Truncated thermalization of incoherent optical waves through supercontinuum generation in photonic crystal fibers

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We revisit the process of optical wave thermalization through supercontinuum generation in photonic crystal fibers. We report theoretically and numerically a phenomenon of 'truncated thermalization': The incoherent optical wave exhibits an irreversible evolution toward a Rayleigh-Jeans thermodynamic equilibrium state characterized by a compactly supported spectral shape. The theory then reveals the existence of a frequency cut-off which regularizes the ultraviolet catastrophe inherent to ensembles of classical nonlinear waves. This phenomenon sheds new light on the mechanisms underlying the formation of bounded supercontinuum spectra in photonic crystal fibers.

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Introduction. The propagation of partially coherent nonlinear optical waves is a subject of growing interest in different fields of investigations, such as, e.g., wave propagation in homogeneous [1–3] or periodic media [4], nonlinear imaging [5], cavity systems [6–8], or nonlinear interferometry [9]. Owing to their highly nonlinear properties, photonic crystal fibers (PCF) offer unique opportunities for the experimental study of incoherent optical waves over large nonlinear propagation lengths. Indeed, light propagation in PCFs nearby a zero-dispersion wavelength (ZDW) is characterized by a dramatic spectral broadening, a phenomenon termed supercontinuum (SC) generation [10]. Although the interpretation of the mechanisms underlying SC generation in PCFs are rather well understood, a satisfactory theoretical description of SC generation is still lacking, due to the multitude of nonlinear effects underlying this process. However, there is a growing interest in developing new theoretical approaches aimed at describing SC generation in more detail, such as soliton induced dispersive wave generation [11–14], the effective three-wave mixing theory [15], or the second-order coherence theory for nonstationary light [16].

From a different perspective, we recently provided a nonequilibrium thermodynamic formulation of SC generation on the basis of the wave turbulence (WT) theory [17–20]. In this framework, the spectral broadening process inherent to SC generation can be interpreted as a consequence of the natural thermalization of the optical field to the state of thermodynamic equilibrium. Indeed, in analogy with the kinetic gas theory, a reversible (Hamiltonian) system of incoherent nonlinear waves exhibits, as a rule, a process of thermalization, which is characterized by an irreversible evolution of the wave toward the thermodynamic equilibrium state, i.e., the Rayleigh-Jeans (RJ) distribution that realizes the maximum of nonequilibrium entropy [21,22]. This irreversible behavior is expressed by the *H* theorem of entropy growth [22], in analogy with the Boltzmann *H* theorem relevant to gas kinetics. The kinetic wave equations then provide a detailed description of the nonequilibrium properties of this process of thermalization.

Our aim in this Brief Report is to show that this irreversible process of thermalization to the RJ spectrum can be truncated within some specific frequency interval. We consider a PCF whose dispersion curve is characterized by two ZDWs, so that the essential properties of light propagation in the fiber can be modeled by the nonlinear Schrödinger (NLS) equation accounting for fourth-order dispersion effects [23]. In these conditions, the kinetic wave theory reveals the existence of an irreversible evolution toward a RJ equilibrium state characterized by a compactly supported spectral shape. This phenomenon of truncated thermalization can shed new light on the mechanisms underlying the formation of bounded spectra in SC generation [12,13,18].

Besides its relevance in the context of SC generation, this phenomenon is also important from a fundamental point of view. It is indeed well known that the thermodynamic equilibrium state of a system of classical waves (i.e., RJ spectrum) is not properly defined, in the sense that it leads to diverging expressions for the energy at equilibrium. This problem refers to the well-known ultraviolet catastrophe, which was originally solved by Planck's law and subsequently by a self-consistent thermodynamic formulation of quantum electromagnetic radiation. On the other hand, when one deals with an ensemble of classical optical waves, such a divergence is regularized by a frequency cutoff, which is often introduced in a rather artificial way [17,18,24], although it can be justified in some particular cases [25]. In this work we show how a genuine frequency cutoff arises naturally in a system of classical waves described by the generalized NLS equation accounting for higher-order dispersion effects.

