Resonant quantum kicked rotor with two internal levels

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We study a system consisting of a quantum kicked rotor with an additional degree of freedom. We show analytically and numerically that this model is characterized by its quantum resonances with ballistic spreading and by the entanglement between the internal and momentum degrees of freedom. We conclude that the model shows certain interesting similarities with the standard quantum walk on the line.

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I. INTRODUCTION

Advances in technology during the last decade have made it possible to obtain samples of atoms at temperatures in the nK range [1] (optical molasses) using resonant or quasiresonant exchanges of momentum and energy between atoms and laser light. The experimental progress that has allowed the construction and preservation of quantum states has also opened the possibility of building quantum computing devices [2–5] and has led the scientific community to think that quantum computers could be a reality in the near future. This progress has been accompanied with the development of the interdisciplinary fields of quantum computation and quantum information. In this scientific framework, the study of simple quantum systems such as the quantum kicked rotor (QKR) [6,7] and the quantum walk (QW) [8] may be useful to understand the quantum behavior of atoms in optical molasses.

The QKR is considered as the paradigm of periodically driven systems in the study of chaos at the quantum level [6]. This system shows behaviors without classical equivalence, such as quantum resonance and dynamical localization, which have posed interesting challenges both in theoretical and experimental [9] terms. The occurrence of quantum resonance or dynamical localization depends on whether the period of the kick T is a rational or irrational multiple of 4π. For rational multiples, the behavior of the system is resonant, while for irrational multiples the average energy of the system grows in a diffusive manner for a short time and then the diffusion stops and localization appears. From a theoretical point of view the two types of values of T determine the spectral properties of the Hamiltonian. For irrational multiples the energy spectrum is purely discrete and for rational multiples it contains a continuous part. Both resonance and localization can be seen as interference phenomena, the first being a constructive interference effect and the second a destructive one. The QKR has been used as a theoretical model for several experimental situations dealing with atomic traps [10–17] and is a matter of permanent attention [18–26].

The quantum walk has been introduced [8,27–34] as a natural generalization of the classical random walk in relation with quantum computation and quantum information processing. In both cases there is a walker and a coin; at every time step the coin is tossed and the walker moves depending on the toss output. In the classical random walk the walker moves to the right or to the left, while in the QW coherent superpositions of right and left and head and tail happen. This feature endows the QW with outstanding properties, such as the linear growth with time of the standard deviation of the position of an initially localized walker, as compared with its classical counterpart, where this growth goes as $t^{1/2}$. This has strong implications in terms of the realization of algorithms based on QWs and is one of the reasons why they have received so much attention. It has been suggested [35] that the QW can be used for universal quantum computation. Some possible experimental implementations of the QW have been proposed by a number of authors [2,3,36–40]. In particular, the development of techniques to trap samples of atoms using resonant exchanges of momentum and energy between atoms and laser light may also provide a realistic frame to implement quantum computers [41].

A parallelism between the behavior of the QKR and a generalized form of the QW was developed in Refs. [21,22] showing that these models have similar dynamics. In those papers, the modified QW was mapped into a one-dimensional Anderson model [42], as had been previously done for the QKR [43]. In the present paper, following the work of Saunders *et al*. [44,45] we propose a modification of the QKR. We study some properties of this modified version of the QKR and find certain similarities between this modified QKR and the QW, which help further the parallelism previously suggested. Essentially, the modified QKR has an additional degree of freedom which describes the internal ground and excited states of a two-level atom. We call this modified system the two-level quantum kicked rotor (2L-QKR). In this system the internal atomic levels are coupled with the momentum of the particle. This coupling produces an entanglement between the internal degrees of freedom and the momentum of the system.

The rest of the paper is organized as follows: In the next section we present the 2L-QKR system. In the third section we obtain the time evolution of the moments. In the fourth section the entanglement between the internal degrees of freedom and momentum is studied. In the last section some conclusions are drawn.

II. TWO-LEVEL QUANTUM KICKED ROTOR

In this section, we present a brief theoretical development to obtain the Hamiltonian of the system studied in this paper, following the work of [44]. The experimental setup which motivates this development consists essentially in submitting a dilute cloud of ultracold atoms, trapped in a magneto-optical trap, to a modulated laser standing wave. This procedure has been realized in a variety of experiences [10–17].

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As a starting point, a Hamiltonian that describes a single two-level atom of mass M is considered. The center-of-mass position coordinate is described by the operator $\widehat{Z}$ and its conjugate momentum by the operator $\widehat{P}$. Its internal ground state is denoted by the vector $|g\rangle$ and its excited state by the vector $|e\rangle$. The internal atomic levels are coupled by two equal-frequency laser traveling waves. These waves propagate in opposite directions so as to form a standing wave. If the laser amplitudes are pulsed periodically, the standing wave with which the atom interacts replicates the impulsive interaction in the usual QKR model.

