Nonunital non-Markovianity of quantum dynamics

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Trace distance is available to capture the dynamical information of the unital aspect of a quantum process. However, it cannot reflect the nonunital part. So the nondivisibility originating from the nonunital aspect cannot be revealed by the corresponding measure based on the trace distance. We provide a measure of nonunital non-Markovianity of quantum processes, which is a supplement to Breuer-Laine-Piilo (BLP) non-Markovianity measure. A measure of the degree of the nonunitality is also provided.

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I. INTRODUCTION

Understanding and characterizing general features of the dynamics of open quantum systems is of great importance to physics, chemistry, and biology [1]. The non-Markovian character is one of the most central aspects of an open quantum process and has attracted increasing attention [2–16]. Markovian dynamics of quantum systems is described by a quantum dynamical semigroup [1,17] and is often taken as an approximation of realistic circumstances with some very strict assumptions. Meanwhile, exact master equations, which describe the non-Markovian dynamics, are complicated [9]. Based on the infinitesimal divisibility in terms of a quantum dynamical semigroup, Wolf *et al*. provided a model-independent way to study the non-Markovian features [2,3]. Later, in the intuitive picture of the backward information flow leading to the increasing of distinguishability in intermediate dynamical maps, Breuer, Laine, and Piilo (BLP) proposed a measure of the degree of non-Markovian behavior based on the monotonicity of the trace distance under quantum channels [4], as shown in Fig. 1. The BLP non-Markovianity has been widely studied and applied in various models [18–23].

Unlike for classical stochastic processes, the non-Markovian criteria for quantum processes are nonunique and even controversial. First, the non-Markovian criteria from the infinitesimal divisibility and the backward information flow are not equivalent [19,20]. Second, several other non-Markovianity measures, based on different mechanisms such as the monotonicity of correlations under local quantum channels, have been introduced [6,13]. Third, even in the framework of backward information flow, trace distance is not the unique monotone distance for the distinguishability between quantum states. Other monotone distances in the space of density operators can be found in Ref. [24], and the statistical distance [25,26] is another widely used one. Different distances should not be expected to give the same non-Markovian criteria. The inconsistency among various non-Markovianity measures reflects different dynamical properties.

In this paper, we show that the BLP non-Markovianity cannot reveal the infinitesimal nondivisibility of quantum processes caused by the nonunital part of the dynamics. Besides non-Markovianity, "nonunitality" is another important dynamical property, which is the necessity for the increase of the purity $\mathrm{Tr}\rho^2$ under quantum channels [27] and for the creation of quantum discord in two-qubit systems under local quantum channels [28]. In the same spirit as BLP non-Markovianity, we define a measure on the nonunitality. As BLP non-Markovianity is the most widely used measure on non-Markovianity, we also provide a measure of the nonunital non-Markovianity, which can be conveniently used as a supplement to the BLP measure when the quantum process is nonunital. We also give an example to demonstrate an extreme case, where the BLP non-Markovianity vanishes while the quantum process is not infinitesimally divisible.

This paper is organized as follows. In Sec. II, we give a brief review of the representation of density operators and quantum channels with a Hermitian orthonormal operator basis and various measures on non-Markovianity. In Sec. III, we investigate the nonunitality and the nonunital non-Markovianity and give the corresponding quantitative measures. In Sec. IV, we apply the nonunital non-Markovianity measure to a family of quantum processes, which are constructed from the generalized amplitude damping channels. Section V is the conclusion.

II. REVIEW OF QUANTUM CHANNELS AND NON-MARKOVIANITY

A. Density operators and quantum channels represented by the Hermitian operator basis

The states of a quantum system can be described by the density operator ρ, which is positive semidefinite and of trace 1. Quantum channels, or quantum operations, are completely positive and trace-preserving (CPT) maps from density operators to density operators and can be represented by Kraus operators, Choi-Jamiołkowski matrices, or transfer matrices [29–32].

