Anomalous switching of optical bistability in a Bose-Einstein condensate

Shuai Yang,[1] M. Al-Amri,[2] and M. Suhail Zubairy[3]

[1]*Institute for Quantum Science and Engineering (IQSE) and Department of Physics and Astronomy, Texas A&M University, College Station, Texas 77843, USA*

[2]*The National Center for Mathematics and Physics, KACST, P.O. Box 6086, Riyadh 11442, Saudi Arabia*

[3]*Institute for Quantum Science and Engineering and Department of Physics and Astronomy, Texas A&M University, College Station, Texas 77843, USA*

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The nonlinear dynamics of the photon number in an optical cavity filled with a cigar-shaped Bose-Einstein condensate is investigated. We find that the way of adding the field is crucial to the switching close to the critical transition point. If the pump field is changed abruptly, the system may jump from one branch to the other even if the pump field intensity has not reached the critical transition point yet. This behavior is similar to the anomalous switching in the dispersive optical bistability.

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I. INTRODUCTION

A combined system with an ultracold atomic ensemble located in a small volume ultrahigh finesse optical cavity is an emerging field with considerable interest recently. In the large detuning limit from the atomic resonance, atom-photon interaction provides an optical lattice to the atoms and affects the mechanical motion of the atoms. In turn, atoms induce a position dependent phase shift on the cavity field. This highly intrinsic nonlocal nonlinearity induces a lot of interesting results such as self-organization of atoms [1–6], optical bistability [7–12], cavity-enhanced super-radiant Rayleigh scattering [13], mapping between the atoms ensemble-cavity system and the canonical optomechanical system [14–16], and analogy of the Dicke quantum phase transition [17,18].

In previous works, the bistable behavior of the intracavity photon numbers is usually studied for the steady state [7–9]. In Ref. [10], the authors proposed a discrete-mode approximation (DMA) method to study the nonlinear dynamics of a cigar-shaped Bose-Einstein condensate in an optical cavity. By selecting the lowest excitation modes of the condensate, the properties of the system can be analyzed in a simple way. The validation of the method is justified by comparing the results from the approximate DMA method and the full description with the Gross-Pitaevski (GP) equation.

In this paper, we investigate the switching behavior of the intracavity photon number from one branch to the other. We find that the way of adding the field is crucial to the switching close to the critical transition point. If the pump field is added adiabatically, the jumping happens exactly at the critical point [10]. If the pump field is added abruptly, the system may jump to the upper branch even if the pump field intensity is less than the critical transition point. This behavior is similar to the anomalous switching of the dispersive optical bistability [19,20]. We analyze the physics of this anomalous switching and examine the effect of the initial condition and the possible damping.

This paper is organized as follows. In Sec. II, we present the system and study the steady-state bistability of the intracavity number. In Sec. III, we focus on the anomalous switching of the system and investigate the physics behind. Then in Sec. IV, the effect of initial condition and damping on the anomalous switching behavior is analyzed. A brief summary is given in Sec. VI.

II. SYSTEM

The system we consider is a pure Bose-Einstein condensate (BEC) of two-level atoms with mass m and transition frequency ω_a located inside a high-Q optical cavity with cavity mode ω_c (see Fig. 1). For the sake of simplicity, we consider the dynamics in the dimension x along the cavity axis. The cavity field mode function is then described simply by $\cos(kx)$, with the wave vector k. The model applies to a cigar-shaped BEC, which is tightly confined in the transverse directions by strong dipole or magnetic trap, such that the transverse size of the condensate is smaller than the waist of the cavity field. An external pumping laser field at frequency ω_p is added along the cavity axis. The atom-pump detuning and the cavity-pump detuning are denoted as $\Delta_a = \omega_a - \omega_p$ and $\Delta_c = \omega_c - \omega_p$, respectively.

