Linear Plotkin bound for entanglement-assisted quantum codes

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(Received 1 November 2012; published 7 March 2013)

The entanglement-assisted (EA) formalism is a generalization of the standard stabilizer formalism, and it can transform arbitrary quaternary classical linear codes into entanglement-assisted quantum error correcting codes (EAQECCs) by using of shared entanglement between the sender and the receiver. Using the special structure of linear EAQECCs, we derive an EA-Plotkin bound for linear EAQECCs, which strengthens the previous known EA-Plotkin bound. This linear EA-Plotkin bound is tighter then the EA-Singleton bound, and matches the EA-Hamming bound and the EA-linear programming bound in some cases. We also construct three families of EAQECCs with good parameters. Some of these EAQECCs saturate this linear EA-Plotkin bound and the others are near optimal according to this bound; almost all of these linear EAQECCs are degenerate codes.

DOI: 10.1103/PhysRevA.87.032309 PACS number(s): 03.67.Pp

I. INTRODUCTION

Since the pioneering works of Shor and Steane [1,2], quantum error-correcting codes (QECCs) have been extensively studied in the literature [3–11]. The most widely studied class of quantum codes are stabilizer (or additive) quantum codes, both binary [3–6] and nonbinary [8,9]. Under the stabilizer formalism [4,6], binary stabilizer codes can be constructed from classical codes over finite fields $\mathbf{F}_2$ or $\mathbf{F}_4$ with certain self-orthogonal (or dual containing) properties, where $\mathbf{F}_2$ is the binary field and $\mathbf{F}_4$ is the quaternary field. The self-orthogonal properties form a barrier to importing all classical codes in QECCs [10–12]. It is pointed out in [12] that "Unfortunately, the need for a self-orthogonal parity check matrix is a substantial obstacle to importing the classical theory in its entirety, especially in the context of modern codes such as low-density parity check (LDPC) codes".

In [12], Brun, Devetak, and Hsieh devised the entanglement-assisted (EA) stabilizer formalism which includes the standard stabilizer formalism as a special case; they showed that if shared entanglement between the encoder and decoder is available, classical linear quaternary (and binary) codes that are not self-orthogonal can be transformed into EAQECCs. Following [12], there has been further study of EAQECCs [13–22], and Refs. [17–22] show that entanglement can improve the performance of EAQECCs.

An $[[n,k,d_{ea};c]]$ EAQECC encodes k information qubits into n channel qubits with the help of c pairs of maximally entangled Bell states. The code can correct up to $\lfloor\frac{d_{ea}-1}{2}\rfloor$ errors acting on the n channel qubits, where d_{ea} is the minimum distance of the code. In [21,22], Lai *et al.* discussed the construction of optimal EAQECCs. An $[[n,k,d_{ea};c]]$ EAQECC is optimal in the sense that d_{ea} is the highest achievable minimum distance for given parameters n,k, and c. To judge the optimality of an $[[n,k,d_{ea};c]]$ EAQECC, people deduced some bounds for EAQECCs, such as the EA-Singleton bound in [12]

$$n+c-k \geqslant 2(d_{ea}-1),$$

and the EA-Hamming bound for nondegenerate EAQECCs [11]

$$2^{n+c-k} \geqslant \sum_{i=0}^{t} 3^i \binom{n}{i},$$

where $t = \lfloor\frac{d_{ea}-1}{2}\rfloor$.

Lai, Brun, and Wilde [22] introduced the dual concept of an EAQECC, presented the EA-linear programming bound and the EA-Plotkin bound for EAQECCs, and compared the tightness of their EA-Plotkin bound with the EA-Singleton bound, the EA-Hamming bound, and the EA-linear programming bound. Their EA-Plotkin bound is as follows.

Lemma 1.1 (EA-Plotkin bound [22]*).* If $Q^{ea} = [[n,k,d_{ea};c]]$, then

$$d_{ea} \leqslant \frac{3n \times 4^k}{4(4^k-1)}.$$

They also commented that: "This Plotkin bound applies to arbitrary EAQECCs. However, note that c does not appear in the bound, and consequently, this bound best describes the characteristics of maximal-entanglement EAQEC codes. And, for large k, the bound is approximately $\frac{3}{4}n$. Hence, this bound is useful only for small values of k."

In this work, we will strengthen this bound in the case of linear EAQECCs, an $[[n,k,d_{ea};c]]$ is a linear EAQECC if it is constructed from a linear code over $\mathbf{F}_4$. And we construct three families of linear EAQECCs from low-dimensional quaternary linear codes, some of these EAQECCs saturate our linear EA-Plotkin bound and the others are near optimal.

This work is structured as follows. Section II reviews the symplectic space, the original EA-stabilizer formalism of [12] and two equivalent EA-stabilizer formalisms given in [18]. The additive EA-stabilizer formalism of [18] allows us to work with additive codes over $\mathbf{F}_4$ rather than subgroups of the Pauli group. In Sec. III, we present our main theorem, discuss tightness of our bound and known bounds for EAQECCs, and improve a proposition, i.e., Proposition 0.3 of [12]. Section IV gives explicit constructions of linear EAQECCs and discusses the optimality of these codes. Section V compares our codes with known EAQECCs in the literature and draws a final remark by putting forward a conjecture.

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II. THE EA-STABILIZER FORMALISM AND ADDITIVE CODES

To prove our main result, we briefly review the basic concepts of symplectic space, Pauli group, and additive code [6,23], the EA-stabilizer formalism of [12], and two equivalent EA-stabilizer formalisms given in [18]. For more details, please see [6,12,24].

Let $\mathbf{F}_2$ be the binary field and $\mathbf{F}_2^{2n}$ be the $2n$-dimensional symplectic space whose elements are denoted as $(a \mid b)$ where $a,b \in \mathbf{F}_2^n$. The symplectic inner product of $(a \mid b)$ and $(a' \mid b')$ is defined to be $((a \mid b),(a' \mid b'))_s = a(b')^T + b(a')^T$. For a subspace S of $\mathbf{F}_2^{2n}$, its symplectic dual is defined as $S^{\perp s} = \{(a|b) \mid ((a|b),(a'|b'))_s = 0 \text{ for any } (a'|b') \in S\}$. A subspace S is called totally isotropic if $S \cap S^{\perp s} = S$ and nonisotropic if $S \cap S^{\perp s} = \{0\}$, see [23]. A totally isotropic subspace is called an isotropic subspace in [12,13], and a nonisotropic subspace is call an symplectic subspace in [12] and an entanglement subspace in [13,14].