In this work, we use the Hermitian operator basis to express operators and represent quantum channels. Let $\{\lambda_\mu \mid \mu = 0,1,\ldots,d^2-1\}$ be a complete set of Hermitian and orthonormal operators on complex space $\mathbb{C}^d$, i.e., λ_μ satisfies $\lambda_\mu^\dagger = \lambda_\mu$ and $\langle\lambda_\mu,\lambda_\nu\rangle := \mathrm{Tr}(\lambda_\mu^\dagger\lambda_\nu) = \delta_{\mu\nu}$. Any operator O on $\mathbb{C}^d$ can be express by a column vector $r := (r_0, r_1, \ldots, r_{d^2-1})^{\mathrm{T}}$

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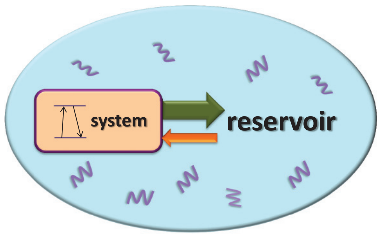

FIG. 1. (Color online) Sketch of the information flow picture for non-Markovianity [4]. According to this scenario, the loss of distinguishability of the system's states indicates the information flow from the system to the reservoir. If the dynamics is Markovian, the information flow is always outward, represented by the thick green arrow. Non-Markovian behaviors occurs when there is inward information flow, represented by the thin orange arrow, bringing some distinguishability back to the system.

through

$$O = \sum_{\mu=0}^{d^2-1} r_\mu(O)\lambda_\mu, \tag{1}$$

with $r_\mu(O) := \langle\lambda_\mu, O\rangle$. Every $r_\mu(O)$ is real if O is Hermitian.

In the meantime, any quantum channel $\mathcal{E} : \rho \mapsto \mathcal{E}(\rho)$ can be represented by $T(\mathcal{E}) : r(\rho) \mapsto r[\mathcal{E}(\rho)]$ via

$$r[\mathcal{E}(\rho)] = T(\mathcal{E})r(\rho), \tag{2}$$

where $T(\mathcal{E})$ is a $d^2 \times d^2$ real matrix with the elements

$$T_{\mu\nu}(\mathcal{E}) := \langle\lambda_\mu, \mathcal{E}(\lambda_\nu)\rangle. \tag{3}$$

Furthermore, one can easily check that

$$T(\mathcal{E}_1 \circ \mathcal{E}_2) = T(\mathcal{E}_1)T(\mathcal{E}_2) \tag{4}$$

for the composition of quantum channels. Here $\mathcal{E}_1 \circ \mathcal{E}_2$ denotes the composite maps $\mathcal{E}_1(\mathcal{E}_2(\rho))$.

Taking into account the normalization of the quantum states, i.e., $\mathrm{Tr}(\rho) = 1$, r_0 can be fixed as $r_0(\rho) = 1/\sqrt{d}$ for any density operator ρ by choosing $\lambda_0 = \mathbb{1}/\sqrt{d}$, with $\mathbb{1}$ being the identity operator. In such a case, λ_μ for $\mu = 1,2,\ldots,d^2-1$ are traceless and generate the algebra su(d). This real parametrization $r_\mu(\rho)$ for density operators is also called a coherent vector or generalized Bloch vector [33–35]. In order to eliminate the degree of freedom for the fixed r_0, we use the decomposition $r = (r_0, \mathbf{r})^{\mathrm{T}}$. Therefore, any density operator ρ can be expressed as

$$\rho = \frac{\mathbb{1}}{d} + \mathbf{r}\cdot\boldsymbol{\lambda}, \tag{5}$$

where $\mathbf{r}$ is the generalized Bloch vector and $\boldsymbol{\lambda}$ represents $(\lambda_1,\lambda_2,\ldots,\lambda_{d^2-1})^{\mathrm{T}}$. Under this frame, quantum channels can be represented by the affine map [17,36]

$$\mathbf{r}(\mathcal{E}(\rho)) = M(\mathcal{E})\mathbf{r}(\rho) + \mathbf{c}(\mathcal{E}), \tag{6}$$

where $M(\mathcal{E})$ is a real matrix with the dimension d^2-1 and the elements of the vector $\mathbf{c}(\mathcal{E})$ read

$$[\mathbf{c}(\mathcal{E})]_\mu = \langle\lambda_\mu, \mathcal{E}(\mathbb{1})\rangle/d \tag{7}$$

for $\mu = 1,2,\ldots,d^2-1$. Comparing Eq. (2) with Eq. (6), one can find that

$$T_{\mu\nu}(\mathcal{E}) = [M(\mathcal{E})]_{\mu\nu} \tag{8}$$

for $\mu,\nu = 1,2,\ldots,d^2-1$. Thus, $T(\mathcal{E})$ can be decomposed into the following subblocks:

$$T(\mathcal{E}) = \left[\begin{array}{c|c} 1 & 0_{1\times(d^2-1)} \\ \hline \sqrt{d}\mathbf{c} & M \end{array}\right]. \tag{9}$$

