Nonparaxial traveling solitary waves in layered nonlinear media

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A large class of exact analytical traveling solitary wave solutions in a variety of inhomogeneous structures consisting of linear and nonlinear layers is obtained. The solutions are related to a spatial resonance condition and describe reflectionless and radiationless beam propagation for arbitrary angles and spatial widths in the nonparaxial regime.

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Wave propagation in inhomogeneous media is one of the most ubiquitous phenomena occurring in almost every branch of physics. Both natural and man-made media are, in general, inhomogeneous, and the study of wave propagation in such media is a research field of continuously increasing interest from the standpoint of applications in modern science and engineering. The inhomogeneity of a medium determines the linear diffraction properties of wave propagation so that appropriately designed spatial structures can have desirable propagation features. In addition, the refractive index of a large variety of materials appears to depend on the wave power resulting in a drastic modification of the linear wave propagation properties for sufficiently high power of the waves. In such nonlinear materials, spatial self-localization of the waves can take place enabling the formation of solitary waves (SWs) and solitons. The combination of the inhomogeneity and the nonlinearity results in wave propagation characteristics that have no counterpart in either linear inhomogeneous or nonlinear homogeneous media. In the context of nonlinear optics, intense research interest has been focused, in theoretical and experimental studies, on photonic structures of varying complexity, such as photonic crystals, waveguide arrays, and periodic or disordered lattices [1]. The effective light control in such structured systems can be achieved by means of engineered heterostructures consisting of waveguide channels, cavities, and other functional elements.

The dominant underlying model governing SW formation and propagation in such structures is the nonlinear Schrödinger equation (NLS), which is known to be completely integrable for one-transverse dimension (space or time) when the propagation medium is homogeneous and has a Kerr type of nonlinearity. Therefore, exact soliton solutions can be derived with the utilization of the inverse scattering transform, and soliton dynamics under the presence of a variety of perturbations including inhomogeneities, dissipation, higher-order diffraction, and nonlocality have been studied with the utilization of perturbation methods [2]. However, *this analytically tractable model is a result of an important assumption with far-reaching consequences with respect to its applicability in several cases of interest*. The NLS equation is derived from the Maxwell equations under the paraxial approximation, which is valid under the following conditions: (i) the beamwidth is much larger than the wavelength (slowly varying envelope approximation), and (ii) the beam propagates along or near negligible angles with respect to the reference axis. These conditions drastically restrict the domain of applicability of the NLS equation with respect to various applications. The progressive miniaturization of the photonic devices on the nanoscale invalidates the slowly varying envelope approximation (i) since the beamwidth may become comparable or even smaller than the wavelength. Additionally, the assumption on the small angle of propagation (ii) strongly restricts the study of wave dynamics (reflection and refraction) at material interfaces and mutual SW collisions, thus, excluding a large number of features with interest for applications. Even if the initially launched beam can be considered paraxial, the evolution under propagation can result in splitting to multiple beams propagating at large angles as well as strong focusing. For the latter, the NLS equation predicts a nonphysical catastrophic beam collapse [3]. Therefore, although there exist a huge number of studies on NLS soliton formation and dynamics in inhomogeneous media, *the investigation of the respective phenomena in nanoscale structures as well as the exploration of novel features of SW dynamics are yet to be followed up in the nonparaxial regime* [4].

For the case of layered media, the original Maxwell equation, under no approximation, leads to a scalar nonlinear Helmholtz (NLH) equation with an intensity-dependent refractive index when the electric field is assumed to be monochromatic and linearly polarized along the y direction (TE polarization),