The Hamiltonian of an atom interacting with two traveling waves is given by

$$\widehat{H} = \widehat{H}_O + \widehat{H}_L, \tag{1}$$

where $\widehat{H}_O$ is the Hamiltonian corresponding to the free two-level atom

$$\widehat{H}_O = \frac{\widehat{P}^2}{2m} + \frac{\hbar\omega_0}{2}(|e\rangle\langle e| - |g\rangle\langle g|) \tag{2}$$

and $\widehat{H}_L$ gives the interaction with the traveling waves

$$\begin{aligned}\widehat{H}_L = {} & \hbar\Omega_1 \cos(k_L\widehat{z} - \omega_L t + \Phi_1)(|e\rangle\langle g| + |g\rangle\langle e|) \\ & + \hbar\Omega_2 \cos(k_L\widehat{z} + \omega_L t + \Phi_2)(|e\rangle\langle g| + |g\rangle\langle e|).\end{aligned} \tag{3}$$

Following the procedure in Ref. [44] we must now perform certain transformations on these expressions to arrive at a Hamiltonian similar to that of the QKR. We start by taking the rotating wave approximation and, taking $\Omega_1 = \Omega_2$, we arrive at

$$\widehat{H}_L = \hbar\Omega \cos(k_L\widehat{z})(e^{-i\omega_L t}|e\rangle\langle g| + e^{i\omega_L t}|g\rangle\langle e|), \tag{4}$$

where the phases Φ have been eliminated by appropriate translation of the space and time coordinates. We must now eliminate the time dependence in the Hamiltonian. To this end, we consider a time dependent unitary transformation of the state space

$$\widehat{U}(t) = e^{i[\omega_L|e\rangle\langle e| - (\omega_0/2)(|e\rangle\langle e| + |g\rangle\langle g|)]t}. \tag{5}$$

Given that the Hamiltonian transforms as $\widehat{H}' = i\hbar\frac{d\widehat{U}}{dt}\widehat{U}^\dagger + \widehat{U}\widehat{H}\widehat{U}^\dagger$, we finally arrive at the transformed Hamiltonian

$$\widehat{H} = \frac{\widehat{P}^2}{2m} + \hbar\Delta|e\rangle\langle e| + \hbar\Omega \cos(k_L\widehat{z})(|e\rangle\langle g| + |g\rangle\langle e|), \tag{6}$$

where $\Delta = \omega_0 - \omega_L$ is the detuning between the laser frequency and the atomic transition frequency. To get the excitation characteristic of the QKR, the amplitude of the standing wave must be pulsed periodically in time. Therefore, we consider the Rabi frequency Ω to depend on time as $\hbar\Omega(t) = K\delta_T(t)$, with

$$\delta_T(t) = \sum_{n=0}^{n=\infty} \delta(t - nT) \tag{7}$$

being a series of periodic Dirac's delta applied at times $t = nT$ with n an integer and T the kick period. We thus arrive at the 2L-QKR Hamiltonian

$$\widehat{H} = \frac{\widehat{P}^2}{2M} + \hbar\Delta|e\rangle\langle e| + K\delta_T(t)\cos(k_L\widehat{z})(|e\rangle\langle g| + |g\rangle\langle e|), \tag{8}$$

which will be the system studied in this paper. To get the Hamiltonian of the QKR one must take a further approximation to remove the excited atomic level from the dynamics. This is done in Ref. [44] by way of the so-called adiabatic approximation which eliminates transitions to the excited level in the far detuned regime. However, we will not take that path here and will keep both levels in the dynamics of the system.

It must be noted that, unlike the usual QKR system where the position coordinate is an angular variable, here we consider that the particle can move along the line. Therefore, the 2L-QKR conjugate position and momentum operators have discrete and continuous components, i.e.,

$$\widehat{z} = \frac{1}{k_L}(2\pi\widehat{l} + \widehat{\theta}), \tag{9}$$

$$\widehat{P} = \hbar k_L(\widehat{k} + \widehat{\beta}), \tag{10}$$

where the eigenvalues of $\widehat{l}$ and $\widehat{k}$ are integers and the eigenvalues of $\widehat{\theta} \in [-\pi,\pi)$ and the eigenvalues of the quasimomentum $\widehat{\beta} \in [-1/2,1/2)$. It is important to point out that the operator $\widehat{\beta}$ commutes with both $\widehat{k}$ and $\widehat{\theta}$. Using Eqs. (9) and (10) to substitute $\widehat{z}$ and $\widehat{P}$ in Eq. (8) yields

$$\begin{aligned}\widehat{H} = {} & \frac{[\hbar k_L(\widehat{k} + \widehat{\beta})]^2}{2M} + \hbar\Delta|e\rangle\langle e| \\ & + K\delta_T(t)\cos(\widehat{\theta})(|e\rangle\langle g| + |g\rangle\langle e|).\end{aligned} \tag{11}$$

It must be noted that Eq. (11) does not depend on the operator $\widehat{l}$ and therefore $\widehat{\beta}$ is a preserved quantity. Then if the initial condition belongs to a subspace corresponding to a well-defined eigenvalue of $\widehat{\beta}$, the dynamics is such that the system remains in said subspace and the evolution of the system will be only determined by the conjugate operators $\widehat{\theta}$ and $\widehat{k}$. Therefore we may restrict ourselves to the study of the evolution constrained to a subspace corresponding to a given eigenvalue of β. In this case the composite Hilbert space for the Hamiltonian, Eq. (11), is the tensor product $\mathcal{H}_s \otimes \mathcal{H}_c$. $\mathcal{H}_s$ is the Hilbert space associated to the discrete momentum on the line and it is spanned by the set $\{|k\rangle\}$. $\mathcal{H}_c$ is the chirality (or coin) Hilbert space spanned by two orthogonal vectors $\{|g\rangle,|e\rangle\}$. In this composite space the system evolves, at discrete time steps $t \in \mathbb{N}$, along a one-dimensional lattice of sites $k \in \mathbb{Z}$. The direction of motion depends on the state of the chirality. Taking this into account it is clear that the Hilbert space of the 2L-QKR (with the preceding restriction) is identical to that of the usual QW on the line.