Recalling that a quantum channel $\mathcal{E}$ is said to be unital if and only if $\mathcal{E}(\mathbb{1}/d) = \mathbb{1}/d$ [36], one can find that the necessary and sufficient condition for a unital map is that $\mathbf{c}(\mathcal{E}) = 0$, namely,

$$\mathbf{c}(\mathcal{E}) = 0 \Longleftrightarrow \mathcal{E} \text{ is unital}. \tag{10}$$

Thus, $\mathbf{c}(\mathcal{E})$ describes the nonunital property of the quantum channel $\mathcal{E}$. The necessary and sufficient condition above could be easily proved by realizing that the Bloch vector of $\mathbb{1}/d$ is a zero vector, i.e., $\mathbf{r} = 0$. Based on the subblock form of $T(\mathcal{E})$, $\mathbf{c}(\mathcal{E}) = 0$ is equivalent to $T(\mathcal{E})$ being a block diagonal, i.e., $T(\mathcal{E}) = \mathrm{diag}[1, M(\mathcal{E})]$.

Whether a quantum channel $\mathcal{E}$ is completely positive (CP) can be reflected by the Choi-Jamiołkowski matrix [30,31],

$$C(\mathcal{E}) := (\mathcal{E} \otimes \mathbb{1})(|\Omega\rangle\langle\Omega|), \tag{11}$$

where $|\Omega\rangle = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}|j\rangle\otimes|j\rangle$ is the maximally entangled state. Here $\{|j\rangle\}$ is a basis in Hilbert space. $\mathcal{E}$ is CP if and only if the Choi-Jamiołkowski matrix is positive. With the Hermitian operator basis, $|\Omega\rangle\langle\Omega|$ is a $d^2 \times d^2$ matrix and can be written in the form [37]

$$|\Omega\rangle\langle\Omega| = \frac{1}{d}\sum_{\nu=0}^{d^2-1} \lambda_\nu \otimes \lambda_\nu^{\mathrm{T}}. \tag{12}$$

Substituting this formula into Eq. (11) and utilizing Eq. (3), one can express the Choi-Jamiołkowski matrix as

$$C(\mathcal{E}) = \sum_{\mu,\nu=0}^{d^2-1} \frac{1}{d} T_{\mu\nu}(\mathcal{E})\lambda_\mu \otimes \lambda_\nu^{\mathrm{T}}. \tag{13}$$

If $\mathcal{E}$ is unital, it can be reduced to

$$C(\mathcal{E}) = \frac{1}{d^2}\left(\mathbb{1}_{d^2\times d^2} + d\sum_{\mu,\nu=1}^{d^2-1} M_{\mu\nu}\lambda_\mu \otimes \lambda_\nu^{\mathrm{T}}\right). \tag{14}$$

B. Nondivisibility and non-Markovianity

Without the presence of correlation between the open system and its environment in the initial states, the reduced dynamics for the open system from $t = 0$ to any $t \geqslant 0$ can be expressed as

$$\mathcal{E}_{t,0} : \rho \mapsto \mathrm{Tr}_E[U(t)(\rho \otimes \rho_E)U(t)^\dagger], \tag{15}$$

which is a quantum channel. This indicates that $\mathcal{E}_{t,0}$ is CPT. The unitary operator $U(t)$ describes the time evolution of the closed total system, and ρ_E is the initial state of the environment. A quantum process $\mathcal{E}_t := \mathcal{E}_{t,0}$ is said to be infinitesimally divisible, also called a time-inhomogeneous or time-dependent Markovian, if it satisfies the following composition law [2]:

$$\mathcal{E}_{t_2,0} = \mathcal{E}_{t_2,t_1} \circ \mathcal{E}_{t_1,0} \tag{16}$$

for any $t_2 \geqslant t_1 \geqslant 0$, where $\mathcal{E}_{t_2,t_1}$ is also completely positive and trace preserving.

Various measures of the degree of the non-Markovian behavior of quantum processes have been proposed and investigated [4,6,11–13]. Almost all of the measures of the non-Markovianity can be classified into three types based on the degree of the violation of the following properties owned by the infinitesimally divisible quantum process.