Model. We consider the dimensionless NLS equation accounting for third- and fourth-order dispersion effects

$$i\partial_z\psi = -\sigma\partial_t^2\psi + i\alpha\partial_t^3\psi + \beta\partial_t^4\psi + |\psi|^2\psi. \qquad (1)$$

We normalized the problem with respect to the nonlinear length $L_0 = 1/(\gamma P)$ and the "healing time" $\tau_0 = (|\beta_2|L_0/2)^{1/2}$, where γ is the nonlinear coefficient, P the average power of the wave, β_2 the second-order dispersion coefficient with $\sigma = \mathrm{sgn}(\beta_2)$. In these units, the normalized dispersion

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parameters read $\alpha = L_0\beta_3/(6\tau_0^3)$, and $\beta = L_0\beta_4/(24\tau_0^4)$, β_3 and β_4 being the third- and fourth-order dispersion coefficients. The NLS equation (1) conserves three important quantities: the normalized power $N = \int |\psi(t)|^2 dt$, the momentum $M = \int \omega |\tilde{\psi}(\omega)|^2 d\omega$, and the total "energy" (Hamiltonian) $H = E + U$, which has a linear (dispersive) kinetic contribution $E(z) = \int k(\omega)|\tilde{\psi}(\omega)|^2 d\omega$ and a nonlinear contribution $U(z) = \frac{1}{2} \int |\psi(t)|^4 dt$, where $k(\omega) = \sigma\omega^2 + \alpha\omega^3 + \beta\omega^4$ is the dispersion relation $[\tilde{\psi}(z,\omega) = \frac{1}{\sqrt{2\pi}} \int \psi(z,t) \exp(-i\omega t)\,dt]$.

Simulations. We recall that for $\alpha = \beta = 0$, Eq. (1) recovers the integrable NLS equation, which does not exhibit wave thermalization [3]. The introduction of third-order dispersion ($\alpha \neq 0$) leads to a process of anomalous thermalization toward an equilibrium state of a different nature than the RJ distribution [2]. The influence of higher-order dispersion on wave thermalization was considered in the context of SC generation through the analysis of the generalized NLS equation [17,18,20]. Here, we revisit these previous works *by truncating the dispersion relation of the generalized NLS equation up to the fourth-order polynomial.*

The initial condition considered in the simulations is a *coherent* cw pump, whose wavelength lies between the ZDWs in the anomalous dispersion regime ($\sigma = -1$). We report in Fig. 1 a typical evolution of the spectrum with respect to the propagation length obtained by solving the NLS equation (1). In this example, we considered a relatively high pump power (200 W), which is in the range of the powers considered in the experiments reported in Refs. [18,20]. In this highly nonlinear regime the optical field exhibits rapid and random temporal fluctuations, which prevent the formation of robust and persistent coherent soliton structures. The optical field then exhibits an incoherent turbulent dynamics, in which coherent soliton and dispersive waves do not play a significant role, as previously discussed in high-power cw-SC generation [17,18,20,26].

In this incoherent regime, the central part of the SC spectrum relaxes toward the RJ distribution, as predicted by the kinetic wave theory [see Fig. 1(b)]. The RJ spectrum is characterized by a double-peaked structure, which results from the presence of two ZDWs in the dispersion curve of the PCF [see Fig. 1(c)] [17,18]. However, a careful analysis reveals the remarkable aspect that this thermalization process is not achieved in a complete fashion: The *tails* of the SC spectrum are characterized by abrupt spectral edges, which are preserved for long propagation lengths and thus do not relax toward the expected RJ spectral tails. Our aim in this paper is to grasp the physical origin of such abrupt spectral edges in the generated SC spectrum.