Let $\mathcal{G}_n$ be the n-fold Pauli group, whose elements are written as $g = i^{\lambda}X(a)Z(b)$ where $\lambda \in \mathbf{Z}_4$ and $(a \mid b) \in \mathbf{F}_2^{2n}$ [6]. The center of $\mathcal{G}_n$ is $Z(\mathcal{G}_n) = \{\pm I, \pm iI\}$, and the quotient group $\bar{\mathcal{G}}_n = \mathcal{G}_n/Z(\mathcal{G}_n)$ is isometry isomorphism to the symplectic space $\mathbf{F}_2^{2n}$ under the map $\tau(i^{\lambda}X(a)Z(b)) = (a \mid b)$ [6]. If $\mathcal{A}$ is a subgroup of $\mathcal{G}_n$, then $\tau(\mathcal{A})$ is a subspace of $\mathbf{F}_2^{2n}$. For a subgroup $\mathcal{A}$, denote its centralizer as $\mathcal{Z}(\mathcal{A})$, then $\tau(\mathcal{Z}(\mathcal{A}))= \tau(\mathcal{A})^{\perp s}$, where $\tau(\mathcal{A})^{\perp s}$ is the symplectic dual space of $\tau(\mathcal{A})$. If $\tau(\mathcal{A})$ is a totally isotropic subspace of F_2^{2n}, $\mathcal{A}$ is called an isotropic subgroup of $\mathcal{G}_n$. If $\tau(\mathcal{A})$ is a nonisotropic subspace of $\mathbf{F}_2^{2n}$, $\mathcal{A}$ is called a symplectic subgroup of $\mathcal{G}_n$ in [12] and an entanglement subgroup in [13], respectively. The EA formalism of [12] is as follow.

Theorem 2.1 [12,22]. Let $\mathcal{S}$ be a subgroup of $\mathcal{G}_n$ of size 2^m, $\mathcal{S}_I$ an isotropic subgroup of size 2^l, and $\mathcal{S}_E$ an entanglement subgroup of size 2^{2c}. If $\mathcal{S}= \mathcal{S}_I \times \mathcal{S}_E$, then $\mathcal{S}$ can be extended into an Abelian subgroup $\tilde{\mathcal{S}}$ of $\mathcal{G}_{n+c}$ with c maximally entangled pairs. $\tilde{\mathcal{S}}$ fixes an EAQECC $\mathcal{Q}^{ea} = [[n,k,d_{ea};c]]$, where $k = n + c - m = n - c - l$, $d_{ea} = \min\{wt(g) \mid g \in \mathcal{Z}(\mathcal{S})\backslash \mathcal{S}_I\}$, and $\mathcal{Z}(\mathcal{S})$ is the centralizer of $\mathcal{S}$. $\mathcal{S}$ is called the EA stabilizer of $\mathcal{Q}^{ea}$.

Let the EA stabilizer of $\mathcal{Q}^{ea} = [[n,k,d_{ea};c]]$ be $\mathcal{S}$. If all nonidentity elements in $\mathcal{S}_I$ have weights greater than d_{ea}, then $\mathcal{Q}^{ea}$ is called a nondegenerate EAQECC, otherwise it is a degenerate one.

Remark 2.1. To simplify statements in the following sections, we always assume that for each $\mathcal{Q}^{ea} = [[n,k,d_{ea};c]]$ EAQECC in this paper, c is the optimal number of entangled bits that $\mathcal{S}$ requires [16]. According to [18,24], in such a case $0 \leqslant k + c \leqslant n$. If $c + k = n$ the EAQECC $[[n,k,d_{ea};n-k]]$ is called a maximal-entanglement EAQECC in [22].

It is very hard to construct EAQECCs using the framework of Theorem 2.1. In [18], using the relationships among $\mathcal{G}_n$, $\mathbf{F}_2^{2n}$, and $\mathbf{F}_4^n$, Li gave two equivalent EA-stabilizer formalisms of Theorem 2.1. To give the two formalisms of [18], we also need the concepts of additive codes over $\mathbf{F}_4$.

Let $\mathbf{F}_4 = \{0,1,\omega,\varpi\}$ be the field of four elements, with $\varpi = 1 + \omega = \omega^2$, $\omega^3 = 1$, and the conjugation is defined by $\overline{x} = x^2$. Let $\mathbf{F}_4^n$ be the n-dimensional row vector space over $\mathbf{F}_4$. For $u = (u_1,u_2,\ldots,u_n)$ and $v = (v_1,v_2,\ldots,v_n) \in \mathbf{F}_4^n$, their trace inner product is defined as $(u,v)_t = tr(u\overline{v}^T) = \sum_1^n(u_j\overline{v_j} + \overline{u_j}v_j)= \sum_1^n(u_jv_j^2 + u_j^2v_j)$, and their Hermitian inner product is defined as $(u,v)_h = \sum_1^n u_j\overline{v_j} = \sum_1^n u_jv_j^2$. An additive code $\mathcal{C}$ of length n over $\mathbf{F}_4$ is a subgroup of $\mathbf{F}_4^n$; if its size is 2^m, it is denoted as $(n,2^m)$ in [6]. If $\mathcal{C}$ is an $(n,2^m)$ additive code, its trace dual is defined as $\mathcal{C}^{\perp t} = \{u \in \mathbf{F}_4^n \mid (u,v)_t = 0 \text{ for all } v \in \mathcal{C}\}$, and $\mathcal{C}^{\perp t}$ is an $(n,2^{2n-m})$ additive code. If $\mathcal{C}$ is an $[n,s]_4$ linear code, its Hermitian dual is defined as $\mathcal{C}^{\perp h} = \{u \in \mathbf{F}_4^n \mid (u,v)_h = 0 \text{ for all } v \in \mathcal{C}\}$, and $\mathcal{C}^{\perp h}$ is an $[n,n-s]_4$ linear code. An $[n,s]_4$ linear code $\mathcal{C}$ is an $(n,2^{2s})$ additive code, and for such a code $\mathcal{C}^{\perp t} = \mathcal{C}^{\perp h}$ is an $(n,2^{2(n-s)})$ additive code. The symplectic space $\mathbf{F}_2^{2n}$ is isometry isomorphism to $\mathbf{F}_4^n$ under the map $\phi((a \mid b)) = \omega a + \varpi b$, and $\bar{\mathcal{G}}_n = \mathcal{G}_n/Z(\mathcal{G}_n)$ is isometry isomorphism to $\mathbf{F}_4^n$ under the map $\phi \circ \tau$.