The evolution of the system is governed by the Hamiltonian given by Eq. (11), so that, as is the case for the usual QKR, the unitary time evolution operator for one temporal period T can be written as the application of two operators, one representing the unitary operator due to the kick and another being the unitary operator of the free evolution [44]

$$\widehat{U} = e^{-i[\hbar\Delta|e\rangle\langle e| + \tau(\widehat{k}+\widehat{\beta})^2]}e^{i\kappa\,\cos\widehat{\theta}\sigma_x}, \tag{12}$$

where σ_x is the Pauli matrix in the x direction,

$$\tau = \frac{k_L^2\hbar}{2M}T, \tag{13}$$

and

$$\kappa = \frac{K}{\hbar}. \tag{14}$$

The unit operator, Eq. (12), in the momentum representation and in the chirality base $\{|e\rangle,|g\rangle\}$ has the following shape:

$$U(\beta)_{jk} = f_{jk}(\beta,\kappa,\tau)\begin{pmatrix} e^{-i\widetilde{\Delta}}\delta_{k-j\,2l} & e^{-i\widetilde{\Delta}}\delta_{k-j\,2l+1} \\ \delta_{k-j\,2l+1} & \delta_{k-j\,2l}\end{pmatrix}, \tag{15}$$

where

$$f_{jk}(\beta,\kappa,\tau) = i^{k-j}J_{k-j}(\kappa)e^{-i(j+\beta)^2\tau}, \tag{16}$$

δ_{kj} is the Kronecker delta, l is an integer number, and

$$\widetilde{\Delta} = T\Delta = \frac{2M}{k_L^2\hbar}\tau\Delta. \tag{17}$$

The wave vector in the momentum representation can be expressed as the spinor

$$\begin{aligned}|\Psi(t)\rangle &\equiv \begin{pmatrix}|\Psi^e(t)\rangle \\ |\Psi^g(t)\rangle\end{pmatrix} \\ &= \sum_{k=-\infty}^{\infty}\int_{-1/2}^{1/2}\begin{pmatrix}a_{k+\beta'}(t) \\ b_{k+\beta'}(t)\end{pmatrix}\delta(\beta-\beta')|k+\beta'\rangle d\beta',\end{aligned} \tag{18}$$

where β is the value of β' for the chosen subspace and

$$\begin{pmatrix}a_{k+\beta}(t) \\ b_{k+\beta}(t)\end{pmatrix} = \begin{pmatrix}\langle k+\beta|\Psi^e(t)\rangle \\ \langle k+\beta|\Psi^g(t)\rangle\end{pmatrix} \tag{19}$$

are the upper and lower components that correspond to the left and right chirality of the QW.

The discrete quantum map is obtained using Eqs. (15) and (18):

$$\begin{pmatrix}a_{k+\beta}(t+T) \\ b_{k+\beta}(t+T)\end{pmatrix} = \sum_{j=-\infty}^{\infty}U(\beta)_{kj}\begin{pmatrix}a_{j+\beta}(t) \\ b_{j+\beta}(t)\end{pmatrix}. \tag{20}$$

The dynamical evolution of the system up to $t = nT$ is obtained applying the above rule, Eq. (20), n times.

A. Resonance $\tau = 2\pi$ in the $\beta = 0$ subspace with $\widetilde{\Delta} = 2m\pi$

In this section we solve analytically the evolution of the system given by the map, Eq. (20). We consider here the principal resonance $\tau = 2\pi$ in the subspace $\beta = 0$. Due to the quasimomentum conservation the value of β does not change. Therefore the accessible momentum spectrum is discrete and from now on the theoretical development is similar to that of the usual QKR in resonance. Additionally we choose $\widetilde{\Delta} = 2m\pi$ with m integer in order to obtain the wave function analytically. We will show afterwards, using numerical calculation, that the qualitative behavior will be similar for arbitrary $\widetilde{\Delta}$. With these conditions the matrix of Eq. (15) only depends on $j-k$. In order to simplify the notation we define

$$U_{k_ak_b}(\kappa) = f_{k_ak_b}(0,\kappa,2\pi)\begin{pmatrix}\delta_{k_a-k_b\,2l} & \delta_{k_a-k_b\,2l+1} \\ \delta_{k_a-k_b\,2l+1} & \delta_{k_a-k_b\,2l}\end{pmatrix}. \tag{21}$$

Using Eq. (20) the initial condition is connected with the wave function at the time $t = nT$ by the equation

$$\begin{aligned}\begin{pmatrix}a_{k_n}(nT) \\ b_{k_n}(nT)\end{pmatrix} = &\sum_{k_{n-1}}\sum_{k_{n-2}}\sum_{k_{n-3}}\cdots \\ &\sum_{k_2}\sum_{k_1}\sum_{k_0}U_{k_nk_{n-1}}(\kappa)U_{k_{n-1}k_{n-2}}(\kappa)\cdots \\ &\times U_{k_2k_1}(\kappa)U_{k_1k_0}(\kappa)\begin{pmatrix}a_{k_0}^0 \\ b_{k_0}^0\end{pmatrix},\end{aligned} \tag{22}$$

where $a_{k_0}^0 = a_{k_0}(0)$ and $b_{k_0}^0 = b_{k_0}(0)$.