In the large detuning limit and in the rotating frame at the pump frequency, the Hamiltonian for the condensate system can be written as [5]

$$\hat{H} = \int dx\, \hat{\Psi}^\dagger(x)\left[-\frac{\hbar^2}{2m}\frac{d^2}{d^2x} + \hbar U_0 \cos^2(kx)\hat{a}^\dagger\hat{a}\right]\hat{\Psi}(x) + \hbar\Delta_c\hat{a}^\dagger\hat{a} + i\hbar\eta(\hat{a}^\dagger - \hat{a}). \tag{1}$$

Here $\hat{\Psi}^\dagger$ and $\hat{a}^\dagger$ are the creation operators for the atoms and the cavity photons, respectively. The atom-cavity photon interaction induces an additional potential $U_0\cos^2(kx)\hat{a}^\dagger\hat{a}$ for the atoms where $U_0 = -g_0^2/\Delta_a$ is the maximal light shift per photon that an atom may experience with g_0 being the atom-photon coupling constant. Here η is the field amplitude of the parallel driving laser. We have omitted the atom-atom interaction and the weak harmonic trapping potential.

The ground state of the condensate with no pumping field is a homogeneous macroscopic state with zero momentum. By absorption and stimulated emission of the cavity photons, the condensate can be excited to the superposition of $\pm 2\hbar k$ momentum states from the ground state. Taking into account the lowest-order perturbation to the uniform condensate wave function, we assume $\hat{\Psi}(x) = \hat{c}_0 + \sqrt{2}\cos(2kx)\hat{c}_2$. Therefore

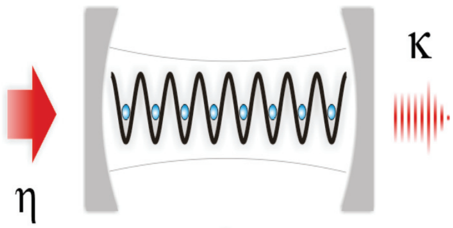

FIG. 1. (Color online) A schematic of a BEC in a cavity.

the Hamiltonian becomes [10]

$$\hat{H} = 4\hbar\omega_r \hat{c}_2^\dagger \hat{c}_2 + \hbar\left(\Delta_c + \frac{NU_0}{2}\right)\hat{a}^\dagger\hat{a} + \frac{\hbar U_0}{2\sqrt{2}}\hat{a}^\dagger\hat{a}(\hat{c}_0^\dagger\hat{c}_2 + \hat{c}_2^\dagger\hat{c}_0) + i\hbar\eta(\hat{a}^\dagger - \hat{a}), \tag{2}$$

where $\hbar\omega_r = \hbar^2k^2/2m$ is the atomic recoil energy and $N = \hat{c}_0^\dagger\hat{c}_0 + \hat{c}_2^\dagger\hat{c}_2$ is the total number of the atoms.

Applying the mean-field approximation $\hat{c}_i \sim \sqrt{N}Z_i$ and $\hat{a} \sim \alpha$, the equation of motion for the condensate can be found as

$$i\hbar\frac{d}{dt}Z = H(\alpha) = (H_0 + |\alpha|^2 H_1)Z, \tag{3}$$

with $Z = (Z_0, Z_2)^T$ and

$$H_0 = \hbar\omega_r\begin{pmatrix}0 & 0\\ 0 & 4\end{pmatrix}, \tag{4a}$$

$$H_1 = \frac{\hbar U_0}{2\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \tag{4b}$$

and the evolution equation of the photon amplitude is

$$\frac{d\alpha}{dt} = -i\left(\Delta_c + \frac{NU_0}{2} + NZ^\dagger\frac{H_1}{\hbar}Z\right)\alpha + \eta - \kappa\alpha. \tag{5}$$

When the cavity damping is much faster than the mechanical motion of the condensate, we can assume that the photon amplitude follows the condensate adiabatically. We then have (with $\dot{\alpha} \simeq 0$)

$$\alpha = \frac{\eta}{i\left(\Delta_c + \frac{NU_0}{2} + NZ^\dagger\frac{H_1}{\hbar}Z\right) + \kappa}. \tag{6}$$

The steady state can be solved as follows. First we take an arbitrary trial photon amplitude α_{tr}, and then solve the ground state of the Hamiltonian $H(\alpha_{tr})$. Next, we substitute the solution Z_s to Eq. (6) and get an output photon amplitude α_{out}. If $\alpha_{\text{out}} = \alpha_{tr}$, we get a self-consistent solution α_s. It has been shown that the steady-state intracavity photon number $n_s = |\alpha_s|^2$ shows bistability, similar to the optical bistability [10,12]. The bistable behavior of the intracavity photon number with respect to the pumping field intensity is shown in Fig. 2, where points with negative slopes correspond to unstable states.