(i) The first type is monotonicity of distance D under CPT maps. That is, $D(\mathcal{E}(\rho_1),\mathcal{E}(\rho_2)) \leqslant D(\rho_1,\rho_2)$ for any quantum channel $\mathcal{E}$, where $D(\rho_1,\rho_2)$ is an appropriate monotone distance under CPT maps of the space of density operators [24], including trace distance, Bures distance, statistical distance, relative entropy, fidelity (although fidelity itself is not a distance, it can be used to construct monotone distances), and so on. Some measures of non-Markovianity by increasing the monotone distance during the mediate dynamical maps $\mathcal{E}_{t_2,t_1}$ have been given and discussed in Refs. [4,12].

The typical measure of this type, which will be used later in this paper, was first proposed by Breuer, Laine, and Piilo in Ref. [4] based on the monotonicity of trace distance [35,36]:

$$D_{\mathrm{tr}}(\rho_1,\rho_2) := \tfrac{1}{2}\mathrm{Tr}|\rho_1 - \rho_2|, \tag{17}$$

where $|O| := \sqrt{O^\dagger O}$. Interpreting the increase of the trace distance during the time evolution as the information flows from the environment back to the system, the definition of the BLP non-Markovianity is defined by

$$\mathcal{N}_{\mathrm{BLP}}(\mathcal{E}_t) := \max_{\rho_1,\,\rho_2} \int_{\sigma>0} dt\, \sigma(t,\rho_1,\rho_2), \tag{18}$$

where

$$\sigma(t,\rho_1,\rho_2) := \frac{d}{dt} D_{\mathrm{tr}}(\rho_1(t),\rho_2(t)) \tag{19}$$

and $\rho_i(t) = \mathcal{E}_t(\rho_i)$ for $i = 1,2$ are two evolving states.

(ii) The second is positivity of the Choi-Jamiołkowski matrix for CPT maps. The Choi-Jamiołkowski matrix $C(\mathcal{E}) \geqslant 0$ if and only if $\mathcal{E}$ is a quantum channel, namely, $\mathcal{E}$ is a CPT map. Some measures of non-Markovianity by the negativity of the Choi-Jamiołkowski matrix for mediate dynamical maps $\mathcal{E}_{t_2,t_1}$ have been given and discussed in Refs. [6,11].

In this work we will use one of these measures, which was proposed by Rivas, Huelga, and Plenio (RHP) in Ref. [6]. They utilize the negativity of the Choi-Jamiołkowski matrix C for the mediate dynamical maps with the definition

$$\mathcal{N}_{\mathrm{RHP}}(\mathcal{E}_t) := \int_0^\infty g(t)dt, \tag{20}$$

where

$$g(t) := \lim_{\epsilon\to 0^+} \frac{\mathrm{Tr}|C(\mathcal{E}_{t+\epsilon,t})| - 1}{\epsilon}. \tag{21}$$

(iii) The last type is monotonicity of correlations E under local quantum channels. That is, $E[(\mathcal{E}\otimes \mathbb{1})(\rho^{AB})] \leqslant E(\rho^{AB})$ for any local quantum channel $\mathcal{E}$, where E is an appropriate measure for the correlations in the bipartite states ρ^{AB}, including entanglement entropy and mutual information. The corresponding measures of non-Markovianity are given and discussed in Refs. [6,13].

III. NONUNITAL NON-MARKOVIANITY

The non-Markovianity measure $\mathcal{N}_{\mathrm{BLP}}$ is able to capture the non-Markovian behavior of the unital aspect of the dynamics. But for the nonunital aspect, it is not capable of capturing the non-Markovian behavior. To show this, we use the Hermitian orthonormal operator basis to express states and quantum channels. Utilizing Eq. (5), the trace distance between two states ρ_1 and ρ_2 is given by

$$D_{\mathrm{tr}}(\rho_1,\rho_2) = \tfrac{1}{2}\mathrm{Tr}|[\mathbf{r}(\rho_1) - \mathbf{r}(\rho_2)]\cdot\boldsymbol{\lambda}|. \tag{22}$$

Therefore, for the two evolving states, we get

$$D_{\mathrm{tr}}(\rho_1(t),\rho_2(t)) = \tfrac{1}{2}\mathrm{Tr}|M(\mathcal{E}_t)[\mathbf{r}(\rho_1) - \mathbf{r}(\rho_2)]\cdot\boldsymbol{\lambda}|, \tag{23}$$

where ρ_1, ρ_2 are initial states of the system.