Now, we can reformulate Theorem 2.1 as the following equivalent Theorems 2.2 and 2.3.

Theorem 2.2 [18]. If S is an m-dimensional subspace of F_2^{2n}, $R(S) = S \cap S^{\perp s}$ is an l-dimensional subspace, and $2c = m - l$, then there is an EAQECC $\mathcal{Q}^{ea} = [[n,k,d_{ea};c]]$, where $k = n + c - m = n - c - l$ and $d_{ea} = \min\{wt_s(\alpha) \mid \alpha \in S^{\perp s} \setminus R(S)\}$. S is called the symplectic EA stabilizer of $\mathcal{Q}^{ea}$.

Theorem 2.3 [18]. If $\mathcal{C}$ is an $(n,2^m)$ additive code and $R(\mathcal{C}) = \mathcal{C} \cap \mathcal{C}^{\perp t}$ is an $(n,2^l)$ additive code, then there is an EAQECC $\mathcal{Q}^{ea} = [[n,k,d_{ea};c]]$, where $k = n - c - l$, $2c = m - l$, and $d_{ea} = \min\{wt(\alpha) \mid \alpha \in \mathcal{C}^{\perp t} \setminus R(\mathcal{C})\}$. $\mathcal{C}$ is called the additive EA stabilizer of $\mathcal{Q}^{ea}$. If $\mathcal{C}$ is a linear code over $\mathbf{F}_4$, $\mathcal{Q}^{ea}$ is called a linear EAQECC.

For a given subspace S of F_2^{2n} in Theorem 2.2 (or additive code $\mathcal{C}$ in Theorem 2.3), using $S^{\perp s}$ (or $\mathcal{C}^{\perp t}$) as an EA stabilizer, one can obtain another EAQECC $\mathcal{Q}^{ea\perp} = [[n,c,d_{ea}^{\perp};k]]$. This EAQECC $\mathcal{Q}^{ea\perp}$ is just the dual code (see [22] for definition) of the EAQECC $\mathcal{Q}^{ea} = [[n,k,d_{ea};c]]$ stabilized by S (or $\mathcal{C}$). Thus, using terminology of symplectic space or additive code, the concept of dual code of an EAQECC can be describe in two equivalent forms as follows.

Corollary 2.4 [24]. (1) Let S be given in Theorem 2.2. Then $S^{\perp s}$ stabilizes an EAQECC $\mathcal{Q}^{ea\perp} = [[n,c,d_{ea}^{\perp};k]]$, where $k = n - c - l, 2c = m - l$, and $d_{ea}^{\perp} = \min\{wt(\alpha) \mid \alpha \in S \setminus R(S)\}$. (2) Let $\mathcal{C}$ be given in Theorem 2.3. Then there is an EAQECC $\mathcal{Q}^{ea\perp} = [[n,c,d_{ea}^{\perp};k]]$, where $k = n - c - l$, $2c = m - l$ and $d_{ea}^{\perp} = \min\{wt(\alpha) \mid \alpha \in \mathcal{C} \setminus R(\mathcal{C})\}$. The code $\mathcal{Q}^{ea\perp} = [[n,c,d_{ea}^{\perp};k]]$ is called the dual code of $\mathcal{Q}^{ea} = [[n,k,d_{ea};c]]$ stabilized by S (or $\mathcal{C}$).

III. STRENGTHENING THE EA-PLOTKIN BOUND FOR LINEAR CODES

In this section, inspired by the result of [25] for linear standard QECCs, we will prove a EA-Plotkin bound for linear EAQECCs, and compare this bound with known upper bounds for EAQECCs. Then, we also improve a proposition of [12] on linear EAQECCs.

To give our EA-Plotkin bound, we also need a concept of equivalence of codes over $\mathbf{F}_4$ [26,27]. Two $[n,s]_4$ linear codes $\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent provided there is a monomial matrix M and an automorphism γ of $\mathbf{F}_4$ such that $\mathcal{C}_2 = \gamma(\mathcal{C}_1M)$. Two equivalent codes over $\mathbf{F}_4$ have the same geometric characteristics [24,26], any $[n,s]_4$ code is equivalent to a code with generator matrix $G = (I_s \quad A)$, where I_s is the identity matrix of size $s \times s$ and A is a matrix of size $s \times (n-s)$, see Theorem 1.6.2 of [27].

Theorem 3.1 (Linear EA-Plotkin bound). If $k \geqslant 1$ and $\mathcal{Q}^{ea} = [[n,k,d_{ea};c]]$ is a linear EAQECC, then $d_{ea} \leqslant \frac{3\cdot 4^k}{8(4^k-1)}(n+c+k)$.

Proof. Let $\mathcal{C} = [n,s]_4$ be the linear EA stabilizer of $\mathcal{Q}^{ea}$ with $r = \dim R(\mathcal{C})$. Then one can deduce $k = n-s-r$ and $c = s-r$ from Theorem 2.3. This implies $s = (n-k+c)/2$ and $r = (n-k-c)/2$. Thus $\mathcal{C}^{\perp h} = [n,n-s]_4$ with $n - s = k + r$.

Since $R(\mathcal{C})$ is an $[n,r]_4$ code, according to the equivalence of quaternary codes given by [26,27], without loss of generality, we can assume $R(\mathcal{C})$ is generated by $G_R = (I_r \quad X)$, where I_r is the identity matrix of size $r \times r$ and X is a matrix of size $r \times (n-r)$, and $\mathcal{C}^{\perp h}$ is generated by $G = \binom{I_r \;\; X}{A_1 \;\; A_2}$. Using elementary row operations over F_4, one can reduce G into G' where

$$G' = \begin{pmatrix} I_r & X \\ 0_{k\times r} & B_1 \end{pmatrix}.$$

Let $(0_{k\times r} \quad B_1)$ generate a code $\mathcal{B}$ and B_1 generates a code $\mathcal{C}_1$. Then $\mathcal{C}_1 = [n-r,k,d_1] = [(n+k+c)/2,k,d_1]$ for some d_1 and $d_{ea} \leqslant d(\mathcal{B}) = d(\mathcal{C}_1) = d_1$. According to the Plotkin bound for quaternary linear codes [27,28], one has $d_1 \leqslant \frac{3\cdot 4^k}{4(4^k-1)} \times (n+c+k)/2$. Thus, we can derive $d_{ea} \leqslant d_1 \leqslant \frac{3\cdot 4^k}{8(4^k-1)}(n+c+k)$.