Kinetic approach: RJ spectrum. The kinetic theory relies on a natural asymptotic closure of the hierarchy of moments equations which is induced by the dispersive properties of the waves [22]. Then assuming that the random nonlinear wave evolves in the weakly nonlinear regime, $|U/E| \ll 1$, the wave statistics turns out to be Gaussian, which leads to an irreversible kinetic equation describing the evolution of the averaged spectrum of the wave, $n(z,\omega)\delta(\Omega) = \langle \tilde{\psi}(\omega + \Omega/2)\,\tilde{\psi}^*(\omega - \Omega/2)\rangle$ [17,21,22]
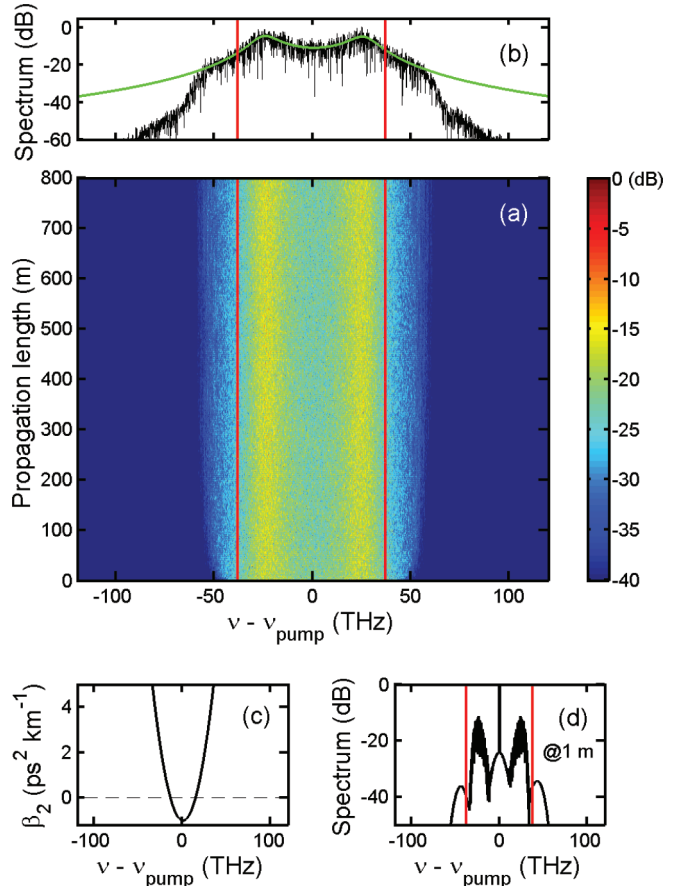
$$\partial_z n(z,\omega) = \text{Coll}[n], \qquad (2)$$

FIG. 1. (Color online) (a) Numerical simulation of the NLS equation (1) showing the evolution of the spectrum $|\tilde{\psi}|^2(\omega,z)$ during the propagation (in dimensional units, $\omega = 2\pi\nu$). (b) Comparison of $|\tilde{\psi}|^2(\omega,z)$ at equilibrium with the corresponding RJ spectrum in green (grey) [Eq. (3)]. (c) Dispersion curve of the PCF. (d) MI spectrum after 1 m of propagation through the PCF. The vertical red (grey) lines denote the frequency bounds, $\omega_\pm$, given by Eq. (7).

where the collision term $\text{Coll}[n] = \frac{1}{\pi} \int d\omega_{1,2,3}\, W\mathcal{N}_{\omega_1\omega_2\omega_3}(\mathbf{n})$ provides a kinetic description of the Kerr effect in Eq. (1), with $\mathcal{N}_{\omega_1\omega_2\omega_3}(\mathbf{n}) = n(\omega)\,n(\omega_1)\,n(\omega_2)\,n(\omega_3)\,[n^{-1}(\omega) + n^{-1}(\omega_1) - n^{-1}(\omega_2) - n^{-1}(\omega_3)]$, and $n(\omega)$ stands here for $n(z,\omega)$. The phase-matching conditions underlying the four-wave mixing are expressed by the Dirac δ functions in $W = \delta(\omega + \omega_1 - \omega_2 - \omega_3)\,\delta[k(\omega) + k(\omega_1) - k(\omega_2) - k(\omega_3)]$. The kinetic equation (2) conserves the densities of power, $N/T_0 = \int n(z,\omega)d\omega$, of "energy" $E/T_0 = \int k(\omega)n(z,\omega)d\omega$, and of "momentum" $M/T_0 = \int \omega n(z,\omega)\,d\omega$ of the wave, where T_0 denotes the numerical time window [17]. The irreversible character of Eq. (2) is expressed by an H theorem of entropy growth, $d\mathcal{S}/dz \geqslant 0$, where the nonequilibrium entropy reads $\mathcal{S}(z) = \int \log[n(z,\omega)]\,d\omega$. The RJ equilibrium spectrum $n^{RJ}(\omega)$ realizing the maximum of $\mathcal{S}[n]$, subject to the constraints of conservation of E, M, and N, is obtained by introducing the corresponding Lagrange multipliers $1/T, \lambda/T$, and $-\mu/T$

$$n^{RJ}(\omega) = \frac{T}{k(\omega) + \lambda\omega - \mu}, \qquad (3)$$

where T denotes the temperature and μ the chemical potential by analogy with thermodynamics [22].