From this equation one can see that the trace distance between any two evolved states is irrelevant to the nonunital part $\mathbf{c}(\mathcal{E}_t)$ of the time evolution. Then, if there are two quantum channels, whose affine maps are $\mathbf{r} \mapsto M\mathbf{r} + \mathbf{c}_1$ and $\mathbf{r} \mapsto M\mathbf{r} + \mathbf{c}_2$, the characteristic of trace distance between the evolving states from any two initial states cannot distinguish these two channels. More importantly, $\mathbf{c}(\mathcal{E}_t)$ may cause the nondivisibility of the quantum process $\mathcal{E}_t$, and this cannot be revealed by $\mathcal{N}_{\mathrm{BLP}}$.

On the other hand, the nonunital part $\mathbf{c}(\mathcal{E}_t)$ has its own physical meaning: $\mathbf{c}(\mathcal{E}_t) \neq 0$ is necessary for the increasing of the purity $\mathcal{P}(\rho) = \mathrm{Tr}(\rho^2)$ [27]. In other words,

$$\mathbf{c}(\mathcal{E}_t) = 0 \implies \mathcal{P}(\mathcal{E}_t(\rho)) \leqslant \mathcal{P}(\rho),\ \forall\, \rho. \tag{24}$$

In addition to the non-Markovian feature, the nonunitality is another kind of general feature of quantum processes. In analogy to the definition of BLP non-Markovianity, we defined the following measure of the degree of the nonunitality of a quantum process:

$$N_{\mathrm{nu}}(\mathcal{E}_t) = \max_{\rho_0} \int_{\frac{d}{dt}\mathcal{P}(\mathcal{E}_t(\rho_0))>0} \left|\frac{d\mathcal{P}(\mathcal{E}_t(\rho_0))}{dt}\right| dt, \tag{25}$$

where ρ_0 is the initial state. Obviously, $N_{\mathrm{nu}}(\mathcal{E}_t)$ vanishes if $\mathbf{c}(\mathcal{E}_t) = 0$.

Since the nonunital aspect of the dynamics, which is not revealed by the trace distance, has its own speciality, we aim to measure the effect of nonunitality on non-Markovian behavior. However, a perfect separation of the nonunital aspect from the total non-Markovianity may be infeasible. Therefore we require a weak version $\mathcal{N}_{\mathrm{nu}}$ for measuring nonunital non-Markovianity to satisfy the following three conditions: (i) $\mathcal{N}_{\mathrm{nu}}$ vanishes if $\mathcal{E}_t$ is infinitesimally divisible, (ii) $\mathcal{N}_{\mathrm{nu}}$ vanishes if $\mathcal{E}_t$ is unital, and (iii) $\mathcal{N}_{\mathrm{nu}}$ should be relevant to $\mathbf{c}(\mathcal{E}_t)$. Based on these conditions, we introduce the following measure:

$$\mathcal{N}_{\mathrm{nu}} := \max_{\varrho_\tau\in\mathcal{X}} \int_{\sigma_{\mathrm{nu}}>0} \sigma_{\mathrm{nu}}(t,\varrho_\tau)dt, \tag{26}$$

where $\mathcal{X} := \{\varrho_\tau \mid 0 \leqslant \tau \leqslant \infty\}$, with $\varrho_\tau := \mathcal{E}_\tau(\mathbb{1}/d)$, is the set of the trajectory states which evolve from the maximally mixed state and

$$\sigma_{\mathrm{nu}}(t,\varrho_\tau) := \frac{d}{dt} D(\mathcal{E}_t(\varrho_0),\mathcal{E}_t(\varrho_\tau)), \tag{27}$$

with $D(\rho_1,\rho_2)$ being an appropriate distance, which will be discussed below. The first condition is guaranteed if we require that D is monotone under any CPT maps, i.e., $D(\mathcal{E}(\rho_1),\mathcal{E}(\rho_2)) \leqslant D(\rho_1,\rho_2)$ for any quantum channel $\mathcal{E}$. For the unital time evolution, the set $\mathcal{X} = \{\mathbb{1}/d\}$ only contains the maximally mixed state, so the above-defined $\mathcal{N}_{\text{nu}}$ vanishes, and the second condition is satisfied. The third condition excludes the trace distance.