$$E_{zz} + E_{xx} + \beta^2(x)E + \gamma(x)|E|^2E = 0, \qquad (1)$$

where the electric field E is normalized to E_0, $\beta \equiv n_0\omega/c$, $\gamma \equiv 2(n_2E_0^2/n_0)\beta^2$ and $n_0(x)$ and $n_2(x)$ are the linear and nonlinear (Kerr-type) refractive indices that are piecewise constant functions of the transverse coordinate x. The NLS equation can be considered as the paraxial limit of the NLH equation. The two equations have fundamental mathematical differences resulting in the description of qualitatively different phenomena of wave propagation. The NLH equation is of hyperbolic type allowing for the description of bi-directional wave propagation in contrast to the NLS equation, which is of parabolic type and describes only unidirectional propagation, thus, excluding the description of wave backscattering. On the other hand, both the numerical solution and the mathematical analysis of the NLH equation are considerably more difficult than the case of the NLS since the latter requires solving an initial value problem, whereas, the former involves the solution of a boundary value problem. Moreover, unlike the NLS equation, which governs the slowly varying envelope evolution, the NLH equation has to be approximated with subwavelength resolution. Additionally, in contrast to the NLS equation, the stability of solutions of the NLH equation is a still unsolved problem. Several papers have been focused on the derivation of nonparaxial unidirectional propagation equations as an attempt to include additional terms to a NLS type of equation in order to extend its range of validity [5]. In an increasing number of papers, bright and dark soliton solutions of the NLH equation have been considered either in bulk media [4,6] or in interfaces [7], and numerical simulations of the NLH equations have shown robust soliton propagation [8]. Both the wide applicability and the difficulties of the NLH equation underpin the importance of knowledge of analytical solutions which can guide our intuition in studying SW propagation under the NLH equation as well as in benchmarking numerical algorithms for its solution. In this paper, we present a large class of analytical solutions corresponding to traveling SWs in a wide variety of layered nonlinear media.

Traveling wave solutions of the NLH Eq. (1) can be sought in the form

$$E(x,z) = u(x - x_0(z))e^{i(k_x x + k_z z)}, \tag{2}$$

where u is real transverse wave profile and k_x and k_z are the transverse and longitudinal wave numbers, respectively. Substitution of Eq. (2) in Eq. (1) results in an ordinary differential equation for the transverse wave profile,

$$u_{xx} + \frac{\beta^2(x) - k^2}{1 + v^2}u + \frac{\gamma(x)}{1 + v^2}u^3 = 0, \tag{3}$$

with $k^2 \equiv k_x^2 + k_z^2$, $v \equiv dx_0/dz = k_x/k_z$ being the transverse wave velocity corresponding to a propagation angle $\theta = \tan^{-1}(k_x/k_z)$ and $u = u(x - x_0(z))$. The above equation corresponds to a dynamical system which is nonautonomous due to the dependence of the piecewise constant coefficients on the transverse coordinate x. For the case of a homogeneous nonlinear medium (where β and γ are constants), there exist a homoclinic (heteroclinic) solution for every x_0 provided that $k^2 > \beta^2$ ($k^2 < \beta^2$) and $\gamma > 0$ ($\gamma < 0$) [4,6],

$$u = \sqrt{2\frac{k^2 - \beta^2}{\gamma}}\,\mathrm{sech}\left(\sqrt{\frac{k^2 - \beta^2}{1 + v^2}}[x - x_0(z)]\right), \tag{4}$$

$$u = \sqrt{\frac{k^2 - \beta^2}{\gamma}}\tanh\left(\sqrt{\frac{\beta^2 - k^2}{2(1 + v^2)}}[x - x_0(z)]\right). \tag{5}$$

These solutions travel transversely with velocity v and correspond to bright (dark) SW solutions of the homogeneous NLH. Note that, in the paraxial limit $v^2 \ll 1$, the solutions given in Eqs. (4) and (5) correspond to the well-known soliton solutions of the NLS equation, whereas, in the nonparaxial regime, the wave profile width dependency on the transverse velocity takes the geometrical dependency of the beamwidth on the propagation angle properly into account [4].

The presence of a transverse medium inhomogeneity, in general, results in the breaking of the translational invariance of the solutions. In such a case, solitons cannot be formed in every transverse position; stable and unstable solitons can be formed only in specific transverse positions determined by the form of the inhomogeneity. This is reflected in the dynamics of the nonautonomous system of Eq. (3) through the breaking of the homoclinic (heteroclinic) orbit: The stable and unstable manifolds corresponding to the respective saddle points are no longer tangent under the presence of a nonautonomous perturbation and intersect at a discrete set of homoclinic (heteroclinic) points. These points correspond to the discrete set of transverse coordinate values where a SW can be formed in the inhomogeneous medium [9,10]. *Such a discreteness excludes the possibility of having a homoclinic (heteroclinic) solution of Eq. (3) for every x_0 and, therefore, the existence of traveling wave solutions.*

Among the wide class of inhomogeneous media, layered structures consisting of alternating (interlaced) linear and nonlinear regions are very important for applications as well as for understanding SW formation and dynamics. A large number of previous studies has been focused on the formation of *standing SW solutions* in finite structures [11–16], infinite periodic structures [17–19], as well as interfaces between semi-infinite (homogeneous or inhomogeneous) structures [20]. Therefore, apart from the expected symmetric SW profiles, novel classes of asymmetric modes have been shown to exist even in symmetric layered structures [11,12,21,22]. Such classes have been obtained either by utilizing explicit boundary conditions on the boundaries between subsequent layers [11] or by utilizing a phase-space method [12,21,22]. The phase-space method combined with topological considerations of the phase space of the constructed standing wave solutions have been shown to provide a sufficient instability condition for their propagation under the paraxial approximation governed by the NLS equation [12,19,21,22].