Remark 3.1. If $R(\mathcal{C}) = \{0\}$, then $c + k = n$ and our bound is reduced into $d_{ea} \leqslant \frac{3n\cdot 4^k}{4(4^k-1)}$ which is the EA-Plotkin bound of [22]. If $R(\mathcal{C}) \neq \{0\}$, then $c + k < n$ and our bound is tighter than the EA-Plotkin bound of [22]. If $k = 1$, this linear EA-Plotkin is the same as the EA-Singleton bond. For $k \geqslant 2$, the linear EA-Plotkin is tighter than the EA-Singleton bond for $n \geqslant 2k$. If $k \geqslant 3$ and $4 \leqslant n \leqslant 15$, the linear EA-Plotkin bound, the EA-linear programming bound, and the EA-Hamming bound match. For $k > 3$ or $k = 3$ and $n \geqslant 16$, we cannot determine the tightness of these three bounds.

For any $[n,s,d]_4$ linear code, using $\mathcal{C}^{\perp h}$ as an EA stabilizer, Brun *et al.* showed that the following Proposition 3.2 holds (the presentation of [12] has some errors; for the correct form of this result, see [14], Proposition 8).

Proposition 3.2 [12,14]. If a classical $[n,s,d]_4$ code exists, then an $[[n,2s-n+c,d;c]]$ EAQECC exists for some nonnegative integer c.

The statement on parameters of the EAQECC is accurate only for nondegenerate EAQECCs; however, it is a little rough for degenerate ones. We will specify their result explicitly in Proposition 3.2$'$, give the value of c and d_{ea} in detail. Then present a method of completely determining d_{ea} of the EAQECC stabilized by $\mathcal{C}^{\perp h}$.

Proposition 3.2$'$. If $\mathcal{C}$ is an $[n,s,d]_4$ linear code and $R(\mathcal{C}) = \mathcal{C} \cap \mathcal{C}^{\perp h}$ is an $[n,r,d']_4$ linear code, then $\mathcal{C}^{\perp h}$ stabilizes a $\mathcal{Q}^{ea} = [[n,2s-n+c,d_{ea};c]] = [[n,s-r,d_{ea};n-s-r]]$ linear EAQECC, where $d_{ea} = \min\{wt(\alpha) \mid \alpha \in \mathcal{C} \setminus R(\mathcal{C})\} \geqslant d$. Especially, if $d' > d$, then $\mathcal{Q}^{ea} = [[n,s-r,d;n-s-r]]$ is a nondegenerate EAQECC.

Proof. Since $\mathcal{C}$ is an $[n,s]_4$ linear code, it is known that $\mathcal{C}^{\perp h}$ is $[n,n-s]_4$ linear code and an $(n,2^{2n-2s})$ additive code. As $R(\mathcal{C})$ is an $(n,2^{2r})$ additive code, according to Theorem 2.3, $\mathcal{C}^{\perp h}$ stabilizes an $[[n,k',d_{ea};c]]$ EAQECC with $2c = 2(n-s) - 2r$, $k' = n + c - 2(n-s) = s - r$, and $d_{ea} = \min\{wt(\alpha) \mid \alpha \in \mathcal{C} \setminus R(\mathcal{C})\}$. It is obvious that $2s - n + c = 2s - n + (n - s - r) = s - r = k'$ and $d_{ea} \geqslant d \geqslant \min\{d,d'\}$, hence the proposition holds.

The minimal distance d_{ea} in Proposition 3.2$'$ can be determined as follows: Let the weight distribution of $\mathcal{C}$ be the sequence $A_0, A_1, \ldots, A_n$, where A_i is the number of vectors in $\mathcal{C}$ of weight i. The polynomial $A(z) = \sum_{i=0}^{n} A_i z^i$ is called the weight polynomial of $\mathcal{C}$. And let the weight polynomial of $R(\mathcal{C})$ be $R(z) = \sum_{i=0}^{n} R_i z^i$. Then $A_i \geqslant R_i$ for $0 \leqslant i \leqslant n$. Suppose $A_j = R_j$ for $0 \leqslant j \leqslant w-1$ and $A_w > R_w$, then $d_{ea} = w$. Moreover, if $d < w$, then $\mathcal{Q}^{ea}$ is a degenerate EAQECC.

IV. CONSTRUCTION OF LINEAR EAQECCS

In this section, we will construct $[[n,2;n-4]]$, $[[n,2;n-6]]$, and $[[n,3;n-5]]$ EAQECCs for each $n \geqslant 7$; some of these EAQECCs saturate the linear EA-Plotkin bound. Our method of constructing EAQECCs is based on proposition 3.2$'$ and the idea of dual EAQECCs of [22]. This construction differs from those of [21,22] for constructing EA repetition codes, $[[n,1,n;n-1]]$ codes with odd n, and $[[n,1,n-1;n-1]]$ codes for even n. First, we make some notations for later use.

Notation 4.1. In the following sections, in each generator matrix of linear codes, we use 2 *and* 3 *to represent* ω *and* ϖ, respectively. For a matrix P, the conjugate transpose of P is denoted as $P^\dagger$, and the juxtaposition $(P,P,\ldots,P)$ of s copies of P is denoted as sP. We judge the optimality of linear EAQECCs only by the linear EA-Plotkin bound.

Let $\mathbf{1}_m = (1,1,\ldots,1)$ and $\mathbf{0}_m = (0,0,\ldots,0)$ be the all one vector and the all zero vector of length m, respectively. Let

$$S_2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 3 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} S_2 & \mathbf{0}_{2\times 1} & S_2 & S_2 & S_2 \\ \mathbf{0}_5 & 1 & \mathbf{1}_5 & 2\mathbf{1}_5 & 3\mathbf{1}_5 \end{pmatrix}.$$

Then $S_2 S_2^\dagger = 0$ and $S_3 S_3^\dagger = 0$. From [27], we know S_2 and S_3 generate the $[5,2,4]_4$ and the $[21,3,16]_4$ Simplex codes, and their weight polynomials are $1 + 15y^4$ and $1 + 63y^{16}$, respectively. We give our constructions in three cases.

