosc,

$$
H = H_{1-\text{HOQBM}} - \lambda Z(t) \sum_k \sum_{\sigma=1}^2 \partial_x u_k^\sigma(0) q(t) \varphi_k^\sigma(t) + O(Z^2).
$$
$$
(7.12)
$$

Hence, one can compute the perturbations of, for example, the exact master equation for one-harmonic oscillator QBM to study the behavior of a movable, partially transmitting mirror. Notice that if $V(Z) = M\Omega_0^2 Z^2(t)/2$, then (7.12) describes a nonlinearly coupled QBM system where the mirosc and the mirror's position are the two oscillators in an open system that couples to the bath provided by the field oscillators. The nonlinearity is only in the mirror's position [i.e., from the $O(Z^2)$ terms above], but the mirosc and the field oscillators still couple to each other bilinearly.

The generalization to N mirrors should be obvious, with the Hamiltonian describing the system being

$$
H = H_{N-\text{HOQBM}} - \lambda \sum_{a=1}^N Z_a(t) \sum_k \sum_{\sigma=1}^2 \partial_x u_k^\sigma(L_a) q(t) \varphi_k^\sigma(t)
$$
$$
+ O(Z_a^2), \quad (7.13)
$$

where the unperturbed position of the ath mirror is at $x = L_a$. In particular, one can compute the perturbations of, for example, the exact master equation for two-harmonic oscillator QBM [69] to study entanglement, decoherence, etc., of a cavity with movable, partially transmitting mirrors.

VIII. SUMMARY AND FURTHER DEVELOPMENTS

In this paper we constructed a microphysics model of moving mirrors interacting with a quantum field. We introduced a harmonic oscillator (a "mirosc") model describing the internal degrees of freedom of the mirror that couples to the incident radiation, thereby providing a mechanism for

the dynamical interplay of the mirror-field system. Since the field can transfer (receive) energy and momentum to (from) the mirosc, collecting them serves the function of a partially reflecting or transmitting mirror. We showed that this mirror-oscillator-field (MOF) system can perfectly reflect or perfectly transmit radiation depending on the values of the mirosc mass m, natural frequency Ω, and coupling strength λ to the field. Perfect reflection can be attained in three ways: (1) $m \to 0$, 2, $\lambda \to \infty$, and (3) the frequency ω of an incident wave is equal to the mirosc natural frequency Ω. Limits (1) and (2) exhibit perfect reflection (or nearly so) among a broad frequency bandwidth, whereas limit (3) strongly reflects modes with frequencies near Ω because of a resonant excitation of the mirosc.

The MOF model reduces to several commonly used models of moving mirrors in a quantum field. We showed that when the mirosc variable $q(t)$ evolves adiabatically ($|\ddot{q}| \ll |\Omega^2 q|$) or when $m \to 0$ but $m\Omega^2 = \kappa$ remains constant, then the MOF model reduces to the Barton and Calogeracos (BC) model [40] of a partially transmitting moving mirror. The free parameter in the BC model γ is related to the mirosc parameters of the MOF model (m,Ω,λ) by $\gamma = \lambda^2/(2m\Omega^2)$. The "auxiliary field" model of Golestanian and Kardar [45,46] arises from the MOF model in the limit that $m \to 0$. In this limit, there is no mirosc dynamics, and $q(t)$ becomes an auxiliary variable. In the quantum theory, $q(t)$ may have any possible realization [see (2.28)], which manifests as a Dirichlet boundary condition on the field at the location of the mirror and thus perfectly reflects incident radiation [see Eq. (2.30)].

The MOF model can also be connected to previous work on moving mirrors forming a perfect cavity, such as that of Law [41]. We also showed that our model reduces to the phenomenological model of a mirror interacting with a cavity field via the radiation pressure exerted on the mirror's surface when a number of assumptions are made (though these may not all be necessary to derive the Nx coupling in other setups). This Nx coupling is often used to describe laboratory setups but may be extended by the MOF model to scenarios where the mirosc does not evolve adiabatically, which may exhibit interesting macroscopic (or perhaps mesoscopic) quantum phenomena. Additionally, Nx-type coupling provides the leading-order corrections to the classical radiation pressure coupling when the cavity is occupied by low photon numbers. The model we present in this paper will remain useful even when the necessary conditions for it to match with models with Nx-type coupling are not met, for example, when the mirror motion is sufficient to excite field quanta to higher modes.

The bulk motion of the mirror in the MOF model, which may be relativistic depending on the application, can be derived from an action or a Hamiltonian. In either formulation, we find that the mirror moves with a time-dependent effective mass M_{eff} that is composed of the mirror's rest mass M and the mirror's total internal energy, which comes from the energy of the mirosc and its interaction with the field. We also showed (in a purely classical setting) that the MOF model seems to admit physical solutions despite the use of a point-particle description for the mirror's motion and despite the interaction between the mirror and field resembling that of a charged particle (which can be plagued by pathologies). We demonstrated that when the field is generated by its interaction

with the mirosc *alone* so that there is no external source J_{ext} and no initial field configuration present, then the mirror will follow an inertial trajectory (i.e., constant velocity), which is the correct expected result, in contradistinction to the radiation reaction on a point charge in electrodynamics where the charge may exhibit runaway motions in the absence of any external force acting on the charge.

As an application of the MOF model, we studied the cooling of a mirror by its interactions with an external field in a purely classical context. We found that when the mirosc is weakly coupled to the field, the mirror, when perturbed, will oscillate around its equilibrium configuration while its displacement amplitude decays slowly in time.

An interesting consequence of our MOF model for moving mirrors is that it relates to models of quantum Brownian motion (QBM) in a straightforward manner. The relation essentially follows because the field can be regarded as a continuum of harmonic oscillators. Hence, for N mirrors held at rest, the MOF model is equivalent to N harmonic oscillators in a bath of oscillators (from the field). For $N = 1, 2$, the master equation for such a system in a general environment has been derived *exactly* [68,69] and even for general N [70]. Consequently, the MOF model can be used to study the superposition of two mirrors, the decoherence by and the disentanglement of moving mirrors via a field, etc., to gain insight into these aspects of macroscopic quantum phenomena. We expect that the rich repository of technical tools and physical insights from the study of QBM can be carried over directly to our MOF model for a broad range of applications involving moving mirrors and quantum fields. For example, QBM results for systems at finite temperature may provide a simple way to incorporate thermal effects into the MOF model. We will begin to explore this theme in a subsequent work [53] on the theory of optomechanics from an open quantum system viewpoint.