In the following, we consider layered structures consisting of interlaced linear and nonlinear parts described by the linear and nonlinear refractive index profiles,

$$[n_0(x), n_2(x)] = \begin{cases} (n_0^l, 0), & x \in U_L, \\ (n_0^{nl}, n_2), & x \in \mathbb{R} - U_L, \end{cases} \tag{6}$$

where $U_L = \bigcup_m [N/2 + (m-1)T, N/2 + (m-1)T + L]$ with $m = 1,2,\ldots,M$ is the union of the linear parts with $T = L + N$ and L and N being the lengths of the linear and the nonlinear layers, respectively. Such profiles describe two types of configurations consisting of a periodic interlaced structure (with spatial period T) and a nonlinear homogeneous part: (a) For a finite M, the structure consists of a finite periodic structure embedded in a homogeneous nonlinear medium. This type includes the case of a linear defect in a nonlinear medium, which has been intensively studied in terms of paraxial propagation of the standing SW [11,12,14–16]. (b) For an infinite M, the structure consists of a semi-infinite periodic structure interfaced with a homogeneous nonlinear medium. The study of waves propagating in an infinite periodic structure can be included in this type by considering waves launched inside the semi-infinite periodic structure sufficiently far from the interface with the homogeneous medium.

The transverse profile of the wave is described by Eq. (3) with $\beta = \beta_l \equiv n_0^l \omega/c$, $\gamma = 0$ and $\beta = \beta_{nl} \equiv n_0^{nl}\omega/c$, $\gamma \neq 0$ in the linear and nonlinear parts, respectively. We focus on the case where $k^2 < \beta_l^2$ so that the solutions of Eq. (3) within the linear parts are periodic, whereas, in the nonlinear parts, we have $k^2 > \beta_{nl}^2$ ($k^2 < \beta_{nl}^2$) and $\gamma > 0$ ($\gamma < 0$) so that the bright (dark) SW solution given by Eq. (4) [Eq. (5)] exists within the nonlinear parts. The total transverse wave profile can be obtained by considering the boundary conditions between subsequent layers. When $k^2 > \beta_{nl}^2$, matching of the solution given by Eq. (4) with the sinusoidal solutions of the linear part in a period T of a structure with $M = 1$ (a linear layer embedded in an infinite homogeneous nonlinear medium) leads to the following relation [11]:

$$\frac{\frac{q_{nl}^2}{q_l^2} + \tan^2\left[q_l \cos\theta\left(\frac{N}{2} - x_0\right)\right]}{\frac{q_{nl}^2}{q_l^2} + \tan^2\left[q_l \cos\theta\left(L + \frac{N}{2} - x_0\right)\right]} = \frac{\cos^2\left[q_l \cos\theta\left(\frac{N}{2} - x_0\right)\right]}{\cos^2\left[q_l \cos\theta\left(L + \frac{N}{2} - x_0\right)\right]}, \tag{7}$$

where $q_{nl}^2 = k^2 - \beta_{nl}^2$ and $q_l^2 = \beta_l^2 - k^2$. This is a nonlinear dispersion relation from which symmetric, antisymmetric, as well as asymmetric standing waves can be obtained corresponding to distinct values of the wave profile center x_0 as given by the solution of Eq. (7). A careful investigation of Eq. (7) shows that it degenerates to an identity relation under the condition,

$$k^2 = \beta_l^2 - \left(\frac{n\pi}{L}\right)^2 (1 + \tan^2\theta), \quad n = 1,2,\ldots . \tag{8}$$