Kinetic approach: Refined analysis. As discussed above through Fig. 1, this process of thermalization to the RJ spectrum (3) is not achieved in a complete way. *Here we show that this effect can be described by a refined analysis of the kinetic equation* (2). As in the numerical simulations, we assume that the pump wave frequency lies in the anomalous dispersion regime ($\sigma = -1$). Using the property [27], we integrate the collision term of the Eq. (2) over ω_1,

$$\partial_z n(z,\omega) = \frac{1}{2\pi} \int \frac{\widetilde{\mathcal{N}}_{\omega_2\omega_3}(\mathbf{n})}{|\omega_2 - \omega| \, |\omega_3 - \omega|} \, \delta[\phi_\omega(\omega_2,\omega_3)] \, d\omega_{2,3},$$

(4)

where $\widetilde{\mathcal{N}}_{\omega_2\omega_3}(\mathbf{n})$ is the functional $\widetilde{\mathcal{N}}_{\omega_1\omega_2\omega_3}(\mathbf{n})$ in which $n_{\omega_1}(z)$ has been changed with $n_{\omega_2+\omega_3-\omega}(z)$, and $\phi_\omega(\omega_2,\omega_3) = \frac{3}{2}\alpha(\omega_2 + \omega_3) + \beta[2(\omega_2^2 + \omega_3^2) + 3\omega_2\omega_3 - \omega(\omega_2 + \omega_3) + \omega^2] - 1$. The function $\phi_\omega(\omega_2,\omega_3)$ is a quadric in the two-dimensional space (ω_2,ω_3). It can be recast into its canonical form with the following change of variables: $[\Omega_2 = \frac{1}{\sqrt{2}}(\omega_2 + \omega_3), \Omega_3 = \frac{1}{\sqrt{2}}(\omega_2 - \omega_3)]$, and then $[\tilde{\Omega}_2 = \Omega_2 + q/(7\beta), \tilde{\Omega}_3 = \Omega_3]$, with $q = 3\alpha/\sqrt{2} - \sqrt{2}\beta\omega$. The kinetic equation (4) then takes the form

$$\partial_z n(z,\omega) = \int \frac{\widetilde{\mathcal{N}}_{\tilde{\Omega}_2 \tilde{\Omega}_3}(\mathbf{n}) \, \delta[(\tilde{\Omega}_2/a_2)^2 + (\tilde{\Omega}_3/a_3)^2 - \rho]}{\pi \, |\tilde{\Omega}_2 + \tilde{\Omega}_3 - r_\omega| \, |\tilde{\Omega}_2 - \tilde{\Omega}_3 - r_\omega|} \, d\tilde{\Omega}_{2,3},$$

(5)

where $a_2 = \sqrt{2/(7\beta)}$, $a_3 = \sqrt{2/\beta}$, and $r_\omega = 6\sqrt{2}\omega/7 + 3\sqrt{2}\alpha/(14\beta)$. It becomes apparent that the condition

$$\rho = 1 - \frac{3}{28}\left(8\beta\omega^2 + 4\alpha\omega - \frac{3\alpha^2}{\beta}\right) \geqslant 0$$

(6)

must be satisfied in Eq. (5). This reveals that the resonant four-wave interaction underlying the Kerr effect can only take place within a specific frequency interval defined by the bounds $\omega \in [\omega_-,\omega_+]$, with

$$\omega_\pm = -\frac{\alpha}{4\beta} \pm \frac{\sqrt{21}}{12\beta}\sqrt{3\alpha^2 + 8\beta}.$$

(7)