The generalization of the MOF model to three spatial dimensions can be made, where the mirror is an extended body having some surface geometry. On this surface, we may place a layer of miroscs that play the role of the electrons in a metal gas or dielectric medium providing the mirror's light degrees of freedom and are responsible for reflection of incident radiation over some bandwidth of the electromagnetic spectrum (e.g., optical, as in many metals). Incorporating the electromagnetic field in the MOF model should also be straightforward as its structure is similar to that of a minimally coupled scalar field in the MOF model (see the appendix of [61]).

In a subsequent paper [54] (the second series on optomechanics back-action effects) we will study the full quantum-mechanical evolution of the MOF system in the context of mirror cooling. Therein, we will derive the exact equations of motion describing the mirror's average position. In the most general case we will show that the mirror motion is described by an integro-differential equation exhibiting non-Markovian dynamics. The equations can be simplified through a series of approximations which directly relate to experimentally engineerable quantities, such as the cavity's quality factor, and the relevant time scales for the mirror's internal dynamics. Given the broad range of applicability, these results can be employed to guide theoretical and experimental investigations ranging from the cooling of the center-of-mass motion of movable mirrors, having broadband reflective properties, to

the manipulation of trapped ions near surfaces, possessing narrow-band reflective properties [12].

While the third series on moving mirror-field superposition and entanglement is under investigation, we mention a related work [60], near completion, on applying a moving mirror analog to gravitational physics, namely, the back reaction of Hawking radiation [15] on the evolution of a black hole. There are controversies in some deep issues related to the end state of black-hole evaporation resulting from the Hawking effect, namely, whether complete evaporation of a black hole means the nonunitary evolution of quantum states (see, e.g., Ref. [71]), which violates the basic tenets of theoretical physics, or, if unitarity is preserved, how so? One key ingredient, namely, the back reaction of the emitted radiation on the space-time, has not been taken into account fully or correctly (for a recent update, see Refs. [72,73] and references therein). There are analog studies on how information is shared in the black-hole (harmonic) atom-quantum field system (see, e.g., Refs. [74,75] and references therein) as well as the moving mirror analog problem [17]. The connection was made between the *s*-wave component of Hawking evaporation and the emission of radiation from moving mirrors by the dynamical Casimir effect. But, like the original calculation by Hawking, the back-reaction effects were not treated rigorously. Since the MOF model offers a large degree of flexibility and tractability, we were able to find exact equations of motion for the mirror incorporating the effects of back reaction [60]. These exact solutions, as well as those from the atom-field analogs, can provide new insights into this basic issue in theoretical physics.

APPENDIX A: ABSENCE OF RUNAWAY SOLUTIONS IN THE MOF MODEL

The structure of (3.2) and (3.4) is reminiscent of a field coupled to a scalar point charge, which here is played by the time-dependent mirosc amplitude $q(t)$. In $3 + 1$ dimensions, such a system exhibits a radiation reaction force on the charge proportional to the third time derivative of the particle's position and exhibits the infamous class of unphysical runaway solutions in the absence of any external influences. Below, we show that no such unphysical solutions manifest in our MOF model here. To show this, we first solve the field equation in

(3.2), which gives

$$\Phi(x^\alpha) = \lambda \int d\tau' \, G(x^\alpha; Z^\mu(\tau')) q(\tau'), \qquad (A1)$$

where we ignore the homogeneous solution and where the retarded Green's function in $1 + 1$ space-time dimensions is

$$G(x^\alpha; x'^\alpha) = \tfrac{1}{2}\theta(t - t')\theta(\sigma(x^\alpha, x'^\alpha)), \qquad (A2)$$

where σ is half of the squared distance between x^α and x'^α as measured by the straight line (i.e., a geodesic) connecting them, namely,

$$\sigma(x^\alpha, x'^\alpha) = \tfrac{1}{2}(x^\alpha - x'^\alpha)(x_\alpha - x'_\alpha). \qquad (A3)$$

The derivative of the field evaluated on the world line is then

$$\begin{aligned}
\partial_\nu \Phi(Z^\mu(\tau)) &\equiv [\partial_\nu \Phi(x^\alpha)]_{x^\alpha = Z^\alpha(\tau)} \\
&= \frac{\lambda}{2} \int d\tau' \, \theta(\tau - \tau') \partial_\nu \sigma(Z^\mu(\tau), Z^\mu(\tau')) \\
&\quad \times \delta(\sigma(Z^\mu(\tau), Z^\mu(\tau'))) q(\tau') \\
&\quad + \frac{\lambda}{2} \int d\tau' \, \partial_\nu(\tau - \tau') \delta(\tau - \tau') \\
&\quad \times \theta(\sigma(Z^\mu(\tau), Z^\mu(\tau'))) q(\tau'). \qquad (A4)
\end{aligned}$$

From (A3) it follows that since the mirror's world line is timelike, then $\sigma(Z^\mu(\tau), Z^\mu(\tau'))$ is always positive except at $\tau' = \tau$, where it vanishes. Hence, the δ function in the first line of (A4) receives a contribution only at coincidence, when $\tau' = \tau$. Likewise, the $\delta(\tau - \tau')$ in the second line of (A4) gives support to the integral at coincidence.