Using the relation

$$\sum_{k_{n-1}}U_{k_nk_{n-1}}(\kappa_1)U_{k_{n-1}k_{n-2}}(\kappa_2) = U_{k_nk_{n-2}}(\kappa_1+\kappa_2) \tag{23}$$

obtained in Appendix A, Eq. (22) is reduced to

$$\begin{aligned}\begin{pmatrix}a_k(nT) \\ b_k(nT)\end{pmatrix} = \sum_j i^{j-k}J_{j-k}(n\kappa)\Bigg\{&\delta_{j-k\,2l}\begin{pmatrix}a_j^0 \\ b_j^0\end{pmatrix} \\ &+\delta_{j-k\,2l+1}\begin{pmatrix}b_j^0 \\ a_j^0\end{pmatrix}\Bigg\},\end{aligned} \tag{24}$$

where l is now an arbitrary integer number.

B. Antiresonance $\tau = 2\pi$ in the $\beta = 0$ subspace with $\widetilde{\Delta} = (2m+1)\pi$

We now find the time evolution of the wave function for $\widetilde{\Delta} = (2m+1)\pi$. Equation (15) shows that in this case the matrix $U(\beta=0)_{jk}$ satisfies the relation

$$\sum_{k_{n-1}}U_{k_nk_{n-1}}(\kappa)U_{k_{n-1}k_{n-2}}(\kappa) = \delta_{k_n\,k_{n-2}}I, \tag{25}$$

where I is the identity matrix. This last expression together with Eq. (20) implies that

$$\begin{pmatrix}a_k(nT) \\ b_k(nT)\end{pmatrix} = \delta_{n\,2l+1}\sum_j U_{kj}(\kappa)\begin{pmatrix}a_j^0 \\ b_j^0\end{pmatrix} + \delta_{n\,2l}\begin{pmatrix}a_k^0 \\ b_k^0\end{pmatrix}. \tag{26}$$

Then it is clear that the 2L-QKR shows a periodic behavior when the parameters of the system take the values here considered. This behavior has no analog in the usual QKR since the parameter $\widetilde{\Delta}$ does not exist in said system. Furthermore, it is interesting to point out that this antiresonance occurs for $\tau = 2\pi$, a value for which the usual QKR is in resonance and does not present periodic behavior.

III. PROBABILITY DISTRIBUTION OF MOMENTUM

The evolution of the variance $\sigma^2 = m_2 - m_1^2$ of the probability distribution of momentum is a distinctive feature of the QKR in resonance. It is known that it increases quadratically in time in the quantum case, but only linearly in the classical case. In this section we study the evolution of the variance of the 2L-QKR, once again restricting ourselves to the $\beta = 0$ subspace and taking $\tau = 2\pi$, which corresponds to the primary resonance of the usual QKR model. We will obtain the variance from the evolution of the first and second moments, defined

as $m_1(t) = \sum_k k P_k(t)$ and $m_2(t) = \sum_k k^2 P_k(t)$, respectively, where $P_k(t) = |a_k(t)|^2 + |b_k(t)|^2$ is the probability to find the particle with momentum $p = \hbar k_L k$ at time t.

We first consider the resonance defined by $\widetilde{\Delta} = 2m\pi$. In this case we are able to calculate the first and second moments analytically using Eq. (24) and the properties of the Bessel functions (see Appendix B), obtainin:

$$m_1(n) = \kappa n \sum_{j=-\infty}^{\infty} \mathrm{Im}\left[a_j^0 b_{j+1}^{0*} - a_j^0 b_{j-1}^{0*}\right] + m_1(0), \quad (27)$$

$$m_2(n) = \frac{(\kappa n)^2}{2}\left(1 + \sum_{j=-\infty}^{\infty} \mathrm{Re}\left[a_j^0 a_{j+2}^{0*}(0) + b_j^0 b_{j+2}^{0*}\right]\right) + \kappa n \sum_{j=-\infty}^{\infty} (2j+1)\mathrm{Im}\left[a_j^0 b_{j+1}^{0*} + a_j^0 b_{j-1}^{0*}\right] + m_2(0), \quad (28)$$

where $\mathrm{Re}[x]$ and $\mathrm{Im}[x]$ are, respectively, the real part and imaginary part of x. $m_1(0)$ and $m_2(0)$ are the moments at time $t = 0$. These last equations show that the behavior of the variance $\sigma^2 = m_2 - m_1^2$ has a quadratic time dependence irrespective of the initial conditions taken.

When $\widetilde{\Delta} = (2n+1)\pi$, it was shown in the previous section that the 2L-QKR has a periodic dynamics and therefore the behavior of the statistical moments will be periodic as well. The case when $\widetilde{\Delta} \neq n\pi$ is cumbersome to solve analytically, so we restrict ourselves to a numerical study. The evolution of the second statistical moment was obtained for different values of $\widetilde{\Delta}$ through numerical iterations of the map given by Eq. (20). It was found, for all the considered values of $\widetilde{\Delta} \neq n\pi$, that the long-time behavior of the second moment (and therefore of the variance) is quadratic after an initial transient. The duration of the initial transient depends on the initial conditions and the value of $\widetilde{\Delta}$. This feature can be appreciated in Fig. 1. The figure shows the time evolution of the second moment for the initial conditions $|\Psi(0)\rangle = |k = 0\rangle|g\rangle$. It can be appreciated that the second moment approaches a quadratic behavior after an oscillatory transient. It was found that the nearer the parameter $\widetilde{\Delta}$ is to $(2n+1)\pi$, the more pronounced this oscillation is.