III. ANOMALOUS SWITCHING

Next we would like to focus on the dynamical properties of the system. There are two different ways to add the pump field. In the first case, the pumping field is turned on adiabatically. The increase of the field intensity is so slow that the condensate and intracavity photon follow all the steady states corresponding to the pump field in the lower branch until the critical point A in Fig. 2. When the pump field exceeds the critical point, the condensate cannot follow the input field adiabatically and therefore jumps to the upper branch. This is because the steady state corresponding to the critical point A in the lower branch is much different from the steady state corresponding to the current pump field in the upper branch. Since there is no damping mechanism for the condensate, the system oscillates around the upper branch [10]. In the second case, instead of adding the pumping field adiabatically, the field is added abruptly. Suppose the condensate is initially in the homogenous state $Z = (1,0)^T$. After solving Eqs. (3) and (6) numerically, we draw the evolution of the intracavity photon number $|\alpha|^2$ with respect to different pumping field intensity, as shown in Fig. 3. If the added field stays in the purely lower branch or lies in the bistable region but not close to the critical point A, the intracavity photon number oscillates around steady state in the lower branch; see Figs. 3(a) and 3(b). If the input field exceeds the critical value of A, it will introduce oscillations between two branches; see Fig. 3(c). These oscillations have been investigated and confirmed by solving the GP equation numerically in Ref. [10].

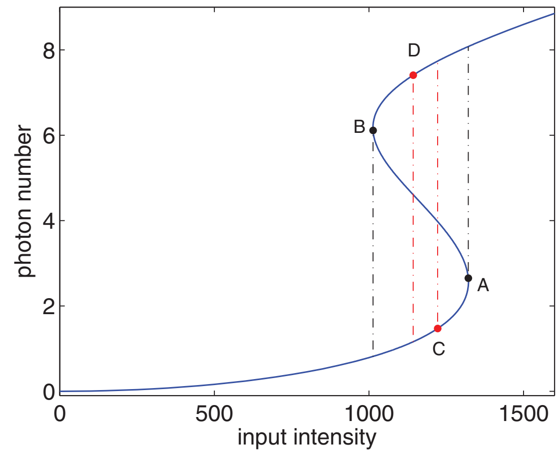

FIG. 2. (Color online) Steady-state intracavity photon number as a function of the input pump intensity. The parameters are $N = 4.8 \times 10^4$, $U_0 = 0.25\omega_r$, $\delta_c = 1.2 \times 10^3\omega_r$, $\kappa = 0.4 \times 10^3\omega_r$. The input pump intensity is also in the unit of ω_r. The critical switching points are $\eta_A = 1322\omega_r$ and $\eta_B = 1013\omega_r$ for the steady state. The critical anomalous switching points are $\eta_C = 1222\omega_r$ and $\eta_D = 1143\omega_r$.

In this paper, we find that the photon number may approach the upper branch with a pump field less than the critical point A if the field is turned on abruptly. This is quite different from the first case where the photon number jumps to the upper branch only if the pump field is greater than the critical value A. The evolution of the intracavity photon number when $\eta = 1250\omega_r$ is shown in Fig. 3(d). Note that the field at the critical point A is $1322\omega_r$. It follows, on comparing the intracavity photon numbers in Fig. 2, that the intracavity photon number reaches the upper branch. Further numerical calculation shows that the switching happens when the added pumping field lies between point C and A in Fig. 2. This phenomenon reminds us of