To evaluate the integrals in Eq. (A4) we will need to determine the behavior of $\delta(\sigma)$ and $\partial_\nu(\tau - \tau')$ when $\tau' \approx \tau$. This follows by expanding (A3) around $s \equiv \tau' - \tau$ near zero, giving

$$\sigma(Z^\mu(\tau), Z^\mu(\tau')) = \frac{s^2}{2} - \frac{s^4}{24} \dot{U}^\alpha(\tau) \dot{U}_\alpha(\tau) + O(s^5), \quad (A5)$$

where we have used the identities $U^\alpha U_\alpha = 1$, $U^\alpha \dot{U}_\alpha = 0$, and $U^\alpha \ddot{U}_\alpha = -\dot{U}^\alpha \dot{U}_\alpha$, which are valid in the proper time parametrization of the world line. Therefore, writing the δ function in Eq. (A4) as a δ function of s and then expanding (A5) for s near zero gives

$$\delta(\sigma(Z^\mu(\tau), Z^\mu(\tau'))) = \frac{\delta(s)}{|s|}\left(1 + \frac{s^2}{6} \dot{U}^\alpha \dot{U}_\alpha + O(s^3)\right).$$

In addition, the second integral in Eq. (A4) is proportional to

$$\int ds \, \partial_\nu s \, \delta(s)\theta(\sigma)q(\tau + s) = \frac{1}{2}q(\tau)[\partial_\nu s]_{s=0}. \quad (A6)$$

The important point to note is that the first integral in Eq. (A4) is potentially divergent. However, we will show now that no divergence actually manifests.

To see this, we observe that (A3) implies $\partial_\nu \sigma(x^\alpha, x'^\alpha) = x_\nu - x'_\nu$, which, when evaluated on the world line and expanding around s equal zero, yields

$$\partial_\nu \sigma(Z^\mu(\tau), Z^\mu(\tau')) = -s U_\nu(\tau) - \frac{s^2}{2} \dot{U}_\nu + O(s^3). \quad (A7)$$

Note also that the above equation implies that $[\partial_\nu s]_{s=0} = -U_\nu(\tau)$ since from (A5) it follows that $\partial_\nu \sigma = s \partial_\nu s + O(s^3)$.

The integral in Eq. (A4) thus becomes

$$\begin{aligned}
\partial_\nu \Phi(Z^\mu(\tau)) &= \frac{\lambda}{2} \int_{-\infty}^{\infty} ds \, \theta(-s)[-s U_\nu(\tau) + O(s^2)] \\
&\quad \times [q(\tau) + O(s)]\frac{\delta(s)}{|s|}[1 + O(s^2)] \\
&\quad + \frac{\lambda}{4} U_\nu(\tau)q(t). \qquad (A8)
\end{aligned}$$

Evaluating the integral over s and using $2\theta(-s) = 1 - \mathrm{sgn}(s)$, we find that

$$\partial_\nu \Phi(Z^\mu(\tau)) = \frac{\lambda}{2} U_\nu(\tau)q(t), \qquad (A9)$$

which is finite. In addition, the derivative of the field above, which is proportional to U_ν, is contracted with $\eta^{\mu\nu} - U^\mu U^\nu$ in Eq. (3.4) to get the force on the mirror, thereby giving *zero*. Hence, the equation of motion for the mirror's world line from (3.4) is simply

$$\dot{U}^\mu(\tau) = 0, \qquad (A10)$$

and the mirror moves inertially. The reason for this trivial dynamics is because the field is not generated by any external sources and because we have ignored the initial configuration of the field [i.e., homogeneous solutions to the field equation (3.2)]. Both of these types of sources will impart a nontrivial dynamics for the mirror's motion.

APPENDIX B: SCATTERING PROPERTIES OF THE TWO-MIRROR MOF MODEL

In this Appendix, we investigate the scattering properties of incident radiation on two mirrors at rest. The equations of motion for the two-mirror MOF model follow from Sec. IV and are given by

$$\partial_t^2 \Phi - \partial_x^2 \Phi = \sum_{a=1}^{2} \lambda_a q_a(t)\delta(x) \qquad (B1)$$

$$m_a \ddot{q}_a + m_a \Omega_a^2 q_a = \lambda_a \Phi(t, 0), \qquad (B2)$$

with $a = 1,2$ in the second equation. Let a monochromatic plane wave of frequency ω be incident from the left so that

$$\Phi_{\omega L}(t, x) = e^{-i\omega t} \psi_{\omega L}(x). \qquad (B3)$$

The part of the mode $\psi_{\omega L}(x)$ can be found using the linearity of the field equation from which the superposition principle allows us to write the contributions from multiple reflections and transmissions off of and through both mirrors as

$$\begin{aligned}
\psi_{\omega L}(x) &= \theta(-x)\left[e^{i\omega x} + \left(R_1 + T_1 R_2 T_1 \sum_{n=0}^{\infty}(R_1 R_2)^n\right) e^{-i\omega x}\right] \\
&\quad + \theta(L - x)\theta(x)\left[T_1 \sum_{n=0}^{\infty}(R_1 R_2)^n e^{i\omega x}\right. \\
&\quad \left. + T_1 R_2 \sum_{n=0}^{\infty}(R_1 R_2)^n e^{-i\omega x}\right] \\
&\quad + \theta(x - L)\left[T_1 T_2 \sum_{n=0}^{\infty}(R_1 R_2)^n e^{i\omega x}\right]. \qquad (B4)
\end{aligned}$$

The geometric series can be summed for $|R_1 R_2| < 1$, whereby

$$\sum_{n=0}^{\infty} (R_1 R_2)^n = \frac{1}{1 - R_1 R_2}. \quad \text{(B5)}$$

To find the reflection and transmission coefficients in terms of the incident frequency ω we assume that the mirosc is in a steady-state evolution and oscillates at the same frequency of the radiation so that

$$q_a(t) = A_a e^{-i\omega t}. \quad \text{(B6)}$$

The field is continuous at the locations of each mirror,

$$\psi_{\omega L}(0^+) = \psi_{\omega L}(0^-), \quad \text{(B7)}$$

$$\psi_{\omega L}(L^+) = \psi_{\omega L}(L^-), \quad \text{(B8)}$$

and the discontinuity of the spatial derivative is to be consistent with the source of the field equation,

$$-\psi'_{\omega L}(0^+) + \psi'_{\omega L}(0^-) = \lambda_1 A_1, \quad \text{(B9)}$$

$$-\psi'_{\omega L}(L^+) + \psi'_{\omega L}(L^-) = \lambda_2 A_2. \quad \text{(B10)}$$