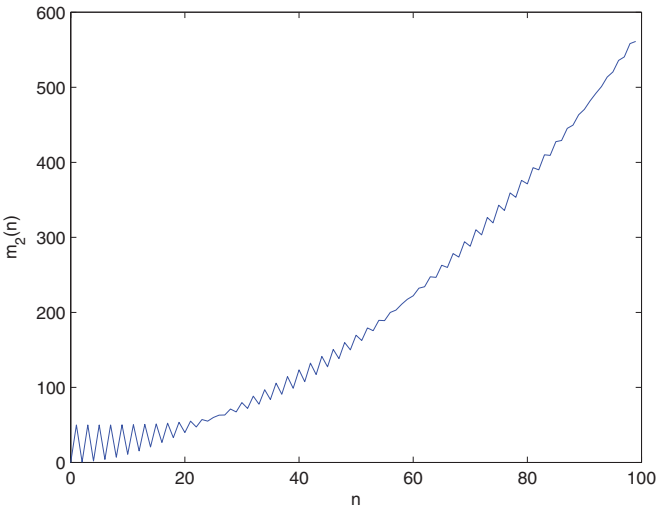

FIG. 1. (Color online) The dimensionless second moment for $\widetilde{\Delta} = 0.97\pi$ as a function of the dimensionless time.

An interesting question is whether the behavior described in the previous sections still holds for values of β different than zero. This question is addressed in Ref. [44] for the system with only one internal energy level, given by the Hamiltonian

$$\widehat{H} = \frac{[\hbar k_L(\widehat{k} + \widehat{\beta})]^2}{2M} + K\delta_T(t)\cos(\widehat{\theta}), \quad (29)$$

which yields the usual QKR Hamiltonian in the $\beta = 0$ subspace.

Considering the case when $\tau = l\pi$, with l an integer, the authors [44] find that the primary resonance condition for generic β is

$$l(1 + 2\beta) = 2m, \quad (30)$$

where m is an integer. This condition is found by imposing that the operator corresponding to the free evolution of the system between two kicks, $U_L = \exp(-i\frac{[\hbar k_L(\widehat{k}+\widehat{\beta})]^2}{2MT\hbar})$, be equal to the identity. Evaluating the previous condition for $\beta = 0$, this yields $l = 2m$, the primary resonance condition in the QKR.

The authors also find the condition for antiresonance for general β as

$$l(1 + 2\beta) = 2m + 1. \quad (31)$$

This condition can be found by imposing that the Floquet operator of the system, U_T, satisfy $U_T^2 = 1$.

For the case of the system considered in this paper, the previous conditions hold when $\Delta = 2n\pi$. In this case the operator $|e\rangle\langle e|$ disappears from the dynamics of the system, and the same derivations that give way to the previous conditions hold, regardless of the presence of the Pauli matrix σ_x. When $\Delta \neq 2n\pi$, the previous derivations no longer hold, given that the operator $|e\rangle\langle e|$ does not commute with σ_x.

IV. ENTANGLEMENT

Considering a quantum system consisting of two quantum subsystems, the phenomenon of entanglement is related to the appearance of correlations between observables belonging to each of the two subsystems. In a quantum system with Hilbert space given by the product of two Hilbert spaces, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, a state $|\Psi\rangle$ in $\mathcal{H}$ is said to be entangled if it cannot be expressed as $|\Psi\rangle = |\Psi\rangle_1 \otimes |\Psi\rangle_2$, with $|\Psi\rangle_1$ and $|\Psi\rangle_2$ being vectors belonging to $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. If $|\Psi\rangle$ could be expressed as a product state, then the expectation of the product of any two observables A_1 and A_2, acting on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, would be $\langle A_1 A_2\rangle = \langle A_1\rangle \langle A_2\rangle$ and they would be uncorrelated.

In this frame, it is interesting to study whether the evolution of a system consisting of the product of two Hilbert spaces, such as the 2L-QKR or the QW, generates an entangled state from an unentangled initial condition. It is in fact possible to quantify, through a certain measure, the amount of entanglement generated by the time evolution for different initial conditions. In the context of QWs several authors [46–58] have been investigating the relationship between the asymptotic coin-position entanglement and the initial conditions of the walk. In order to compare the model

considered in this paper with the QW, we investigate the asymptotic chirality-momentum entanglement in the 2L-QKR.

Before addressing the problem of entanglement, an interesting question regarding the similarity between the QW and the 2L-QKR is whether the evolution of the latter can be put in terms of a coin and a shift operator as in the QW. If the value of β is given, the value of the momenta of the particle in the 2L-QKR is specified by the discrete eigenvalues of $\widehat{k}$, which can be identified with the position along the $\mathbb{Z}$ grid in the quantum walk. The internal states in the 2L-QKR are identified with the chirality states in the quantum walk model $|L\rangle$ and $|R\rangle$. Within this framework, the one step time evolution of both systems can be put in the form of Eq. (20),

$$\begin{pmatrix} a_k(t+T) \\ b_k(t+T) \end{pmatrix} = \sum_{j=-\infty}^{\infty} U_{kj} \begin{pmatrix} a_j(t) \\ b_j(t) \end{pmatrix}. \tag{32}$$