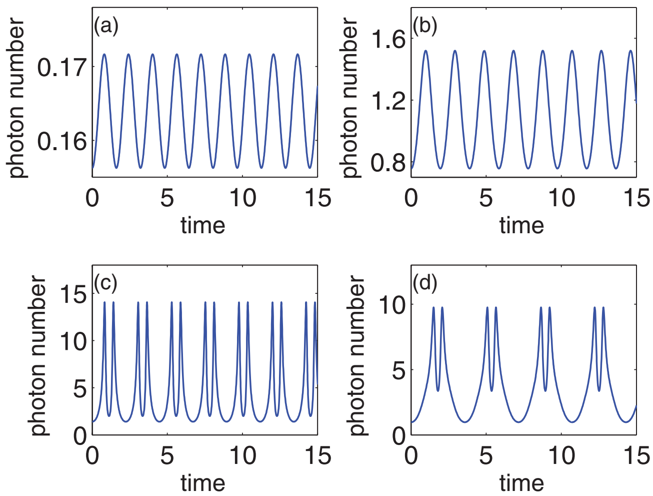

FIG. 3. (Color online) Intracavity photon number as a function of time. Initially there is no pump field and the condensate is the homogeneous state. Then the pump field is turned on immediately with values (a) $500\omega_r$, (b) $1100\omega_r$, (c) $1500\omega_r$, and (d) $1250\omega_r$. The parameters are $N = 4.8 \times 10^4$, $U_0 = 0.25\omega_r$, $\delta_c = 1.2 \times 10^3\omega_r$, $\kappa = 0.4 \times 10^3\omega_r$. The time is in the unit of $1/\omega_r$.

the anomalous switching of the dispersive bistability for the traditional two-level atomic media in the good cavity limit [19,20].

In order to understand this anomalous switching behavior, we proceed by introducing the quadratures of the mechanical oscillators $X = \sqrt{N/2}(Z_0^* Z_2 + Z_0 Z_2^*)$, $P = i\sqrt{N/2}(Z_2^* Z_0 - Z_0^* Z_2)$. The corresponding equations of motion are

$$\frac{dX}{dt} = 4\omega_r P, \tag{7a}$$

$$\frac{dP}{dt} = -4\omega_r X - \frac{\sqrt{N}U_0}{2}|\alpha|^2, \tag{7b}$$

and the photon amplitude is (with $X = \frac{2\sqrt{N}}{U_0} Z^\dagger \frac{H_1}{\hbar} Z$)

$$\alpha = \frac{\eta}{i\left(\Delta_c + \frac{NU_0}{2} + \frac{\sqrt{N}U_0}{2}X\right) + \kappa}. \tag{8}$$

In deriving the above equations, we have used the fact that $|Z_1|^2 \ll |Z_0|^2 \simeq 1$. Then the evolution of the generalized displacement would be

$$\frac{d^2X}{d^2t} = -(4\omega_r)^2 X - \frac{2\sqrt{N}U_0\omega_r\eta^2}{\left(\Delta_c + \frac{NU_0}{2} + \frac{\sqrt{N}U_0}{2}X\right)^2 + \kappa^2} = -\frac{dV(X)}{dX}, \tag{9}$$

where

$$V(X) \equiv \int^X ds\left[(4\omega_r)^2 s + \frac{2\sqrt{N}U_0\omega_r\eta^2}{\left(\Delta_c + \frac{NU_0}{2} + \frac{\sqrt{N}U_0}{2}s\right)^2 + \kappa^2}\right]. \tag{10}$$

The condensate can then be viewed as a nonlinear spring. The dynamic properties will be determined by the potential $V(X)$ and the initial condition. Figure 4 shows the potential function for different pump field intensity. The solid black line corresponds to the case $\eta = 500\omega_r$. We find one and only one minimum close to the origin. This minimum denotes the steady state in the purely lower branch. The potential with $\eta = 1500\omega_r$ is shown as the magenta dotted line. The single minimum is far away from the origin and corresponds to the steady state in the purely upper branch. The red dash-dotted line ($\eta = 1100\omega_r$) and blue dashed line ($\eta = 1250\omega_r$) lie in the bistable region. The potential is double-well like and has two minimum points. The one close to the origin means the steady state in the lower branch while the other indicates the upper branch. The peak in the middle represents the unstable state.