In this paper, we use the Bures distance, which is defined as

$$D_{\text{B}}(\rho_1,\rho_2) = \sqrt{2[1 - F(\rho_1,\rho_2)]}, \tag{28}$$

where

$$F(\rho_1,\rho_2) = \text{Tr}|\sqrt{\rho_1}\sqrt{\rho_2}| = \text{Tr}\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}} \tag{29}$$

is the Uhlmann fidelity [38,39] between ρ_1 and ρ_2. Here $|O| = \sqrt{O^\dagger O}$. The Bures distance is an appropriate distance for $\mathcal{N}_{\text{nu}}$ because it obeys the monotonicity under CPT maps [24] and is relevant to $\mathbf{c}(\mathcal{E}_t)$. As only the monotonicity of distance is relevant here, for simplicity, we can also take the square of the Bures distance, or the opposite value of the Uhlmann fidelity, as a simple version of monotone "distance" [12]. Quantum relative entropy [40] $S(\rho_1\|\rho_2) = \text{Tr}[\rho_1(\ln\rho_1 - \ln\rho_2)]$, or its symmetric version $S_{\text{sym}}(\rho_1\|\rho_2) := S(\rho_1\|\rho_2) + S(\rho_2\|\rho_1)$, is another qualified candidate for the distance. Noting that when the support of ρ_1 is not within the support of ρ_2, namely, $\text{supp}(\rho_1) \nsubseteq \text{supp}(\rho_2)$, $S(\rho_1\|\rho_2)$ will be infinite, so in such cases, quantum relative entropy will bring singularity to the measure of non-Markovianity. Also, the Hellinger distance [41] is qualified. Although all of these distances are monotone under CPT maps, they may have different characteristics in the same dynamics (see Ref. [42]).

The difference between nonunital non-Markovian measures defined by Eq. (26) and the BLP-type measures, including those which use other alternative distances, is the restriction on the pairs of initial states. Compared with the BLP-type measures relying on any pair of initial states, the nonunital non-Markovianity measure only relies on the pairs consisting of the maximally mixed state and its trajectory states. On the one hand, this restriction makes the nonunital non-Markovianity measure vanish when the quantum processes are unital, whether they are Markovian or non-Markovian; on the other hand, this restriction reflects the fact that the nonunital non-Markovianity measure reveals only part of the information concerning non-Markovian behaviors.

IV. EXAMPLE

To illustrate the nonunital non-Markovian behavior, we give an example in this section. We use the generalized amplitude damping channel (GADC) as a prototype to construct a quantum process. The GADC can be described by $\mathcal{E}(\rho) = \sum_i E_i\rho E_i^\dagger$, with the Kraus operators $\{E_i\}$ given by [36,43]

$$E_1 = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\eta} \end{pmatrix}, \tag{30}$$

$$E_2 = \sqrt{p}\begin{pmatrix} 0 & \sqrt{1-\eta} \\ 0 & 0 \end{pmatrix}, \tag{31}$$

$$E_3 = \sqrt{1-p}\begin{pmatrix} \sqrt{\eta} & 0 \\ 0 & 1 \end{pmatrix}, \tag{32}$$

$$E_4 = \sqrt{1-p}\begin{pmatrix} 0 & 0 \\ \sqrt{1-\eta} & 0 \end{pmatrix}, \tag{33}$$

where p and η are real parameters. Note that for any $p \in [0,1]$ and any $\eta \in [0,1]$, the corresponding $\mathcal{E}$ is a quantum channel. For a two-level system, the Hermitian orthonormal operator basis can be chosen as $\boldsymbol{\lambda} = \boldsymbol{\sigma}/\sqrt{2}$, where $\boldsymbol{\sigma} = \{\sigma_x,\sigma_y,\sigma_z\}$ is the vector of Pauli matrices. With the decomposition in Eq. (5), the affine map for the Bloch vector is given by $\mathbf{r}(\mathcal{E}(\rho)) \mapsto M(\mathcal{E})\mathbf{r}(\rho) + \mathbf{c}(\mathcal{E})$ [36], where

$$M(\mathcal{E}) = \begin{pmatrix} \sqrt{\eta} & 0 & 0 \\ 0 & \sqrt{\eta} & 0 \\ 0 & 0 & \eta \end{pmatrix}, \tag{34}$$

$$\mathbf{c}(\mathcal{E}) = \left(0,0,\frac{(2p-1)(1-\eta)}{\sqrt{2}}\right)^{\text{T}}. \tag{35}$$

The GADC is unital if and only if $p = 1/2$ or $\eta = 1$. When $\eta = 1$, $M(\mathcal{E}) = \mathbb{1}$, the map is an identity map.

A quantum process can be constructed by making parameters p and η be dependent on time t. For simplicity, we take $p_t = \cos^2\omega t$ and $\eta_t = e^{-t}$, where ω is a constant real number. This is a legitimate quantum process because $\mathcal{E}_t$ is a quantum channel for every $t \geqslant 0$ and $\mathcal{E}_{t=0}$ is the identity map.