In the case of the QW, if H_{ij} are the components of the coin operator in the $\{|L\rangle,|R\rangle\}$ basis, then U_{kj} is given by [59]

$$U_{kj} = \begin{pmatrix} H_{11} & H_{12} \\ 0 & 0 \end{pmatrix} \delta_{kj+1} + \begin{pmatrix} 0 & 0 \\ H_{21} & H_{22} \end{pmatrix} \delta_{kj-1}, \tag{33}$$

where the 2×2 matrices act on the inner states and the action of the shift operator is given by the Kronecker deltas. From Eq. (33) it is seen that the QW evolution only connects points in the $\mathbb{Z}$ grid with their immediate neighbors. However, the 2L-QKR evolution given by Eq. (15) connects every point in the momentum grid with each other, as the matrix U_{jk} in this case has nonzero elements for any pair jk. Therefore the 2L-QKR operator cannot be put in the simple form of the QW with a coin and a shift operator. However, since U_{jk} for the 2L-QKR is proportional to $J_{j-k}(\kappa)$, then U_{j-k} goes to zero rapidly in the limit of large $|j-k|$ when κ is small. This is due to the fact that

$$J_{j-k} \sim \frac{1}{\Gamma(j-k)} \left(\frac{\kappa}{2}\right)^{j-k} \tag{34}$$

for non-negative $j-k$ when $0 < \kappa \ll \sqrt{j-k+1}$. Therefore, in this limit, when the interaction κ is small, the map of the 2L-QKR is more similar to the quantum walk map in this respect. For larger κ, that is, stronger interaction, more momentum modes are coupled by the evolution of the system.

Even though not formally identical to the QW, the unitary evolution of the 2L-QKR generates entanglement between chirality and momentum degrees of freedom in a manner similar to that of the aforementioned system. This entanglement will be characterized [46,57] by the von Neumann entropy of the reduced density operator, called entropy of entanglement. The quantum analog of the Gibbs entropy is the von Neumann entropy

$$S_N(\rho) = -\mathrm{tr}(\rho \ln \rho), \tag{35}$$

where $\rho = |\Psi(t)\rangle \langle\Psi(t)|$ is the density matrix of the quantum system. Owing to the unitary dynamics of the 2L-QKR, the system remains in a pure state, and this entropy vanishes. In spite of this chirality and momentum are entangled, and the entanglement can be quantified by the associated von Neumann entropy for the reduced density operator:

$$S = -\mathrm{tr}(\rho_c \log_2 \rho_c), \tag{36}$$

where $\rho_c = \mathrm{tr}_k(\rho)$ is the reduced density matrix that results from taking the partial trace over the momentum space. The reduced density operator can be explicitly obtained using the wave function, Eq. (18), in the subspace $\beta = 0$ and its normalization properties

$$\rho_c = \begin{pmatrix} P_g(n) & Q(n) \\ Q^*(n) & P_e(n) \end{pmatrix}, \tag{37}$$

where

$$P_g(n) = \sum_{j=-\infty}^{\infty} |a_k(nT)|^2, \tag{38}$$

$$P_e(n) = \sum_{j=-\infty}^{\infty} |b_k(nT)|^2, \tag{39}$$

$$Q(n) = \sum_{j=-\infty}^{\infty} a_k(nT) b_k^*(nT). \tag{40}$$

$P_e(n)$ and $P_g(n)$ may be interpreted as the time-dependent probabilities for the system to be in the excited and the ground states, respectively. In order to investigate the entanglement dependence on the initial conditions, we consider the localized case, that is, the initial state of the rotor is assumed to be sharply localized with vanishing momentum and arbitrary chirality, thus

$$\begin{pmatrix} a_k(0) \\ b_k(0) \end{pmatrix} = \begin{pmatrix} \cos\frac{\gamma}{2} \\ \exp i\varphi \ \sin\frac{\gamma}{2} \end{pmatrix} \delta_{k0}, \tag{41}$$

where $\gamma \in [0,\pi]$ and $\varphi \in [0,2\pi]$ define a point on the unit three-dimensional Bloch sphere. Equation (24) takes the following form:

$$\begin{pmatrix} a_k(nT) \\ b_k(nT) \end{pmatrix} = i^k J_k(n\kappa) \left\{ \delta_{k\,2l} \begin{pmatrix} \cos\frac{\gamma}{2} \\ \exp i\varphi \ \sin\frac{\gamma}{2} \end{pmatrix} + \delta_{k\,2l+1} \begin{pmatrix} \exp i\varphi \ \sin\frac{\gamma}{2} \\ \cos\frac{\gamma}{2} \end{pmatrix} \right\}. \tag{42}$$

Substituting Eq. (42) into Eqs. (38)–(40) and using the properties of the Bessel functions, we obtain

$$P_g(n) = \frac{1}{2}\left[1 + J_0(2n\kappa)\cos\gamma\right], \tag{43}$$

$$P_e(n) = \frac{1}{2}\left[1 - J_0(2n\kappa)\cos\gamma\right], \tag{44}$$

$$Q(n) = \frac{\sin\gamma}{2}\left[\cos\varphi - i\,\sin\varphi J_0(2n\kappa)\right]. \tag{45}$$

The eigenvalues of the density operator ρ_c, Eq. (37), as a function of $P_g(n)$, $P_e(n)$, and $Q(n)$ is

$$\lambda_\pm = \frac{1}{2}[1 \pm \sqrt{1 - 4(P_g(n)\,P_e(n) - |Q(n)|^2)}], \tag{46}$$

and the reduced entropy as a function of these eigenvalues is

$$S(n) = -\lambda_+ \log_2 \lambda_+ - \lambda_- \log_2 \lambda_-. \tag{47}$$

Therefore the dependence of the entropy on the initial conditions is expressed through the angular parameters φ and γ. This means that, given certain initial conditions, the degree of entanglement of the chirality and momentum degrees of freedom is determined.