Finally notice that by introducing the following parametrization of the ellipse $[\tilde{\Omega}_2 = \tilde{a}_2 \cos(\theta), \tilde{\Omega}_3 = \tilde{a}_3 \sin(\theta)]$, with $\tilde{a}_{2,3} = a_{2,3}\sqrt{\rho}$, and making use again of the property [27], Eq. (5) can be recast in the following compact form:

$$\partial_z n(z,\omega) = \frac{\tilde{a}_2 \tilde{a}_3}{2\pi |\rho|} \int_0^{2\pi} \frac{\widetilde{\mathcal{N}}_{\cos(\theta) \sin(\theta)}(\mathbf{n})}{\mathcal{F}_\omega(\theta)} \, d\theta,$$

(8)

where $\mathcal{F}_\omega(\theta) = |\tilde{a}_2 \cos(\theta) + \tilde{a}_3 \sin(\theta) - r_\omega| \, |\tilde{a}_2 \cos(\theta) - \tilde{a}_3 \sin(\theta) - r_\omega|$.

Discussion. There is an appreciable discrepancy between the simulation of the NLS equation (1) reported in Fig. 1 and the frequency interval $\omega_\pm$ predicted by the kinetic theory in Eq. (7). Actually, the pump power considered in Fig. 1 is quite elevated, so that the modulational instability (MI) sidebands are located far away from the carrier pump frequency. As a result, the cascade of MI sidebands generated in the early stage of propagation spreads beyond the frequency interval predicted by the theory, as clearly illustrated in Fig. 1(d). Since the MI
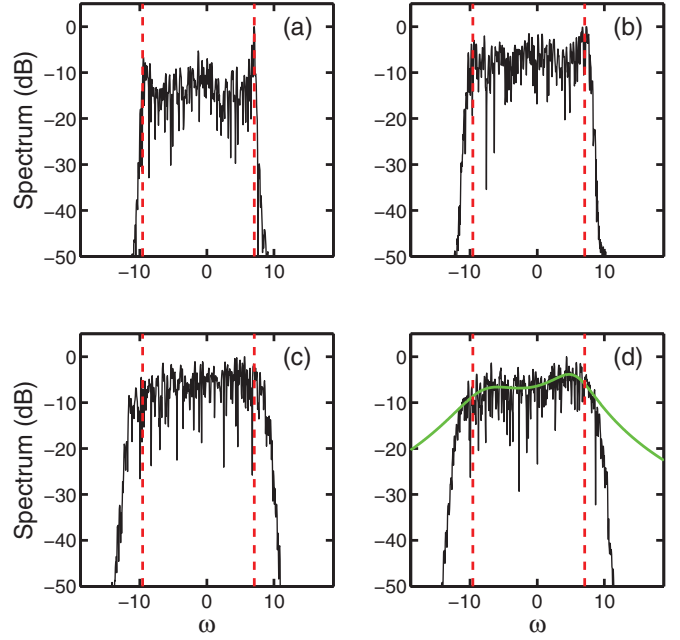

FIG. 2. (Color online) Spectra $|\tilde{\psi}|^2(\omega,z)$ obtained by solving the NLS equation (1) for $\sigma = -1$, $\alpha = 0.1$, and $\beta = 0.02$: (a) $z = 200$, (b) $z = 10^4$, (c) $z = 5 \times 10^5$, (d) $z = 10^6$. After a long transient, the wave relaxes toward a truncated RJ distribution [Eq. (3), green (grey) curve] in (d). The dashed vertical lines denote the frequencies $\omega_\pm$ in Eq. (7).

process is inherently a coherent nonlinear phase-matching effect, it is not described by the kinetic equation (2), whose validity is restricted to the weakly nonlinear regime ($|U/E| \ll 1$). This explains why the numerical simulations reported in previous works [17,18] did not give evidence of a signature of this phenomenon of truncated thermalization.