The mirosc amplitudes A_1, A_2 satisfy the mirosc equations of motion so that we have six equations for the six unknowns $\{R_a, T_a, A_a\}$ (note the subscript $a = 1, 2$). Thus, the reflection and transmission coefficients are

$$R_1 = \frac{i\lambda_1^2}{2m_1\omega(\Omega_1^2 - \omega^2) - i\lambda_1^2}, \quad \text{(B11)}$$

$$T_1 = 1 + R_1, \quad \text{(B12)}$$

$$R_2 = \frac{i\lambda_2^2 e^{2i\omega L}}{2m_2\omega(\Omega_2^2 - \omega^2) - i\lambda_2^2}, \quad \text{(B13)}$$

$$T_2 = 1 + R_2 e^{-2i\omega L}, \quad \text{(B14)}$$

and the amplitudes of oscillation for the miroscs are

$$A_1 = \frac{\lambda_1 T_1}{m_1(\Omega_1^2 - \omega^2)}\left(\frac{1 + R_2}{1 - R_1 R_2}\right), \quad \text{(B15)}$$

$$A_2 = \frac{\lambda_2 T_2}{m_2(\Omega_2^2 - \omega^2)}\left(\frac{T_1 e^{i\omega L}}{1 - R_1 R_2}\right). \quad \text{(B16)}$$

One can check that the identities $|R_a|^2 + |T_a|^2 = 1$ are indeed satisfied. The incident-field mode (B4) can then be written as

$$\psi_{\omega L}(x) = \theta(-x)\left[e^{i\omega x} + \frac{R_1 + R_2 + 2R_1 R_2}{1 - R_1 R_2} e^{-i\omega x}\right]$$
$$+ \theta(L - x)\theta(x)\left[\frac{T_1}{1 - R_1 R_2}(e^{i\omega x} + R_2 e^{-i\omega x})\right]$$
$$+ \theta(x)\left[\frac{T_1 T_2}{1 - R_1 R_2} e^{i\omega x}\right]. \quad \text{(B17)}$$

When the mirror at $x = 0$ is perfectly transmitting and the mirror at $x = L$ is perfectly reflecting, the field mode is

$$\psi_{\omega L}(x) = \theta(L - x)(e^{i\omega x} - e^{i\omega(2L - x)}), \quad \text{(B18)}$$

which vanishes as $x \to L$, as expected. In the complementary case when the mirror at $x = 0$ is perfectly reflecting, the field mode incident from the left is

$$\psi_{\omega L}(x) = \theta(-x)(e^{i\omega x} - e^{-i\omega x}), \quad \text{(B19)}$$

as also expected. Hence, the MOF model describes the partially reflecting and transmitting properties of two, and generally more, mirrors.

APPENDIX C: CALCULATION DETAILS OF CLASSICAL MIRROR COOLING FOR ARBITRARY COUPLING STRENGTH

Our first step will be to solve (5.2) for the field and eliminate its appearance in the remaining equations of motion. Assuming that there is no initial field present [76], so that Φ is generated by J_{ext} and by interactions with the remaining degrees of freedom, then the solution to (5.2) is

$$\Phi(x^\alpha) = \int d^2x' \, G(x^\alpha; x'^\alpha) J_{\text{ext}}(x'^\alpha)$$
$$+ \lambda \int dt' \, G(x^\alpha; t', L + Z(t')) q(t'), \quad \text{(C1)}$$

where the retarded Green's function $G(x^\alpha; x'^\alpha)$ for the field subject to Dirichlet boundary conditions at the fixed mirror is given by

$$G(t, x; t', x') = \tfrac{1}{2}\theta(t - t')\big[\theta\big(\tfrac{1}{2}(t - t')^2 - \tfrac{1}{2}(x - x')^2\big)$$
$$- \theta\big(\tfrac{1}{2}(t - t')^2 - \tfrac{1}{2}(x + x')^2\big)\big]. \quad \text{(C2)}$$

Note that if $x = x' = L > 0$, then

$$G(t, L; t', L) = \tfrac{1}{2}[\theta(t - t') - \theta(t - t' - 2L)]. \quad \text{(C3)}$$

Substituting (C1) into the remaining equations, (5.3) and (5.4), gives

$$\ddot{q}(t) + \Omega^2 q(t) = \frac{\lambda}{m} F_{\text{ext}}(t, L + Z(t))$$
$$+ \frac{\lambda^2}{m} \int dt' \, G(t, L + Z(t); t', L + Z(t')) q(t'), \quad \text{(C4)}$$

$$\ddot{Z}(t) + \Omega_0^2 Z(t)$$
$$= \frac{\lambda}{M} q(t) \partial_x F_{\text{ext}}(t, L + Z(t))$$
$$+ \frac{\lambda^2}{M} q(t) \int dt' \, \partial_x G(t, L + Z(t); t', L + Z(t')) q(t'), \quad \text{(C5)}$$

where

$$F_{\text{ext}}(x^\alpha) \equiv \int d^2x' \, G(x^\alpha; x'^\alpha) J_{\text{ext}}(x'^\alpha) \quad \text{(C6)}$$

is the propagated external source for the field.

Next, we solve for the mirosc variable $q(t)$. At this point we can take advantage of the assumption that $Z(t) \ll L$ so that the typical amplitude of the mirror's motion is much smaller than the size of the cavity. This implies we can write the solution for the oscillator perturbatively as $q = q_0 + q_1 + \cdots$, where $q_n = O(Z^n)$. The equation of motion for the leading-order mirosc dynamics is

$$\ddot{q}_0(t) + \Omega^2 q_0(t) - \frac{\lambda^2}{m} \int dt' \, G(t, L; t', L) q_0(t')$$
$$= \frac{\lambda}{m} F_{\text{ext}}(t, L). \quad \text{(C7)}$$

The solution to (C7) is given by (again, ignoring homogeneous solutions)

$$q_0(t) = \lambda \int_{-\infty}^{\infty} dt' \, D(t-t') F_{\text{ext}}(t',L), \qquad \text{(C8)}$$

where the kernel $D(\tau)$ is found to be

$$D(\tau) = -\int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\omega e^{-i\omega\tau}}{2m\omega(\omega^2 - \Omega^2) + i\lambda^2(1 - e^{2i\omega L})} \qquad \text{(C9)}$$

$$\equiv \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, e^{-i\omega\tau} \tilde{D}(\omega). \qquad \text{(C10)}$$