Case A. Construction of $[[n,2;n-4]]$ *EAQECC.* In this case, we discuss the construction of $[[n,2;n-4]]$ code from $[n,3]_4$ code $\mathcal{C}_n$ with $\dim R(\mathcal{C}_n) = 1$. Let $G_{3,4} = \begin{pmatrix} 1111 \\ 0123 \\ 0213 \end{pmatrix}$. Then $G_{3,4} G_{3,4}^\dagger = \begin{pmatrix} 000 \\ 010 \\ 001 \end{pmatrix}$. Hence $G_{3,4}$ generates a code $\mathcal{C}_4$ with $\dim R(\mathcal{C}_4) = 1$, and $R(\mathcal{C}_4)$ has weight polynomial $1 + 3z^4$.

For each given matrix $A_{2,l}$ of size $2 \times l$, denote $A_{3,l} = \binom{0_{1\times l}}{A_{2,l}}$ and construct $G_{3,n} = (G_{3,4} \mid A_{3,l})$ for $n = 4 + l$. By choosing a suitable $A_{2,l}$, we can make $G_{3,n}$ generates a code $\mathcal{C}_n$, such that $R(\mathcal{C}_n)$ is generated by the first row of $G_{3,n}$ and its weight polynomial is $1 + 3y^4$.

Theorem 4.1. (1) If $t \geqslant 2$ and $n = 5t + i$ for $0 \leqslant i \leqslant 1$, then there are $[[n,2,n-t-2;n-4]]$ EAQECCs; these codes are near optimal. (2) If $t \geqslant 1$ and $n = 5t + i$ for $2 \leqslant i \leqslant 4$, then there are $[[n,2,n-t-2;n-4]]$ EAQECCs; these codes saturate the linear EA-Plotkin bound.

Proof. Let $A_{2,3}=\begin{pmatrix}001\\112\end{pmatrix}$, $A_{2,4}=\begin{pmatrix}1011\\0123\end{pmatrix}$, $A_{2,6}=\begin{pmatrix}111100\\223311\end{pmatrix}$, $A_{2,7}=\begin{pmatrix}1111001\\0033111\end{pmatrix}$.

(1) For $t\geqslant 2$, construct $A_{2,5t-4}=(A_{2,6}\mid(t-2)S_2)$, $A_{2,5t-3}=(A_{2,7}\mid(t-2)S_2)$, $G_{3,5t}=(G_{3,4}\mid A_{3,5t-4})$, $G_{3,5t+1}=(G_{3,4}\mid A_{3,5t-3})$. Since the codes with generator matrices $G_{3,10}$ and $G_{3,11}$ have weight polynomials $W_{10}(y)=1+3y^4+12y^6+12y^7+18y^8+12y^9+6y^{10}$ and $W_{11}(y)=1+3y^4+6y^7+30y^8+12y^9+6y^{10}+6y^{11}$, respectively, then for $n=5t+i$ and $0\leqslant i\leqslant 1$, the code with generator matrices $G_{3,n}$ has the weight polynomial $W_n(y)=1+3y^4+[W_{10+i}(y)-1-3y^4]y^{4(t-2)}$. Thus, $\mathcal{C}_n^{\perp h}$ stabilizes an $[[n,2,n-t-2;n-4]]$ EAQECC. For $n=5t+i$ and $0\leqslant i\leqslant 1$, the distances of these $[[n,2,n-t-2;n-4]]$ codes are one less than the linear EA-Plotkin bound, hence they are near optimal codes at least.

(2) For $t\geqslant 1$, construct $A_{2,5t-2}=(A_{2,3}\mid(t-1)S_2)$, $A_{2,5t-1}=(A_{2,4}\mid(t-1)S_2)$, and $A_{2,5t}=(tS_2)$. Construct $G_{3,5t+i}=(G_{3,4}\mid A_{3,5t+i-4})$ for $2\leqslant i\leqslant 4$. The codes with generator matrices $G_{3,7}$, $G_{3,8}$, $G_{3,9}$ have the weight polynomials

$$W_7(y)=1+21y^4+12y^5+18y^6+12y^7,$$
$$W_8(y)=1+3y^4+12y^5+30y^6+12y^7+6y^8,$$
$$W_9(y)=1+3y^4+18y^6+24y^7+18y^8$$

respectively. Hence, for $n=5t+i$ and $2\leqslant i\leqslant 4$, the code with generator matrices $G_{3,n}$ has weight polynomials $W_n(y)=1+3y^4+[W_{5+i}(y)-1-3y^4]y^{4(t-1)}$ for $2\leqslant i\leqslant 4$. Thus, $\mathcal{C}_n^{\perp h}$ EA stabilizes an EAQECC $[[n,2,n-t-2;n-4]]$. These EAQECCs saturate the linear EA-Plotkin bound.

Case B. Construction of $[[n,2;n-6]]$ *EAQECC.* In this case, we discuss construction of $[[n,2;n-6]]$ code from $[n,4]_4$ code $\mathcal{C}_n$ with $\dim R(\mathcal{C}_n)=2$.

Let $G_{4,6}=\begin{pmatrix}111100\\001111\\010101\\000011\end{pmatrix}$. Then $G_{4,6}G_{4,6}^T=\begin{pmatrix}0000\\0000\\0011\\0010\end{pmatrix}$. The code $\mathcal{C}_6$ generated by $G_{4,6}$ satisfies $\dim R(\mathcal{C}_6)=2$; $R(\mathcal{C}_6)$ is generated by the first two rows of $G_{4,6}$ and its weight polynomial is $W_{2,6}(z)=1+9y^4+6y^6$.

For a given matrix $A'_{2,l}$, denote $A_{4,l}=\begin{pmatrix}0_{2\times l}\\A'_{2,l}\end{pmatrix}$. For each $n=l+6\geqslant 6$, we will construct a matrix $G_{4,n}=(G_{4,6}\mid A_{4,l})$ with suitable $A'_{2,l}$ such that $G_{4,n}$ generates a code $\mathcal{C}_n$, $R(\mathcal{C}_n)$ is generated by the first two rows of $G_{4,n}$ and with weight polynomial $W_{2,n}(z)=1+9y^4+6y^6$. Thus we have the following theorem.

Theorem 4.2. If $t\geqslant 1$ and $n=5t+i$ for $1\leqslant i\leqslant 5$, then there are $[[n,2,n-t-3;n-6]]$ EAQECCs. The $[[n,2,n-t-3;n-6]]$ EAQECCs for $1\leqslant i\leqslant 2$ are near optimal; the $[[n,2,n-t-3;n-6]]$ EAQECCs for $3\leqslant i\leqslant 5$ saturate the linear EA-Plotkin bound.