This condition leads to the existence of a solution for every x_0. The resulting continuous family of solutions, for every n, is parametrized by x_0 and includes a symmetric, an antisymmetric, and an infinite number of asymmetric modes. Therefore, it is the fulfillment of Eq. (8) that allows for the existence of a solitary wave solution of Eq. (3) determining the wave profile in the transverse direction for every $x_0(z) = vz + x_0(0)$ and, therefore, for the *existence of traveling SWs in contrast to the general case where only standing SWs at fixed x_0 exist.* [18] The condition represents a *spatial resonance* according to which an integer number of half-periods of the sinusoidal wave profile in the linear part is contained within the length of the linear part L. Moreover, this condition ensures the existence of continuous, with respect to x_0, families of solutions for finite configurations corresponding to $M > 1$ as well as semi-infinite configurations corresponding to an infinite M. A geometric construction of the respective solutions can be considered in the phase space (u,u_x) of the dynamical system given by Eq. (3). The homoclinic (heteroclinic) asymptotic orbits corresponding to bright (dark) solitary wave solutions within the nonlinear parts are symmetric with respect to both axes $u = 0$ and $u_x = 0$ of the phase space, whereas, the phase space of the solutions within the linear parts consists of ellipses centered around the origin $(u,u_x) = (0,0)$. As a result, under the condition (8), any solution starting from a point of the homoclinic (heteroclinic) orbit at the boundary between a nonlinear and a linear part of the structure returns to the same point (for n, even) or to its symmetric point with respect to the origin (for n, odd) after evolving with respect to x in the linear part of length L [18].

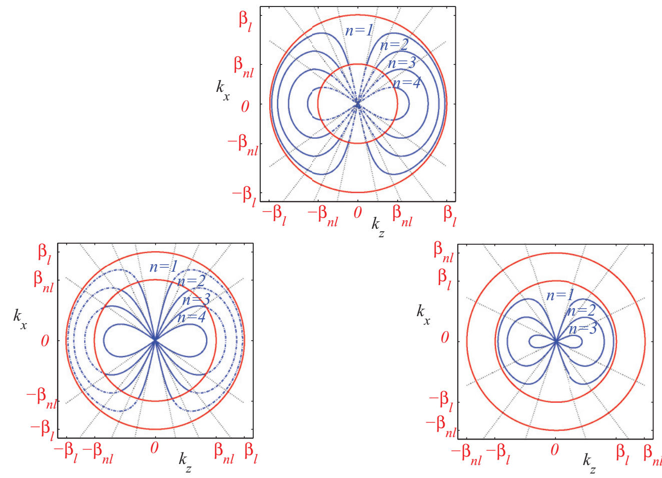

FIG. 1. (Color online) Dispersion curves of the nonparaxial traveling SWs. The solid-line parts of the lemniscoid (blue) curves depict the dispersion relation for the solutions. The solid-line (red) circles bound the dispersion curves, and their radii depend on the linear refractive indices of the linear (β_l) and nonlinear (β_{nl}) parts. The straight dotted lines (black) depict the asymptotes of the dispersion curves at the origin. (Top) bright SW and (bottom) dark SW when $\beta_l > \beta_{nl}$ (left) and $\beta_l < \beta_{nl}$ (right).

Equation (8) is the dispersion relation between k_x and k_z for the respective nonparaxial traveling SWs. Figure 1 depicts the dispersion curves for the case of bright and dark SWs along with their domains of existence. The linear refractive index profile across the structure determines the boundaries of the domains through the values of β_l and β_{nl}. For the case of bright SWs, a certain contrast between the linear and the nonlinear layers is necessary. Different integer values of n correspond to different traveling SWs with n, by construction, being the number of nodes (zeros) of the solution within a linear part. The number of different SWs n_{max}, in each case, depends on the values of the linear refractive index in the layers as well as on the length of the linear layer L. The length of the nonlinear layer N determines the spatial extent of the solutions. The angles of the asymptotes of the lemniscoids described by Eq. (8) at the origin are given by $\tan\theta_{as} = \pm\sqrt{(L\beta_l/n\pi)^2 - 1}$ and determine, along with the other constraints, the maximum angle of propagation for each SW. Depending on the position of the lemniscoid curves with respect to the circles of radii β_l and β_{nl}, the maximum angle can be further restricted as in the case of bright SWs shown in Fig. 1(top), whereas, a nonzero minimum angle of propagation can occur as in the case of dark SWs shown in Fig. 1(bottom, left) where paraxial propagation ($k_x \simeq 0$) cannot take place for some of the SWs. In general, an increasing L results in larger maximum angles of propagation for each n as well as a larger n_{max}. At the limit of large L, the dispersion curves densely fill the respective domains of existence, and their asymptotes approach the $k_z = 0$ axis.