First, let us consider the $\mathcal{N}_{\text{BLP}}$ for this quantum process. For any two initial states ρ_1 and ρ_2, we have the trace distance

$$\begin{aligned} D_{\text{tr}}(\mathcal{E}_t(\rho_1),\mathcal{E}_t(\rho_2)) &= \frac{1}{2}\text{Tr}\left|M(\mathcal{E}_t)[\mathbf{r}(\rho_1) - \mathbf{r}(\rho_2)]\cdot\frac{\boldsymbol{\sigma}}{\sqrt{2}}\right| \\ &= \frac{1}{\sqrt{2}}|M(\mathcal{E}_t)[\mathbf{r}(\rho_1) - \mathbf{r}(\rho_2)]|, \end{aligned} \tag{36}$$

where $|\mathbf{r}| = \sqrt{\mathbf{r}\cdot\mathbf{r}}$ is the Euclidean length of the vector $\mathbf{r}$, and we used the equality

$$(\mathbf{a}\cdot\boldsymbol{\sigma})(\mathbf{b}\cdot\boldsymbol{\sigma}) = (\mathbf{a}\cdot\mathbf{b})\mathbb{1} + i\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{b}) \tag{37}$$

for Pauli matrices. Denoting $\mathbf{r}(\rho_1) - \mathbf{r}(\rho_2)$ by $(x,y,z)^{\text{T}}$, we get

$$D_{\text{tr}}(\mathcal{E}_t(\rho_1),\mathcal{E}_t(\rho_2)) = \frac{e^{-t/2}}{\sqrt{2}}\sqrt{x^2 + y^2 + e^{-t}z^2}, \tag{38}$$

which implies $\frac{d}{dt}D_{\text{tr}}(\mathcal{E}_t(\rho_1),\mathcal{E}_t(\rho_2)) \leqslant 0$ for every time point $t \geqslant 0$ and for any real numbers x, y, and z. Thus, the BLP non-Markovianity vanishes, i.e., $\mathcal{N}_{\text{BLP}}(\mathcal{E}_t) \equiv 0$, although $\mathcal{E}_t$ may be not infinitesimally divisible, which will become clear later.

In order to investigate whether $\mathcal{E}_t$ is infinitesimally divisible or not, we shall apply $\mathcal{N}_{\text{nu}}$ in the above model. The trajectory of the maximally mixed state under $\mathcal{E}_t$ reads

$$\mathcal{E}_t(\varrho_0) = \frac{1}{2}\mathbb{1} + \mathbf{c}_t\cdot\frac{\boldsymbol{\sigma}}{\sqrt{2}} = \frac{1}{2}\begin{pmatrix} 1 + W_t & 0 \\ 0 & 1 - W_t \end{pmatrix}, \tag{39}$$

where

$$W_t := (2p_t - 1)(1 - \eta_t) = \cos(2\omega t)(1 - e^{-t}). \tag{40}$$

Taking these trajectory states as the initial states, we get the corresponding evolving states:

$$\mathcal{E}_t(\varrho_\tau) = \frac{1}{2}\mathbb{1} + (M_t\mathbf{c}_\tau + \mathbf{c}_t)\cdot\frac{\boldsymbol{\sigma}}{\sqrt{2}} \tag{41}$$

$$= \frac{1}{2}\begin{pmatrix} 1 + W_t + \eta_t W_\tau & 0 \\ 0 & 1 - W_t - \eta_t W_\tau \end{pmatrix}. \tag{42}$$

Then the fidelity reads

$$F(\mathcal{E}_t(\varrho_0),\mathcal{E}_t(\varrho_\tau)) = \tfrac{1}{2}(h_+ + h_-), \tag{43}$$