Proof. Let $A'_{2,1}=\begin{pmatrix}1\\2\end{pmatrix}$, $A'_{2,2}=\begin{pmatrix}10\\21\end{pmatrix}$, $A'_{2,3}=\begin{pmatrix}101\\213\end{pmatrix}$, $A'_{2,4}=\begin{pmatrix}1011\\2131\end{pmatrix}$, and $A'_{2,5t}=(tS_2)$ for $t\geqslant 1$. Construct $A'_{2,5+i}=(A'_{2,i}\mid S_2)$ for $1\leqslant i\leqslant 5$ and $G_{4,5+j}=(G_{4,6}\mid A_{4,j-1})$ for $2\leqslant j\leqslant 5$. Let $A'_{2,5t+i}=(A'_{2,5+i}\mid(t-1)S_2)$ and $G_{4,5t+i}=(G_{4,6}\mid A_{4,5(t-1)+i-1})$ for $t\geqslant 2$ and $1\leqslant i\leqslant 5$.

Let $\mathcal{C}_n$ be the code generated by $G_{4,n}$. It is easy to check that $R(\mathcal{C}_n)$ is generated by the first two rows of $G_{4,n}$ and with weight polynomial $W_{2,n}(y)=1+9y^4+6y^6$. For $6\leqslant n\leqslant 10$, the weight polynomials of $\mathcal{C}_n$'s are as follows:

$$W_6(y)=1+9y^2+24y^3+99y^4+72y^5+51y^6,$$
$$W_7(y)=1+9y^3+69y^4+54y^5+90z^6+33y^7,$$
$$W_8(y)=1+30y^4+48y^5+96z^6+48y^7+33y^8,$$
$$W_9(y)=1+9y^4+21y^5+90z^6+54y^7+60y^8+21y^9,$$
$$W_{10}(y)=1+9y^4+39y^6+72y^7+90y^8+24y^9+21y^{10}.$$

For $t\geqslant 2$, $n=5t+i$, and $1\leqslant i\leqslant 5$, the weight polynomial of $\mathcal{C}_n$ is $W_n(y)=1+9y^4+6y^6+[W_{5+i}(y)-1-9y^4-6y^6]y^{4(t-1)}$. Thus, one can derive the minimal weight d_{ea} of $\mathcal{C}_n\setminus R(\mathcal{C}_n)$ is $n-t-3$. Hence $\mathcal{C}_n^{\perp h}$ EA stabilizes an $[[n,2,n-t-3;n-6]]$ EAQECC.

For $n=5t+i$, $t\geqslant 1$, and $1\leqslant i\leqslant 5$, it is easy to check that the $[[n,2,n-t-3;n-6]]$ EAQECCs for $1\leqslant i\leqslant 2$ have minimal distances that are one less than the linear EA-Plotkin bound, hence are near optimal codes at least; the $[[n,2,n-t-3;n-6]]$ EAQECCs for $3\leqslant i\leqslant 5$ saturate the linear EA-Plotkin bound. Thus, the theorem holds.

Case C. Construction of $[[n,3;n-5]]$ *EAQECC.* In this case, we discuss construction of $[[n,3;n-5]]$ code from $[n,4]_4$ code $\mathcal{C}_n$ with $\dim R(\mathcal{C}_n)=1$. Let $G=G'_{4,6}=\begin{pmatrix}111100\\012310\\001110\\000011\end{pmatrix}$. Then $GG^{\dagger}=\begin{pmatrix}0000\\0001\\0011\\0110\end{pmatrix}$, and the code $\mathcal{C}_6$ generated by $G'_{4,6}$ satisfies $\dim R(\mathcal{C}_6)=1$, and $R(\mathcal{C}_6)$ has the weight polynomial $W_{1,6}(z)=1+3z^4$.

Denote $D_{4,21}=\begin{pmatrix}0_{1\times 21}\\S_3\end{pmatrix}$, $B_{4,l}=\begin{pmatrix}0_{1\times l}\\B_{3,l}\end{pmatrix}$ for a given matrix $B_{3,l}$. For each $n=l+6+21t\geqslant 7$, we will construct a matrix $G_{4,n}=(G_{4,6}\mid B_{4,l}\mid tD_{4,21})$ with suitable $B_{3,l}$, such that $G_{4,n}$ generates a code $\mathcal{C}_n$ and $R(\mathcal{C}_n)$ with the weight polynomial $W_{1,n}(z)=1+3z^4$, and the minimal weight d_{ea} of $\mathcal{C}_n\setminus R(\mathcal{C}_n)$ is as large as possible. Using $\mathcal{C}_n^{\perp h}$ as the EA stabilizer, one can obtain an $[[n,3,d_{ea};n-5]]$ EAQECC. Our results are the following Theorem 4.3; for its proof please see the Appendix.

Theorem 4.3 (1) If $7\leqslant n\leqslant 19$, then there are the following EAQECCs: $[[n,3,n-4;n-5]]$ for $7\leqslant n\leqslant 9$, $[[n,3,n-5;n-5]]$ for $10\leqslant n\leqslant 14$, and $[[n,3,n-6;n-5]]$ for $15\leqslant n\leqslant 19$.

(2) If $t\geqslant 1$, $-1\leqslant i\leqslant 1$, and $n=21t+i$, then there is an $[[n,3,16t+i-2;n-5]]$ EAQECC.

(3) If $t\geqslant 1$, $2\leqslant i\leqslant 6$, and $n=21t+i$, then there is an $[[n,3,16t+i-3;n-5]]$ EAQECC.

(4) If $t\geqslant 1$, $7\leqslant i\leqslant 9$, and $n=21t+i$, then there is an $[[n,3,16t+i-4;n-5]]$ EAQECC.

(5) If $t\geqslant 1$, $10\leqslant i\leqslant 14$, and $n=21t+i$, then there is an $[[n,3,16t+i-5;n-5]]$ EAQECC.

(6) If $t\geqslant 1$, $15\leqslant i\leqslant 19$, and $n=21t+i$, then there is an $[[n,3,16t+i-6;n-5]]$ EAQECC.

The $[[14,3,9;9]]$, $[[18,3,12;13]]$, and $[[19,3,13;14]]$ EAQECCs in (1), the codes $[[21t+6,3,16t+3;21t+1]]$, $[[21t+14,3,16t+9;21t+9]]$, $[[21t+18,3,16t+12;21t+13]]$, and $[[21t+19,3,16t+13;21t+14]]$ for $t\geqslant 1$ in (3)–(6) are optimal codes and saturate the linear EA-Plotkin bound, the others in (1)–(6) are near optimal.