The equation of motion for the first-order perturbative correction to the mirosc dynamics is

$$\ddot{q}_1(t) + \Omega^2 q_1(t) - \frac{\lambda^2}{m} \int dt' \, G(t,L;t',L) q_1(t')$$

$$= \frac{\lambda}{m} \partial_x F_{\text{ext}}(t,L) Z(t)$$

$$+ \frac{\lambda^2}{m} \int dt' [Z(t)\partial_x + Z(t')\partial_{x'}] G(t,L;t',L) q_0(t'). \quad \text{(C11)}$$

The right side of (C11) simplifies somewhat since (C2) implies that

$$\partial_x G(t,L;t',L) = \tfrac{1}{2}\delta(t-t'-2L) = \partial_{x'} G(t,L;t',L), \qquad \text{(C12)}$$

so (C11) can be written as

$$\ddot{q}_1(t) + \Omega^2 q_1(t) - \frac{\lambda^2}{m} \int dt' \, G(t,L;t',L) q_1(t')$$

$$= \frac{\lambda}{m} \partial_x F_{\text{ext}}(t,L) Z(t) + \frac{\lambda^2}{2m}[Z(t) + Z(t-2L)] q_0(t-2L). \qquad \text{(C13)}$$

Thus, the solution to (C13) is given by

$$q_1(t) = \int_{-\infty}^{\infty} dt' \, D(t-t') \left\{ \lambda \partial_x F_{\text{ext}}(t',L) Z(t') \right.$$

$$\left. + \frac{\lambda^2}{2}[Z(t') + Z(t'-2L)] q_0(t'-2L) \right\}. \qquad \text{(C14)}$$

Finally, we expand the equation of motion for the world line to leading order in $Z(t)$ to find

$$M\ddot{Z}(t) + M\Omega_0^2 Z(t) = \mathcal{F}[Z(t)], \qquad \text{(C15)}$$

where $\mathcal{F}[Z(t)]$ accounts for the external forces and back reaction from the cavity field and mirosc and is given by

$$\mathcal{F}[Z(t)] = \lambda q_0(t) \left(\partial_x F_{\text{ext}}(t,L) + \frac{\lambda}{2} q_0(t-2L) \right)$$

$$+ \lambda q_0(t) \left(\partial_x^2 F_{\text{ext}}(t,L) - \frac{\lambda}{2} \dot{q}_0(t-2L) \right) Z(t)$$

$$+ \lambda q_0(t) \left(-\frac{\lambda}{2} \dot{q}_0(t-2L) Z(t) \right.$$

$$- \frac{\lambda}{2} q_0(t-2L) \dot{Z}(t-2L)$$

$$\left. - \frac{\lambda}{2} \dot{q}_0(t-2L) Z(t-2L) \right)$$

$$+ \lambda q_1(t) \left(\partial_x F_{\text{ext}}(t,L) + \frac{\lambda}{2} q_0(t-2L) \right)$$

$$+ \frac{\lambda^2}{2} q_0(t) q_1(t-2L). \qquad \text{(C16)}$$

APPENDIX D: CALCULATION DETAILS OF CLASSICAL MIRROR COOLING FOR WEAK COUPLING

Let the external source of the field be given by

$$J_{\text{ext}}(x^\alpha) = A \cos \Omega_D t, \qquad \text{(D1)}$$

so that

$$F_{\text{ext}}(x^\alpha) = A \int_{t_i}^{\infty} dt' \int_0^{\infty} dx' \, G(t,x;t',x') \cos \Omega_D t', \quad \text{(D2)}$$

where we take the initial time to be at $t = t_i$ and at the end of the calculation we take the limit $t_i \to -\infty$. Performing the space-time integration gives

$$F_{\text{ext}}(t,L) = \alpha e^{-i\Omega_D t} + \text{c.c.}, \qquad \text{(D3)}$$

where $\alpha = A(e^{i\Omega_D L} - 1)/(2\Omega_D^2)$. The first two spatial derivatives of F_{ext} evaluated at $x = L$ are similarly evaluated, giving

$$\partial_x F_{\text{ext}}(t,L) = \alpha' e^{-i\Omega_D t} + \text{c.c.}, \qquad \text{(D4)}$$

$$\partial_x^2 F_{\text{ext}}(t,L) = i\Omega_D \alpha' e^{-i\Omega_D t} + \text{c.c.}, \qquad \text{(D5)}$$

where $\alpha' = iAe^{i\Omega_D L}/(2\Omega_D)$. In addition, we can also derive the explicit form for $q_0(t)$ given the expression for the external source,

$$q_0(t) = \lambda\alpha \tilde{D}(\Omega_D) e^{-i\Omega_D t} + \text{c.c.} \qquad \text{(D6)}$$

Using these expressions, we shall evaluate the time average of (5.5) over the pump period, $T = 2\pi/\Omega_D$. The time average of the terms independent of $q_1(t)$ in Eq. (5.6) are easily evaluated. However, the time average of the terms in Eq. (5.5) containing $q_1(t)$ requires some elaboration. First, we write $q_1(t)$ in Eq. (C14), with $Z(t)$ and $D(t-t')$ replaced by their Fourier transforms,

$$q_1(t) = \lambda \int_{-\infty}^{\infty} \frac{d\nu \, d\omega}{2\pi \, 2\pi} \int dt' \, Z(\nu) \tilde{D}(\omega) e^{-i\nu t' - i\omega(t-t')}$$

$$\times \left[\partial_x F_{\text{ext}}(t',L) + \frac{\lambda}{2}(1 + e^{i2\nu L}) q_0(t'-2L) \right].$$

Evaluating the t' integral and then integrating over ω gives

$$q_1(t) = \lambda \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} Z(\nu) e^{-i\nu t} \left\{ e^{-i\Omega_D t} \tilde{D}(\nu + \Omega_D) \right.$$

$$\times \left[\alpha' + \frac{\lambda^2}{2}(1 + e^{i2\nu L})\alpha \tilde{D}(\Omega_D) e^{i2\Omega_D L} \right]$$

$$+ e^{i\Omega_D t} \tilde{D}(\nu - \Omega_D)$$

$$\left. \times \left[\alpha'^* + \frac{\lambda^2}{2}(1 + e^{i2\nu L})\alpha^* \tilde{D}^*(\Omega_D) e^{-i2\Omega_D L} \right] \right\}. \quad \text{(D7)}$$