It is seen from Eqs. (43)–(45) that the occupation probabilities and the coherence Q tend to a certain limit when

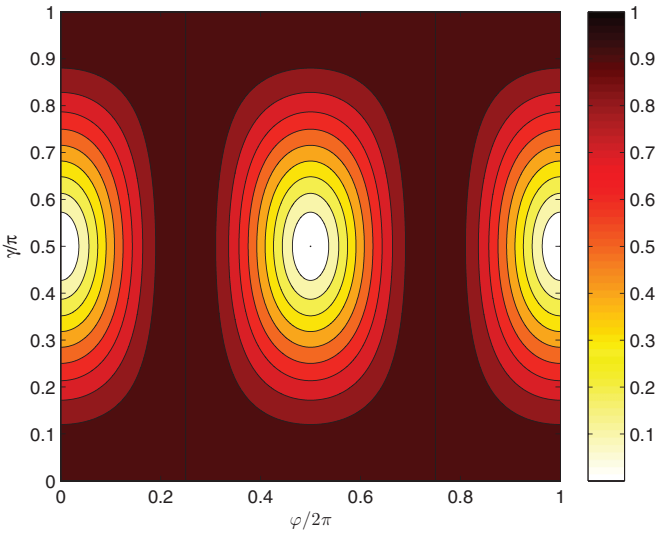

FIG. 2. (Color online) The dimensionless entanglement entropy as a function of the dimensionless initial conditions [see Eq. (41)]. The gray scale (color online) corresponds to different values of the entropy between zero and one.

$n \to \infty$. In this limit $J_0(2n\kappa) \to 0$ and both of the occupation probabilities tend to $1/2$, irrespective of the initial conditions. However, in the asymptotic regime, dependence on the initial conditions is still maintained by Q, and therefore by the entropy as well. Thus, in the asymptotic regime we have

$$\lambda_\pm \to \Lambda_\pm = \tfrac{1}{2}\left[1 \pm \cos\varphi \, \sin\gamma\right], \tag{48}$$

and the asymptotic value of the entropy $S(n) \to S_0$ is

$$S_0 = -\Lambda_+ \log_2 \Lambda_+ - \Lambda_- \log_2 \Lambda_-. \tag{49}$$

Figure 2 shows the dependence of the asymptotic entanglement entropy on the parameters φ and γ that determine the initial conditions given by Eq. (41). For the initial condition $\varphi = \pi/2$ and/or $\gamma = \pi$ on the Bloch sphere, $Q \to 0$ and both eigenvalues are $\Lambda_\pm = 1/2$. In this case the asymptotic entanglement entropy, Eq. (49), has its maximum value $S_0 = 1$. When initial conditions with $\gamma = \pi/2$ and $\phi = 0,\pi$ are chosen, the asymptotic entanglement entropy goes to zero.

We have found that, for an un-entangled initial condition of the form

$$|\Psi(0)\rangle = \left\{\cos\left(\frac{\gamma}{2}\right)|e\rangle + \sin\left[\frac{\gamma}{2}\exp(i\phi)\right]|e\rangle\right\}|\beta = 0, k = 0\rangle,$$

the evolution of the system generates asymptotically a state whose entanglement depends on the parameters that determine the initial condition. This dependence is quantified in Fig. 2. It was also seen that for certain specific initial conditions the evolution can yield a state with either maximum or no entanglement at all. This behavior is similar to what is seen for the QW [46,47], although in that case it is found that there is no localized initial condition which yields an un-entangled asymptotic state. The qualitative similarity, between the QW and the 2L-QKR, for the dependence on the initial condition of the asymptotic entanglement is not surprising due to the fact that both systems have the same Hilbert space and the quantum evolution maps of both systems are similar. However, quantitative aspects of these results differ due to the particular details of each model.

Finally, it is interesting to point out that this kind of entanglement has been used [58] to define a QW temperature which characterizes the local equilibrium between the chirality and position degrees of freedom. This kind of idea could be extended to the system studied in this paper to define a thermal equilibrium between the internal states and the momentum degrees of freedom.

V. CONCLUSION

We have studied a modified QKR model with an additional degree of freedom, the 2L-QKR. This system exhibits quantum resonances with a ballistic spreading of the variance of the momentum distribution, and entanglement between the internal and momentum degrees of freedom only depending on the initial conditions. These results were established analytically and numerically for different values of the parameter space of this system. The above two behaviors also characterize the QW on the line and help further the similarities previously studied between the two systems. It is important to point out that these phenomena (ballistic spreading of the variance, dependence of the asymptotic entanglement on the initial conditions) could be of great importance for future quantum computing applications. In particular, the ballistic spreading of the variance could be used to render quantum search algorithms significantly faster than their classical counterparts. The fact that these behaviors are present in several quantum systems augments the possibility of the realization of quantum computing devices. The model studied in this paper, a variant of the model studied by [44], could be realized as a modification on the existing experimental model of the latter.

We have also found that, although our system exhibits characteristics similar to those found in the usual QKR model, there are still features, such as the existence of the antiresonance described in Sec. II B, which have no analog in the usual QKR model. These characteristics of the 2L-QKR render the system as an interesting candidate for further study within the framework of quantum computation.