V. DISCUSSION AND CONCLUDING REMARKS

In this paper we have derived an EA-Plotkin bound for linear EAQECCs, and constructed three families of EAQECCs with good parameters; some of these codes also saturate this linear EA-Plotkin bound and the others are near optimal according to this bound. Almost all the linear EAQECCs we constructed are degenerate, except the $[[7,2,4;3]]$, $[[8,2,5;4]]$, $[[7,2,3;1]]$, $[[8,2,4;2]]$, $[[7,3,3;2]]$, and $[[8,3,4;3]]$ codes. For $n \geqslant 9$, all our $[[n,2;n-4]]$, $[[n,2;n-6]]$, and $[[n,3;n-5]]$ EAQECCs are degenerate codes; the $[[7,2,4;3]]$, $[[7,2,3;1]]$, $[[8,2,4;2]]$, $[[7,3,3;2]]$, and $[[8,3,4;3]]$ EAQECCs are nondegenerate codes; the $[[8,2,5;4]]$ and $[[8,3,4;3]]$ EAQECCs have been obtained by [21], and the $[[8,2,5;4]]$ code is degenerate.

Our results implicate that some optimal EAQECCs can be constructed from "poor" classical codes; the resulting EAQECCs are degenerate codes. Hence, the current idea (given in [12]) of constructing good EAQECCs from good classical codes may be an illusion in some cases, and the assertion "the performance of EAQECC constructed from classical code is determined by the performance of the classical code" in [13] is not always true. The reasons are as follows. The classical codes $\mathcal{C}$ and $\mathcal{C}^{\perp h}$ we used for constructing EAQECCs are "poor" codes, and both of their distances cannot exceed 4. For example, according to Theorem 4.1, there is a $[[49,2,38;45]] = [[5\times 9+4,2,4\times 9+2;5\times 9]]$ degenerate EAQECC, which is constructed from classical codes $\mathcal{C}$ with $\mathcal{C} = [49,3,4]_4$ and $\mathcal{C}^{\perp h} = [49,46,2]_4$. According to Grassl's table on optimal quaternary linear codes [29], an optimal $[49,3]_4$ code has minimum distance 36; using a $[49,3,36]_4$ optimal quaternary linear code, one can only obtain a $[[49,2,36;45]]$ or a $[[49,3,36;46]]$. It is not difficult to check that an optimal nondegenerate $[[49,2;45]]$ EAQECC has $d \leqslant 37$ [30]. Generally, for $n = 5t+49$ and $t \geqslant 0$, one can check that an optimal $[[5t+49,2,4t+38;5t+45]]$ must be a degenerate EAQECC. All this evidence shows that EAQECCs have some properties different from those of classical codes and standard QECCs.

We have checked that all known EAQECCs in [21,22] with $k \geqslant 1$, linear or nonlinear, also obey the linear EA-Plotkin bound. We guess that this linear EA-Plotkin bound may hold for all EAQECCs with $k \geqslant 1$. So, we put forward the following conjecture.

Conjecture: If there is an $[[n,k,d_{ea};c]]$ EAQECC, then $d_{ea} \leqslant \frac{3\cdot 4^k}{8(4^k-1)}(n+c+k)$.

ACKNOWLEDGMENTS

This work is supported by National Natural Science Foundation of China under Grants No. 11071255 and No. 61075054.

APPENDIX: PROOF OF THEOREM 4.3

To prove Theorem 4.3, we give our discussion in three steps. First, we construct $B_{3,i}$ for $1 \leqslant i \leqslant 21$, such that $G_{4,n} = (G'_{4,6} \mid B_{4,i})$ generates a code $\mathcal{C}_n$ and $R(\mathcal{C}_n)$ is a one-dimensional code with weight polynomial $W_{1,n}(z) = 1+3z^4$. The matrices $B_{3,i}$ for $1 \leqslant i \leqslant 21$ are as follows:

$$B_{3,1} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \quad B_{3,2} = \begin{pmatrix} 11 \\ 03 \\ 12 \end{pmatrix}, \quad B_{3,3} = \begin{pmatrix} 111 \\ 023 \\ 133 \end{pmatrix},$$

$$B_{3,4} = \begin{pmatrix} 1011 \\ 0113 \\ 1122 \end{pmatrix}, \quad B_{3,19} = \begin{pmatrix} 0110111010101111011 \\ 1031123100011221113 \\ 1120313211300132322 \end{pmatrix},$$

$$B_{3,5} = \begin{pmatrix} 11111 \\ 01301 \\ 33112 \end{pmatrix}, \quad B_{3,18} = \begin{pmatrix} 111111001101111111 \\ 122303112012001031 \\ 012211123002132332 \end{pmatrix},$$

$$B_{3,6} = \begin{pmatrix} 110110 \\ 201011 \\ 122320 \end{pmatrix}, \quad B_{3,17} = \begin{pmatrix} 11000111101010111 \\ 31011301310131212 \\ 20133123121200123 \end{pmatrix},$$

$$B_{3,7} = \begin{pmatrix} 0111101 \\ 1330211 \\ 0122123 \end{pmatrix}, \quad B_{3,16} = \begin{pmatrix} 0111110101111101 \\ 1321011013010113 \\ 0110021222311333 \end{pmatrix},$$

$$B_{3,8} = \begin{pmatrix} 11110111 \\ 30311102 \\ 11223333 \end{pmatrix}, \quad B_{3,15} = \begin{pmatrix} 101101110011111 \\ 012312331120012 \\ 200133223113102 \end{pmatrix},$$

$$B_{3,9} = \begin{pmatrix} 110101111 \\ 131211012 \\ 332102302 \end{pmatrix}, \quad B_{3,14} = \begin{pmatrix} 11110011010111 \\ 31231102111313 \\ 11302122200223 \end{pmatrix},$$

$$B_{3,10} = \begin{pmatrix} 1011111110 \\ 1100122321 \\ 3031232103 \end{pmatrix}, \quad B_{3,13} = \begin{pmatrix} 0111101011111 \\ 1010313102213 \\ 0103221321323 \end{pmatrix},$$

$$B_{3,11} = \begin{pmatrix} 11101011110 \\ 22111130231 \\ 12233313331 \end{pmatrix}, \quad B_{3,12} = \begin{pmatrix} 110010111111 \\ 021131210103 \\ 111230303321 \end{pmatrix},$$

$$B_{3,20} = \begin{pmatrix} 12310221122301303001 \\ 02121230320221132213 \\ 21012331211012203203 \end{pmatrix},$$

$$B_{3,21} = \begin{pmatrix} 011011111111100011101 \\ 001132030202211113113 \\ 100121123321212231203 \end{pmatrix}.$$

Second, using the matrices $B_{3,i}$ for $1 \leqslant i \leqslant 21$, we construct $G_{4,n}$ for $n \geqslant 7$ as follows.