In the weak-coupling limit, $\lambda^2 \ll m\Omega^3$, the effect of the cavity field on the (forced) mirror motion is sufficiently small that the mirror will continue to oscillate at a frequency nearly equal

to Ω_0. Consequently, we expect $Z(\nu)$ to be sharply peaked for frequencies $\nu \sim \Omega_0$. Since L is inversely proportional to the cavity period and $\nu \sim \Omega_0$, then it follows that $\nu L \ll 1$ and $\Omega_D \gg \nu$. Therefore, expanding all terms but $Z(\nu)$ in the integrand of (D7) for ν near zero [77],

$$q_1(t) \approx \lambda Z(t) \{ e^{-i\Omega_D t} D(\Omega_D) [\alpha' + \lambda^2 \alpha \tilde{D}(\Omega_D) e^{i2\Omega_D L}]$$
$$+ e^{i\Omega_D t} D(-\Omega_D) [\alpha'^* + \lambda^2 \alpha^* \tilde{D}^*(\Omega_D) e^{-i2\Omega_D L}] \}.$$
$$(D8)$$

Last, since the mirror moves on a time scale $(\sim 1/\Omega_0)$ much longer than the round-trip travel time of light in the cavity $(\sim 2L)$, then $\Omega_0 L \ll 1$, and we may expand all delay terms about their instantaneous values so that, for example, $Z(t - 2L) = Z(t) - 2L\dot{Z}(t) + O((\Omega_0 L)^2)$. Using (D8), one may then easily compute the time average of the terms depending on $q_1(t)$ in Eq. (5.6). Putting everything together and remembering to expand the delay terms as discussed above, we find that (5.5) becomes

$$M\ddot{Z}(t) + \Gamma(L)\dot{Z}(t) + M[\Omega_0^2 - \Delta\Omega^2(L)]Z(t) = F_{\text{rad}}(L),$$
$$(D9)$$

which is simply the equation for a forced, damped harmonic oscillator with mass M, frequency $[\Omega_0^2 - \Delta\Omega^2(L)]^{1/2}$, and damping coefficient $\Gamma(L)$. Notice that the latter two quantities depend explicitly on the length of the cavity. After time averaging, the explicit form for the radiation pressure is given by

$$F_{\text{rad}}(L) = \lambda^2 \alpha \tilde{D}(\Omega_D) \left(\alpha'^* + \frac{\lambda^2}{2} \alpha^* \tilde{D}^*(\Omega_D) e^{-i2\Omega_D L} \right),$$

the frequency shift is given by

$$M\Delta\Omega^2(L)$$
$$= -i\Omega_D \lambda^2 \alpha \tilde{D}(\Omega_D) \left(\alpha'^* + \frac{3\lambda^2}{2} \alpha^* \tilde{D}^*(\Omega_D) e^{-i2\Omega_D L} \right)$$
$$+ \lambda^2 \tilde{D}(\Omega_D)(\alpha' + \lambda^2 \alpha \tilde{D}(\Omega_D) e^{i2\Omega_D L})(\alpha'^*$$
$$+ \lambda^2 \alpha^* \tilde{D}^*(\Omega_D) \cos 2\Omega_D L),$$

and the damping coefficient is given by

$$\Gamma(L) = \frac{\lambda^4}{2} |\alpha \tilde{D}(\Omega_D)|^2 \cos 2\Omega_D L. \qquad (D10)$$

[1] F. Marquardt and S. M. Girvin, Physics **2**, 40 (2009).

[2] C. M. Caves, Phys. Rev. Lett. **45**, 75 (1980).

[3] H. B. G. Casimir, Proc. K. Ned. Akad. Wet. **51**, 793 (1948).

[4] H. B. G. Casimir and D. Polder, Phys. Rev. **73**, 360 (1948).

[5] For a review on the dynamical Casimir effect, see, e.g., V. V. Dodonov, Phys. Scr. **82**, 038105 (2010); D. A. R. Dalvit, P. A. Maia Neto, and F. Diego Mazzitelli, in *Casimir Physics*, edited by D. Dalvit *et al.*, Lecture Notes in Physics No. 834 (Springer, Berlin, 2011), Chap. 13.

[6] See, e.g., C. Eberlein, Phys. Rev. A **53**, 2772 (1996), and references therein.

[7] C. M. Wilson, G. Johansson, A. Pourkabirian, M. Simoen, J. R. Johansson, T. Duty, F. Nori, and P. Delsing, Nature (London) **479**, 376 (2011).

[8] A. D. Armour, M. P. Blencowe, and K. C. Schwab, Phys. Rev. Lett. **88**, 148301 (2002); M. P. Blencowe, Phys. Rep. **395**, 159 (2004); Contemp. Phys. **46**, 249 (2005).

[9] P. T. Rakich, C. Reinke, R. Camacho, P. Davids, and Z. Wang, Phys. Rev. X **2**, 011008 (2012).

[10] J. D. Thompson, B. M. Zwickl, A. M. Jayich, F. Marquardt, S. M. Girvin, and J. G. E. Harris, Nature (London) **452**, 72 (2008); J. C. Sankey, C. Yang, B. M. Zwickl, A. M. Jayich, and J. G. E. Harris, Nat. Phys. **6**, 70721 (2010).

[11] K. Hammerer, M. Wallquist, C. Genes, M. Ludwig, F. Marquardt, P. Treutlein, P. Zoller, J. Ye, and H. J. Kimble, Phys. Rev. Lett. **103**, 063005 (2009); M. Wallquist, K. Hammerer, P. Zoller, C. Genes, M. Ludwig, F. Marquardt, P. Treutlein, J. Ye, and H. J. Kimble, Phys. Rev. A **81**, 023816 (2010).

[12] G. Hétet, L. Slodička, M. Hennrich, and R. Blatt, Phys. Rev. Lett. **107**, 133002 (2011).