In order to analyze our theoretical predictions in more detail, we decrease the injected pump power so as to maintain the (cascaded) MI sidebands within the frequency interval (7). Intensive numerical simulations of the NLS equation (1) in this regime of reduced pump power have been performed. This study reveals that the nonlinear dynamics slows down in a dramatic way, so that the expected process of thermalization requires huge nonlinear propagation lengths and huge CPU time computations. This results from the fact that the parameters α and β decrease as the pump power decreases, so that Eq. (1) approaches the integrable limit of the NLS equation, which does not exhibit any thermalization effect [3]. We report in Fig. 2 the wave spectra at different propagation lengths obtained by solving the Eq. (1) with $\alpha = 0.1$ and $\beta = 0.02$. In the early stage of propagation, $z \sim 200$, the spectrum remains confined within the frequency interval $[\omega_-,\omega_+]$ predicted by the theory [Eq. (7)], although the spectrum exhibits a completely different spectral profile than the expected RJ distribution. As a matter of fact, the process of thermalization requires enormous propagation lengths, as illustrated in Fig. 2(d), which shows that the wave spectrum eventually relaxes toward a truncated RJ distribution. Here, the Lagrangian multipliers (μ,λ,T) have been calculated from the conserved quantities (N,M,E), without using adjustable parameters [28]. A unique triplet (μ,λ,T) is obtained from
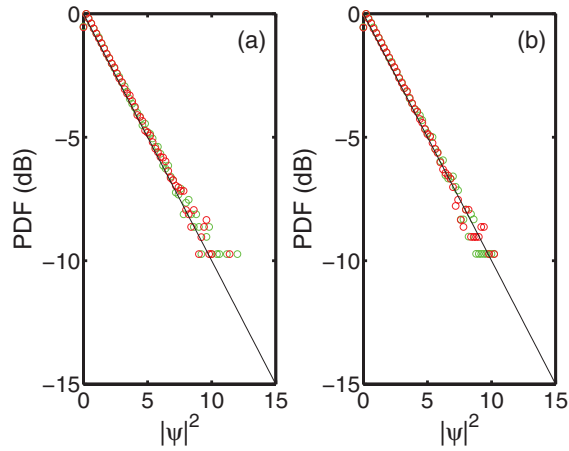
FIG. 3. (Color online) PFD of the wave intensity, $I = |\psi|^2$, corresponding to the simulation of Fig. 1 at $z = 200$ green (grey), $z = 10^4$ red (a); $z = 5 \times 10^5$ green (grey), $z = 10^6$ red (b): The random wave eventually enters the kinetic regime of Gaussian statistics.

a set (N, M, E), a feature which is consistent with the fact that a "closed" Hamiltonian system should exhibit a unique thermodynamic equilibrium state.

Note, however, in Fig. 2(d) that, despite the good agreement, the whole spectrum spans a frequency band which exceeds the frequency interval $[\omega_-, \omega_+]$ predicted by the kinetic theory in Eq. (7). Indeed, in the first stage of evolution [see Figs. 2(a)–2(c) for $200 < z < 5 \times 10^5$], the SC spectrum exhibits a slow process of spectral broadening, so that the corresponding SC edges spread beyond the frequency bound $[\omega_-, \omega_+]$. Such a discrepancy decreases in a significant way as the system becomes weakly nonlinear, i.e., $|U/E|$ decreases. Accordingly, this discrepancy can be ascribed to a deviation

from Gaussian statistics of the incoherent wave. Indeed, we report in Fig. 3 the PDF of the wave intensity calculated at different propagation lengths. A deviation from Gaussian statistics is visible for $z < 10^4$, which can merely explain the slow process of spectral broadening beyond the frequency interval (7) predicted by the theory. This conclusion is corroborated by the analysis of the *kurtosis of the intensity distribution*, $K(z) = \langle I^2 \rangle(z)/(2\langle I \rangle^2) - 1$ (data not shown). The value of $K(z)$ and the variance of its fluctuations are shown to slowly decay during the propagation to zero. Then as the system evolves, it eventually reaches a kinetic regime of Gaussian statistics, which is subsequently preserved in the further evolution. It is interesting to underline that, once the state of Gaussian statistics is reached, *the incoherent wave does not exhibit any spectral broadening (for $z > 5 \times 10^5$), while its spectral profile slowly relaxes toward the truncated RJ distribution*, as described by the kinetic theory.

Finally note that we have performed the same numerical study for $\alpha = 0$ (data not shown), for which the frequency bounds (7) reduce to $\omega_\pm = \pm\sqrt{7/(6\beta)}$. This study confirms the process of relaxation toward a spectrally truncated RJ distribution for the incoherent wave.

Conclusion. We have reported a phenomenon of truncated thermalization which is characterized by the formation of a RJ distribution with a compactly supported spectral shape. From a broader perspective, this work is related to the wider question underlying the physics of dynamical thermalization of nonlinear systems, in line with the Fermi-Pasta-Ulam problem [29].

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