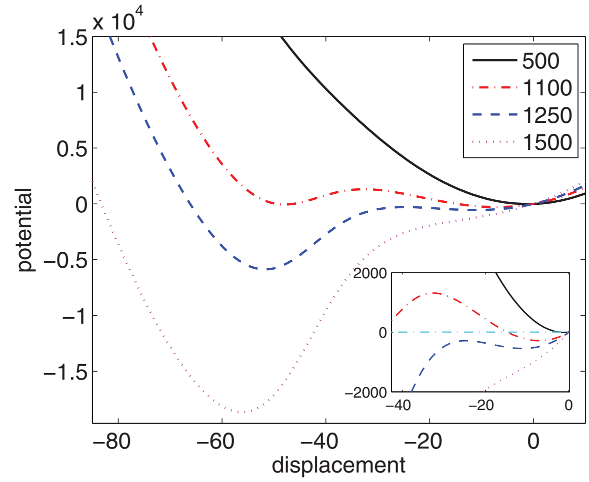

FIG. 4. (Color online) The potential $V(x)$ (in the unit of ω_r^2). The lines from top to bottom correspond to different pump fields, (1) $500\omega_r$, (2) $1100\omega_r$, (3) $1250\omega_r$, and (4) $1500\omega_r$. The parameters are $N = 4.8 \times 10^4$, $U_0 = 0.25\omega_r$, $\delta_c = 1.2 \times 10^3\omega_r$, $\kappa = 0.4 \times 10^3\omega_r$.

The evolution of the condensate can then be understood as a pointlike ball sliding in a one-dimension smooth bowl. The shape of the bowl is determined by the potential function $V(X)$. At $t = 0$, the shape of the bowl is harmoniclike. The initial homogenous condensate corresponds to a rest ball placed at the $X = 0$. If the field is added adiabatically, the shape of the bowl changes gradually and the ball will stay at the bottom of the bowl. Then the double well appears and the ball stays at the bottom of the right well. If the field is increased further, the left well gets deeper and the right well is raised. At the critical point, the minimum of the right well coincides with the unstable peak of the barrier in the middle. Therefore the right well disappears. The ball falls down to the left well and oscillates in the left well.

If the field is added abruptly, the shape of the bowl will be changed immediately. The ball is then released to the bowl from the initial position at $X = 0$. As the bowl is frictionless, the total energy is conserved. If the added pump field is weak, the bowl has only one minimum close to the origin, as shown in the black solid line, the ball will oscillate around this minimum; see Fig. 3(a). If the added pump field corresponds to a value in the bistable region, the potential is a double-well. There exists two cases. If the pump rate is on the left of point C in Fig. 2, the peak in the middle of the double well is higher than the initial position (the red dash-dotted line). Therefore, the ball will be confined in the right side of the double well. Correspondingly, the photon number will oscillate around the lower branch as shown in Fig. 3(b). If the pump field is increased between C and A, the right well gets deeper and the barrier in the center would be lower than the initial position (the blue dashed line). In this case, the ball will not be trapped in the right well and will reach the left well too. This corresponds to the anomalous switching as shown in Fig. 3(d). Note that the barrier is not much lower than $V(X) = 0$. Therefore, the velocity of the ball when passing the barrier is small and the period of the whole oscillation is long. If the field is strong, the double wells disappear and the single minimum corresponding to the steady state in the upper branch emerges far away from the origin (the magenta dotted line). The ball will oscillate between the initial position to the other side of the bowl at the same level as the initial position. It is shown in Fig. 3(c) that the photon number oscillates between the upper branch and lower branch.

IV. EFFECT OF INITIAL CONDITION AND DAMPING RATE

With the above picture in mind, we can expect that the initial condition also plays an important role in the anomalous switching, as it determines the total energy of the system. If the initial state corresponds to a steady state in the lower branch, the initial position in the $V(X)$ line might be lower than the barrier of potential when the field is turned on, there is no anomalous switching; see Fig. 5(a). On the other hand, if the initial state corresponds to a steady state in the higher branch, the anomalous switching to the lower branch may happen close to critical point B; see Fig. 5(b). In Fig. 2, it shows that the photon number switches to the higher branch if the pump field is on the right of C when the condensate is initially in the homogenous state. On the other hand, if the condensate is initially prepared in the steady state of $\eta = 1600\omega_r$, the anomalous switching from the higher branch to the lower branch happens when the pump field is between B and D.