[13] L. Parker, Phys. Rev. **183**, 1057 (1969).

[14] Ya. B. Zeldovich, Pis'ma Zh. Eksp. Teor. Fiz. **12**, 443 (1970) [JETP Lett. **12**, 307 (1970)].

[15] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).

[16] W. G. Unruh, Phys. Rev. D **14**, 870 (1976).

[17] S. A. Fulling and P. C. W. Davies, Proc. R. Soc. London, Ser. A **348**, 393 (1976).

[18] P. C. W. Davies and S. A. Fulling, Proc. R. Soc. London, Ser. A **356**, 237 (1977).

[19] A. Abramovici *et al.*, Science **256**, 325 (1992); http://www.ligo.caltech.edu.

[20] A. Giazotto, Nucl. Instrum. Methods Phys. Res., Sect. A **289**, 518 (1990); http://www.virgo.infn.it.

[21] B. Wilke *et al.*, Classical Quantum Gravity **19**, 1377 (2002); http://geo600.aei.mpg.de.

[22] R. L. Forward, Phys. Rev. D **17**, 379 (1978).

[23] A. Schliesser, R. Rivière, G. Anetsberger, O. Arcizet, and T. J. Kippenberg, Nat. Phys. **4**, 415 (2008); A. Schliesser, O. Arcizet, R. Rivière, G. Anetsberger, and T. J. Kippenberg, *ibid.* **5**, 509 (2009); Y. S. Park and H. Wang, *ibid.* **5**, 489 (2009); S. Groblacher, J. B. Hertzberg, M. R. Vanner, G. D. Cole, S. Gigan, K. C. Schwab, and M. Aspelmeyer, *ibid.* **5**, 485 (2009); T. Rocheleau, T. Ndukum, C. Macklin, J. B. Hertzberg, A. A. Clerk, and K. C. Schwab, Nature (London) **463**, 72 (2009); R. Rivière, S. Deleglise, S. Weis, E. Gavartin, O. Arcizet, A. Schliesser, and T. J. Kippenberg, Phys. Rev. A **83**, 063835 (2011); J. Chan, T. P. M. Alegre, A. H. Safavi-Naeini, J. T. Hill, A. Krause, S. Groeblacher, M. Aspelmeyer, and O. Painter, Nature (London) **478**, 89 (2011).

[24] A. D. O'Connell, M. Hofheinz, M. Ansmann, R. C. Bialczak, M. Lenander, E. Lucero, M. Neeley, D. Sank, H. Wang, M. Weides, J. Wenner, J. M. Martinis, and A. N. Cleland, Nature (London) **464**, 697 (2010).

[25] T. Rocheleau, T. Ndukum, C. Macklin, J. B. Hertzberg, A. A. Clerk, and K. C. Schwab, Nature (London) **463**, 72 (2009).

[26] V. Braginsky and S. P. Vyatchanin, Phys. Lett. A **293**, 228 (2002).

[27] A. Buonanno and Y. Chen, Phys. Rev. D **69**, 102004 (2004).

[28] A. Buonanno and Y. Chen, Phys. Rev. D **65**, 042001 (2002).

[29] H. Müller-Ebhardt, H. Rehbein, R. Schnabel, K. Danzmann, and Y. Chen, Phys. Rev. Lett. **100**, 013601 (2008).

[30] V. B. Braginsky and F. Ya. Khalili, *Quantum Measurement* (Cambridge University Press, Cambridge, 1992); Rev. Mod. Phys. **68**, 1 (1996).

[31] C. M. Caves, Phys. Rev. D **23**, 1693 (1981).

[32] W. G. Unruh, in *Quantum Optics, Experimental Gravitation, and Measurement Theory*, edited by P. Meystre and M. O. Scully (Plenum, New York, 1982), p. 647.

[33] H. J. Kimble, Y. Levin, A. B. Matsko, K. S. Thorne, and S. P. Vyatchanin, Phys. Rev. D **65**, 022002 (2001).

[34] W. Marshall, C. Simon, R. Penrose, and D. Bouwmeester, Phys. Rev. Lett. **91**, 130401 (2003).

[35] M. Pinard, A. Dantan, D. Vitali, O. Arcizet, T. Briant, and A. Heidmann, Europhys. Lett. **72**, 747 (2005); M. J. Hartmann and M. B. Plenio, Phys. Rev. Lett. **101**, 200503 (2008); M. Bhattacharya, P. L. Giscard, and P. Meystre, Phys. Rev. A **77**, 030303(R) (2008).

[36] S. Reynaud, P. Maia Neto, A. Lambrecht, and M. T. Jaekel, Europhys. Lett. **54**, 135 (2001); Int. J. Mod. Phys. A **17**, 1003 (2002); B. Lamine, R. Herve, A. Lambrecht, and S. Reynaud, Phys. Rev. Lett. **96**, 050405 (2006).

[37] L. Diosi, Phys. Lett. A **105**, 199 (1984); Phys. Lett. **120**, 377 (1987); Phys. Rev. A **40**, 1165 (1989); J. Phys. A **40**, 2989 (2007); R. Penrose, Gen. Relativ. Gravitation **28**, 581 (1996); Philos. Trans. R. Soc. London, Ser. A **356**, 1927 (1998); in *Mathematical Physics 2000*, edited by A. Fokas *et al*. (Imperial College Press, London, 2000), pp. 266–282; S. L. Adler, *Quantum Theory as an Emergent Phenomenon* (Cambridge University Press, Cambridge, 2004).

[38] G. T. Moore, J. Math. Phys. (NY) **11**, 2679 (1970).

[39] M. T. Jaekel and S. Reynaud, Phys. Lett. A **180**, 9 (1993).

[40] G. Barton and A. Calogeracos, Ann. Phys. (NY) **238**, 227 (1995); A. Calogeracos and G. Barton, *ibid*. **238**, 268 (1995).

[41] C. K. Law, Phys. Rev. A **51**, 2537 (1995).

[42] V. V. Dodonov and A. B. Klimov, Phys. Rev. A **53**, 2664 (1996).