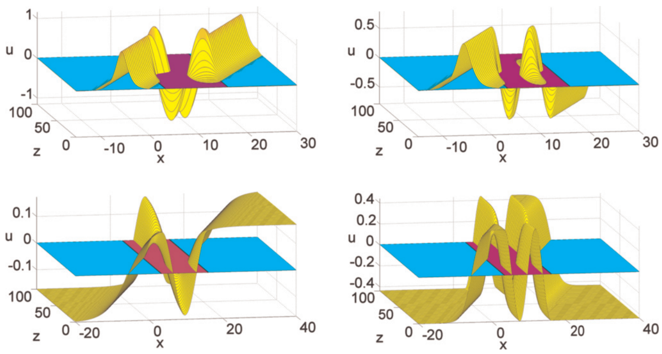

FIG. 2. (Color online) Traveling SWs in a structure consisting of a single linear layer ($M = 1$) with $L = 4\pi$ and $N = 2\pi$. The velocity is $v = 0.2$ and left: $n = 2$; right: 3. top: $\gamma = 2$, $\beta_{nl}^2 = 0.3$, and $\beta_L^2 = 1.5$; bottom: $\gamma = -2$, $\beta_{nl}^2 = 1.8$, and $\beta_l^2 = 2$.

The propagation of traveling SWs in the case of a structure consisting of a single linear layer is depicted in Fig. 2. The respective solutions describe *reflectionless transmission* through the linear layer. For $\gamma > 0$ and $k^2 > \beta_{nl}^2$, corresponding to solutions in the nonlinear layers given by Eq. (4), a bright traveling SW propagates through the linear layer with the transmitted wave having the same (n, even) or reverse (n, odd) polarity on the other side of the linear layer as shown in Fig. 2(top). For $\gamma < 0$ and $k^2 < \beta_{nl}^2$, corresponding to solutions in the nonlinear layers given by Eq. (5), the respective dark traveling SWs have either the opposite (n, even) or the same (n, odd) sign of asymptotic values as shown in Fig. 2(bottom). In all cases, the transmitted wave differs from the incident wave only as to a possible sign reversal depending on n and a displacement of length L along the transverse dimension.

The case of traveling SWs in a structure consisting of three linear layers is depicted in Fig. 3. In such cases where $M > 1$, the sign reversal of the transmitted wave requires both n and M being odd numbers, whereas, the total transverse displacement of the transmitted wave with respect to the incident wave is ML. The case of an infinite M, corresponding to a structure consisting of a semi-infinite periodic structure interfaced with a semi-infinite homogeneous nonlinear part, can be readily considered; a bright or dark incident SW is transmitted with no reflection in the periodic structure with an increased averaged velocity $\bar{v} = v(1 + L/N)$. Inside the periodic part and sufficiently far from the interface, the SW has the form of a *traveling breather* as in the case of an infinite periodic layered structure.

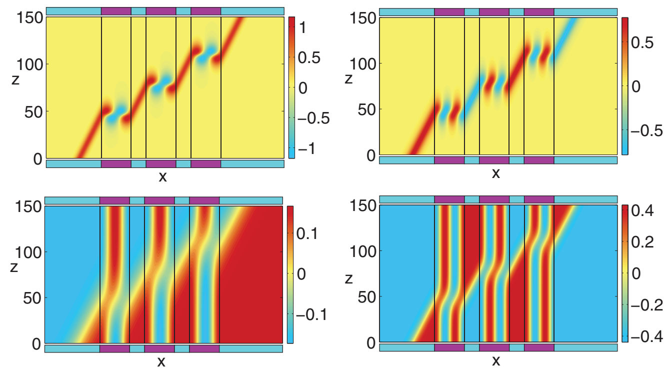

FIG. 3. (Color online) Traveling SWs in a structure consisting of three linear layers ($M = 3$). All other parameters are the same with those of Fig. 2.

In conclusion, a large class of exact analytical traveling solitary wave solutions in a variety of inhomogeneous structures consisting of linear and nonlinear layers has been obtained. The solutions are related to a spatial resonance condition and describe reflectionless and radiationless SW propagation for arbitrary angles and spatial widths in the nonparaxial regime, allowing for their applicability on the nanoscale. The generality of the results facilitates the experimental observation of the respective solutions in planar dielectric structures having the form of finite or infinite waveguide arrays for layer dimensions ranging from several wavelengths to a subwavelength. The presented solutions can be directly extended to even larger classes of inhomogeneous photonic structures including other types of nonlinearities, medium anisotropy, magnetic materials, and metamaterials. Moreover, structures having layers with gain in order to compensate lossy metallic layers commonly occurring in nano-optics applications can also be considered. The respective traveling SW solutions are expected to have significant potentiality for optical control and signal-processing functionality. Finally, these solutions are applicable and provide physical intuition for the formation of traveling SWs in layered media occurring in other branches of physics beyond nonlinear optics.

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