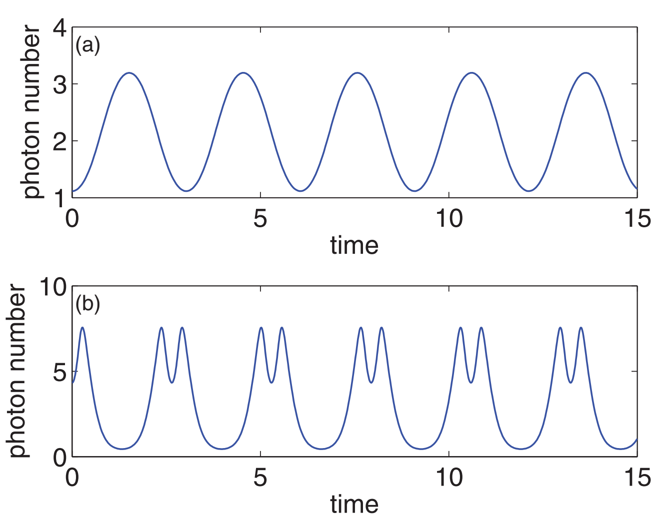

FIG. 5. (Color online) Intracavity photon number as a function of time for different initial conditions. (a) The initial state is the steady state corresponding to $\eta = 800\omega_r$. Then the pump field is tuned sharply to $\eta = 1250\omega_r$. (b) The initial state is the steady state corresponding to $\eta = 1600\omega_r$. Then the pump field is tuned to $\eta = 1100\omega_r$ abruptly. The other parameters are $N = 4.8 \times 10^4$, $U_0 = 0.25\omega_r$, $\delta_c = 1.2 \times 10^3\omega_r$, $\kappa = 0.4 \times 10^3\omega_r$. The time is in the unit of $1/\omega_r$.

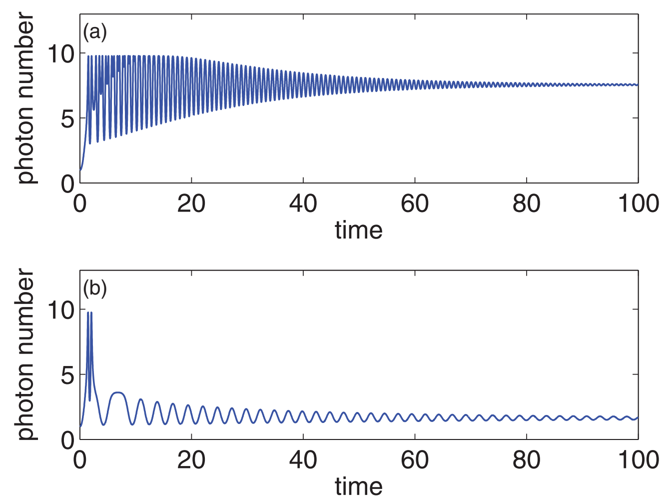

FIG. 6. (Color online) Intracavity photon number as a function of time with damping. Initially there is no pump field and the condensate is the homogeneous state. Then the pump field is turned on immediately with $\eta = 1250\omega_r$. The damping rates are (a) $0.05\omega_r$ and (b) $0.02\omega_r$. The other parameters are $N = 4.8 \times 10^4$, $U_0 = 0.25\omega_r$, $\delta_c = 1.2 \times 10^3\omega_r$, $\kappa = 0.4 \times 10^3\omega_r$. The time is in the unit of $1/\omega_r$.

In the above analysis, we have neglected the damping effect. This needs further investigation. Notice that the omitted harmonic tapping potential couples the $\pm 2\hbar k$-momentum modes to other modes and results in the damping of the above model [15,21]. In this case we have to add a damping term in Eq. (7a). The damping introduces a loss in the total energy. Therefore, in the pointlike ball picture, the surface of the bowl is frictional and the ball keeps losing energy after it is released. If the ball reaches one of the wells and loses all the kinetic energy before it overcomes the peak of the barrier, the ball then stays in that well and finally is frozen at the bottom corresponding to the steady state. Figure 6 shows the evolution of the intracavity photon number for different damping rates. We can find that the system finally approaches different branches even if it starts from the same initial states.

V. CONCLUSION

In conclusion, we have studied the nonlinear dynamics of the photon number in an optical cavity filled with a cigar-shaped Bose-Einstein condensate. We find that the way of adding the field is crucial to the switching close to the critical transition point. If the pump field is changed abruptly, the system may jump from one branch to the other even if the pump field intensity has not reached the critical transition point. The physics of this anomalous switching is that the oscillation introduced by the abrupt change of the pumping field may overcome the barrier between the two basins corresponding to the two bistable states. Different initial conditions and damping rates may affect this anomalous switching behavior.

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