where

$$h_+ := \sqrt{(1+W_t)(1+W_t+\eta_t W_\tau)}, \tag{44}$$

$$h_- := \sqrt{(1-W_t)(1-W_t-\eta_t W_\tau)}. \tag{45}$$

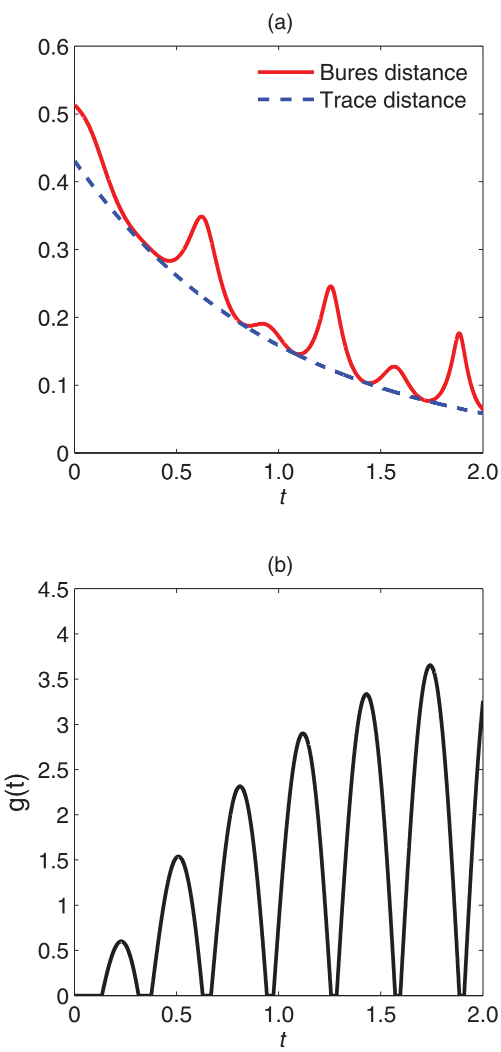

FIG. 2. (Color online) (a) Evolution of trace distance and the Bures distance between two evolving states of a two-level system under the variant generalized amplitude damping channel, initially from the maximal mixed state $\varrho_0 = \mathbb{1}/2$ and its trajectory state $\varrho_\tau = \mathcal{E}_\tau(\varrho_0)$. (b) The evolution of $g(t)$ defined by Eq. (21), whose integral with respect to time t is the RHP measure for non-Markovianity. In these plots, the parameters are taken as $\tau = 10$ and $\omega = 5$.

To compare with the behavior of the trace distance, we also get $D_{\rm tr}(\mathcal{E}_t(\varrho_0),\mathcal{E}_t(\varrho_\tau)) = |\eta_t W_\tau|/2$. With the expressions $\eta_t = e^{-t}$ and $p_t = \cos^2\omega t$, it is

$$D_{\rm tr}(\mathcal{E}_t(\varrho_0),\mathcal{E}_t(\varrho_\tau)) = \frac{e^{-t}}{2}|\cos 2\omega\tau|(1-e^{-\tau}). \tag{46}$$

In Fig. 2(a), we can see that while the trace distance between the evolving states $\mathcal{E}_t(\varrho_0)$ and $\mathcal{E}_t(\varrho_\tau)$ monotonously decreases with time t, the Bures distance increases during some intermediate time intervals. From Eq. (46), one can see that although $D_{\rm tr}(\mathcal{E}_t(\varrho_0),\mathcal{E}_t(\varrho_\tau))$ depends on W_τ, it does not depend on W_t. Actually, from Eq. (38) one could find that for any two initial states, the trace distance between the evolving states is independent of W_t. In this sense, the BLP non-Markovianity treats a family of quantum processes, which only differ by p_t, as the same one. Meanwhile, $\mathcal{N}_{\rm nu}$ reveals the effects of p_t on the infinitesimal nondivisibility and is capable of measuring it.

In order to compare with the BHP measure, we also calculate $g(t)$ defined by Eq. (21). We get

$$g(t) = \tfrac{1}{2}[|1 - f(t)| + |f(t)| - 1], \tag{47}$$

with

$$f(t) := -\omega\sin(2\omega t)(1-e^{-t}) + \cos^2(\omega t). \tag{48}$$

The mediate dynamical maps $\mathcal{E}_{t+\epsilon,t}$ with infinitesimal ϵ are not completely positive when $g(t) > 0$. From Fig. 2(b), we can see that the increase in the Bures distance occurs in the regimes where $g(t) > 0$, which coincides with the monotonicity of the Bures distance under CPT maps.

V. CONCLUSION

In conclusion, we have shown that the measure for non-Markovianity based on trace distance cannot reveal the infinitesimal nondivisibility caused by the nonunital part of the dynamics. In order to reflect the effects of the nonunitality, we have constructed a measure of the nonunital non-Markovianity and have also defined a measure of the nonunitality, in the same spirit as the BLP non-Markovianity measure.

Like non-Markovianity, the nonunitality is another interesting feature of quantum dynamics. With the development of quantum technologies, we need novel theoretical approaches for open quantum systems. It is expected that some quantum information methods would help us understand some generic features of quantum dynamics. We hope this work may draw attention to studying more dynamical properties from the informational perspective.

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