[43] R. Schutzhold, G. Plunien, and G. Soff, Phys. Rev. A **57**, 2311 (1998); G. Schaller, R. Schutzhold, G. Plunien, and G. Soff, *ibid*. **66**, 023812 (2002).

[44] B. L. Hu and A. Matacz, Phys. Rev. D **49**, 6612 (1994).

[45] R. Golestanian and M. Kardar, Phys. Rev. Lett. **78**, 3421 (1997); Phys. Rev. A **58**, 1713 (1998).

[46] M. Kardar and R. Golestanian, Rev. Mod. Phys. **71**, 1233 (1999).

[47] C. H. Wu and D. S. Lee, Phys. Rev. D **71**, 125005 (2005).

[48] C. D. Fosco, F. C. Lombardo, and F. D. Mazzitelli, Phys. Rev. D **76**, 085007 (2007).

[49] L. H. Ford and A. Vilenkin, Phys. Rev. D **25**, 2569 (1982).

[50] N. Obadia and R. Parentani, Phys. Rev. D **64**, 044019 (2001).

[51] J. Haro and E. Elizalde, Phys. Rev. Lett. **97**, 130401 (2006); Phys. Rev. D **76**, 065001 (2007).

[52] An analogous endeavor undertaken by one of us and many others in the early 1970s is quantum field theory (QFT) in curved space-time [78], which is the theory behind cosmological particle creation and Hawking-Unruh effects. It contains as a subset, QFT with boundaries, which deals with effects due to geometry, topology, such as the Casimir effect [3] and due to finite size of the system [79]. In its latest reincarnation this is the subject of QFT under external conditions; see, e.g., [73].

[53] C. H. Chou, C. R. Galley, B. L. Hu, and S. Y. Lin (in preparation).

[54] R. O. Behunin, C. R. Galley, and B. L. Hu (unpublished).

[55] J. Pendry, J. Phys. Condens. Matter **9**, 10301 (1997).

[56] A. I. Volokitin and B. N. J. Persson, J. Phys. Condens. Matter **11**, 345 (1999).

[57] G. V. Dedkov and A. A. Kyasov, Phys. Solid State **44**, 1809 (2002).

[58] T. G. Philbin and U. Leonhardt, New J. Phys. **11**, 033035 (2009).

[59] B. L. Hu, Phys. Lett. A **90**, 375 (1982).

[60] C. R. Galley, R. O. Behunin, and B. L. Hu (unpublished).

[61] A. Raval, B. L. Hu, and J. Anglin, Phys. Rev. D **53**, 7003 (1996).

[62] Note that any idealization requires careful consideration of the actual physical conditions: A two-level idealization gives an apt description only for a resonant type of coupling [80], while a harmonic atom (oscillator model) includes multiple-level activities when coupled with a field.

[63] B. Huttner and S. M. Barnett, Phys. Rev. A **46**, 4306 (1992).

[64] Actually, to describe the nonequilibrium dynamics of this quantum theory one should use an initial value formulation of the path integral. Such a prescription is given by the "in-in" (Schwinger-Keldysh or "closed-time path") or the influence functional formalisms (read, e.g., Chaps. 5 and 6 in Ref. [81] and bibliography therein). The latter method was used for treating moving mirror problems by, e.g., [48].

[65] P. D. Nation, M. P. Blencowe, A. J. Rimberg, and E. Buks, Phys. Rev. Lett. **103**, 087004 (2009).

[66] T. Zhang, Y. Wang, W.-L. Liu, J.-G. Cheng, Z.-T. Song, S.-L. Feng, H. C.-W. Lai-Wa, and C.-L. Choy, Chin. Phys. Lett. **22**, 694 (2005).

[67] Care must be taken when using a cutoff regularization in a gauge theory, such as quantum electrodynamics (QED), since a cutoff prescription does not necessarily preserve the symmetries of the theory. In that case, one must use a scheme that does such as dimensional regularization [82,83]. In QED, a high-frequency cutoff breaks the gauge symmetry and induces a mass for the photon [84].

[68] B. L. Hu, J. P. Paz, and Y. Zhang, Phys. Rev. D **45**, 2843 (1992).

[69] C. H. Chou, T. Yu, and B. L. Hu, Phys. Rev. E **77**, 011112 (2008).

[70] C. H. Fleming, A. Roura, and B. L. Hu, arXiv:1106.5752.

[71] J. Preskill, in *Black Holes, Membranes, Wormholes and Superstrings*, edited by K. Kalara and D. V. Nanopoulos (World Scientific, Singapore, 1993).

[72] B. L. Hu and A. Roura, Phys. Rev. D **76**, 124018 (2007).

[73] http://benasque.org/2011qfext/.

[74] S. Y. Lin and B. L. Hu, Phys. Rev. D **73**, 124018 (2006).

[75] S. Y. Lin and B. L. Hu, Classical Quantum Gravity **25**, 154004 (2008).

[76] The homogeneous solution for the field that we are neglecting here would play the role of a Langevin forcing term when plugged into the remaining equations of motion and would account for the initial state of the field. In a future work this term will be considered and will account for the field's thermal and quantum fluctuations.

[77] We do not expand $Z(v)$ itself since we wish to find the equation of motion for the time-averaged $Z(t)$ that is self-consistent with the evolution of the field and mirosc.

[78] N. D. Birrell and C. P. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).

[79] D. J. O'Connor, C. R. Stephens, and B. L. Hu, Ann. Phys. (NY) **190**, 310 (1989).

[80] The difference in the decoherent behavior between the resonant type of coupling of a two-level atom with a field and that of a harmonic oscillator with a harmonic bath is discussed in B. L. Hu, Chaos Solitons Fractals **16**, 391 (2003); See also C. Anastopoulos and B. L. Hu, Phys. Rev. A **62**, 033821 (2000), and references therein.

[81] E. Calzetta and B. L. Hu, *Nonequilbrium Quantum Field Theory* (Cambridge University Press, Cambridge, 2008).

[82] C. G. Bollini and J. J. Giambiagi, Phys. Lett. B **40**, 566 (1972).

[83] G. t'Hooft and M. Veltman, Nucl. Phys. B **44**, 189 (1972).

[84] M. Peskin and D. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, Reading, MA, 1995).