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APPENDIX A

Starting from Eq. (21) the following expression is obtained:

$$\sum_{k_1} U_{k_1 k_2}(\kappa) U_{k_0 k_1}(\kappa) = i^{k_2 - k_0} \sum_{k_1} J_{\nu_2}(\kappa) J_{\nu_1}(\kappa) \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}, \tag{A1}$$

where

$$E_1 = e^{-i2\tilde{\Delta}}\delta_{\nu_1\, 2l}\delta_{\nu_2\, 2l'} + e^{-i\tilde{\Delta}}(1 - \delta_{\nu_1\, 2l})(1 - \delta_{\nu_2\, 2l'}),$$
$$E_2 = e^{-i2\tilde{\Delta}}\delta_{\nu_1\, 2l}(1 - \delta_{\nu_2\, 2l'}) + e^{-i\tilde{\Delta}}\,\delta_{\nu_2\, 2l'}(1 - \delta_{\nu_1\, 2l}),$$
$$E_3 = e^{-i\tilde{\Delta}}\,\delta_{\nu_2\, 2l'}(1 - \delta_{\nu_1\, 2l}) + \delta_{\nu_1\, 2l}(1 - \delta_{\nu_2\, 2l'}),$$
$$E_4 = e^{-i\tilde{\Delta}}(1 - \delta_{\nu_1\, 2l})(1 - \delta_{\nu_2\, 2l'}) + \delta_{\nu_1\, 2l}\delta_{\nu_2\, 2l'},$$

and with $\nu_1 = k_1 - k_0$, $\nu_2 = k_2 - k_1$. In the above equations, three different types of sums are involved, which can be carried out using the properties of the Bessel functions (Ref. [60], p. 992, Eq. **8.530**).

$$\sum_{k_1} J_{k_2-k_1}(\kappa) J_{k_1-k_0}(\kappa) = J_{\mu_2}(2\kappa), \tag{A2}$$

$$\sum_{k_1} J_{k_2-k_1}(\kappa) J_{k_1-k_0}(\kappa) \delta_{k_1-k_0\,2l} = \frac{1}{2}[J_{\mu_2}(2\kappa) + \delta_{k_2k_0}], \tag{A3}$$

$$\sum_{k_1} J_{k_2-k_1}(\kappa) J_{k_1-k_0}(\kappa) \delta_{k_1-k_0\,2l} \delta_{k_2-k_1\,2l'} = \frac{1}{2}\delta_{\mu_2\,2(l+l')}[J_{\mu_2}(2\kappa) + \delta_{k_2k_0}], \tag{A4}$$

where $\mu_2 = k_2 - k_0$. Substituting the above equations into Eq. (A1) and defining $p = l + l'$,

$$\sum_{k_1} U_{k_1k_2}(\kappa_1) U_{k_0k_1}(\kappa_2) = \frac{e^{-i\tilde{\Delta}}}{2}\left[\begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix} + \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}\right], \tag{A5}$$

where

$$\begin{aligned} F_1 &= i^{\mu_2} J_{\mu_2}(2\kappa)\delta_{\mu_2\,2p}(1 + e^{-i\tilde{\Delta}}), \\ F_2 &= i^{\mu_2} J_{\mu_2}(2\kappa)(1 + e^{-i\tilde{\Delta}})(1 - \delta_{\mu_2\,2p}), \\ F_3 &= i^{\mu_2} J_{\mu_2}(2\kappa)(1 + e^{i\tilde{\Delta}})(1 - \delta_{\mu_2\,2p}), \\ F_4 &= i^{\mu_2} J_{\mu_2}(2\kappa)\delta_{\mu_2\,2p}(1 + e^{i\tilde{\Delta}}), \\ G_1 &= \delta_{k_2k_0}(e^{-i\tilde{\Delta}} - 1), \\ G_2 &= \delta_{k_2k_0}(e^{i\tilde{\Delta}} - 1). \end{aligned}$$

APPENDIX B

The probability $P_k(n)$ of finding the system with momentum k at a time $t = nT$ is obtained using Eq. (24).

$$\begin{aligned} P_k(n) &= |a_k(n)|^2 + |b_k(n)|^2 \\ &= \frac{1}{2}\sum_{j,l} f_{jl}\left[a_j^0 a_l^{0*} + b_j^0 b_l^{0*}\right] + \frac{1}{2}\sum_{j,l} \mathrm{Re}\left\{f_{jl}\left[a_j^0 b_l^{0*}\right]\right\}, \end{aligned} \tag{B1}$$

where

$$f_{jl} = i^{l-j}[J_{k-j}(n\kappa)J_{k-l}(n\kappa) + J_{k-j}(-n\kappa)J_{k-l}(-n\kappa)]$$

and a_k^0 and b_k^0 are given by the initial conditions of the system. To calculate the moments $m_1(n)$ and $m_2(n)$ we need the following sums:

$$I_{jl}^{(1)} = i^{l-j}\sum_{k=-\infty}^{\infty} k J_{k-j}(\kappa) J_{k-l}(\kappa) = j\delta_{jl} - \frac{i\kappa}{2}(\delta_{lj+1} - \delta_{lj-1}) \tag{B2}$$

and

$$\begin{aligned} I_{jl}^{(2)} &= i^{l-j}\sum_{k=-\infty}^{\infty} k^2 J_{k-j}(\kappa) J_{k-l}(\kappa) \\ &= \frac{\kappa^2}{2}\left[\delta_l j - \frac{1}{2}(\delta_{lj+2} + \delta_{lj-2})\right] \\ &\quad + i\kappa\left[\frac{1}{2}(\delta_{lj+1} + \delta_{lj-1}) + j(\delta_{lj+1} - \delta_{lj-1})\right] + l^2\delta_{jl}. \end{aligned} \tag{B3}$$

Using these expressions together with Eq. (B1) and the definition of the moments we obtain the first and second moments, Eqs. (27) and (28).

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