(1) Let $G_{4,n} = (G'_{4,6} \mid B_{4,n-6})$ for $7 \leqslant n \leqslant 19$.

(2) If $t \geqslant 1$, $-1 \leqslant i \leqslant 6$, and $n = 21t+i$, let $G_{4,21t+i} = (G'_{4,6} \mid B_{4,15+i} \mid (t-1)D_{4,21})$.

3) If $t \geqslant 1$, $7 \leqslant i \leqslant 19$, and $n = 21t+i$, let $G_{4,21t+i} = (G'_{4,6} \mid B_{4,i-6} \mid tD_{4,21})$.

Let $\mathcal{C}_n$ be the code generated by $G_{4,n}$ and the weight polynomial of $\mathcal{C}_n$ be $W_n(z)$ for $n \geqslant 7$. Then $R(\mathcal{C}_n)$ is a one-dimensional code with weight polynomial $W_{1,n}(z) = 1+3z^4$, and all the weight polynomials of these $W_n(z)$ can be derived from $W_i(z)$ for $7 \leqslant i \leqslant 27$. It is not difficult to check that $W_i(z)$ for $7 \leqslant i \leqslant 27$ are as follows:

$$W_7(z) = 1+15z^3+45z^4+90z^5+66z^6+39z^7,$$
$$W_8(z) = 1+24z^4+72z^5+60z^6+72z^7+27z^8,$$

$$W_9(z) = 1 + 3z^4 + 39z^5 + 66z^6 + 78z^7 + 42z^8 + 27z^9,$$
$$W_{10}(z) = 1 + 3z^4 + 63z^6 + 72z^7 + 36z^8 + 72z^9 + 9z^{10},$$
$$W_{11}(z) = 1 + 3z^4 + 18z^6 + 57z^7 + 54z^8 + 78z^9 + 36z^{10} + 9z^{11},$$
$$W_{12}(z) = 1 + 3z^4 + 21z^7 + 78z^8 + 66z^9 + 30z^{10} + 57z^{11},$$
$$W_{13}(z) = 1 + 3z^4 + 39z^8 + 78z^9 + 48z^{10} + 60z^{11} + 21z^{12} + 6z^{13},$$
$$W_{14}(z) = 1 + 3z^4 + 54z^9 + 75z^{10} + 72z^{11} + 24z^{12} + 18z^{13} + 9z^{14},$$
$$W_{15}(z) = 1 + 3z^4 + 15z^9 + 54z^{10} + 78z^{11} + 42z^{12} + 51z^{13} + 12z^{14},$$
$$W_{16}(z) = 1 + 3z^4 + 15z^{10} + 84z^{11} + 60z^{12} + 48z^{13} + 33z^{14} + 12z^{15},$$
$$W_{17}(z) = 1 + 3z^4 + 48z^{11} + 54z^{12} + 69z^{13} + 54z^{14} + 18z^{15} + 9z^{17},$$
$$W_{18}(z) = 1 + 3z^4 + 51z^{12} + 84z^{13} + 36z^{14} + 60z^{15} + 21z^{16},$$
$$W_{19}(z) = 1 + 3z^4 + 63z^{13} + 90z^{14} + 54z^{15} + 18z^{16} + 27z^{17},$$
$$W_{20}(z) = 1 + 3z^4 + 33z^{13} + 54z^{14} + 72z^{15} + 42z^{16} + 33z^{17} + 12z^{18} + 6z^{19},$$
$$W_{21}(z) = 1 + 3z^4 + 51z^{14} + 60z^{15} + 36z^{16} + 72z^{17} + 21z^{18} + 12z^{19},$$
$$W_{22}(z) = 1 + 3z^4 + 51z^{15} + 60z^{16} + 78z^{17} + 42z^{18} + 15z^{19} + 6z^{20},$$
$$W_{23}(z) = 1 + 3z^4 + 30z^{15} + 39z^{16} + 72z^{17} + 42z^{18} + 42z^{19} + 21z^{20} + 6z^{22},$$
$$W_{24}(z) = 1 + 3z^4 + 24z^{16} + 75z^{17} + 54z^{18} + 54z^{19} + 18z^{20} + 15z^{21} + 12z^{22},$$
$$W_{25}(z) = 1 + 3z^4 + 66z^{17} + 51z^{18} + 42z^{19} + 36z^{20} + 30z^{21} + 21z^{22} + 6z^{23},$$
$$W_{26}(z) = 1 + 3z^4 + 51z^{18} + 72z^{19} + 36z^{20} + 72z^{21} + 21z^{22},$$
$$W_{27}(z) = 1 + 3z^4 + 84z^{19} + 51z^{20} + 60z^{21} + 36z^{22} + 21z^{24}.$$

For $n = 21t + i > 27$, from the construction of $G_{4,n}$, we can deduce that the weight polynomial $W_n(z)$ of $\mathcal{C}_n$ must be $W_{21t+i}(z) = 1 + 3z^4 + (W_{21+i}(z) - 1 - 3z^4)z^{16(t-1)}$ for $-1 \leqslant i \leqslant 6$, and $W_{21t+i}(z) = 1 + 3z^4 + (W_i(z) - 1 - 3z^4)z^{16t}$ for $7 \leqslant i \leqslant 19$.

Third, from the weight polynomial $W_n(z)$ of $\mathcal{C}_n (n \geqslant 7)$, one can deduce the minimal weight $d_{ea}(n)$ of $\mathcal{C}_n \setminus R(\mathcal{C}_n)$. For $n = 21t + i$, the $d_{ea}(n)$ are

$$d_{ea}(21t + i) = 16t + i - 2 \quad \text{for} \quad t \geqslant 1, \quad -1 \leqslant i \leqslant 1,$$
$$d_{ea}(21t + i) = 16t + i - 3 \quad \text{for} \quad t \geqslant 1, \quad 2 \leqslant i \leqslant 6,$$
$$d_{ea}(21t + i) = 16t + i - 4 \quad \text{for} \quad t \geqslant 1, \quad 7 \leqslant i \leqslant 9,$$
$$d_{ea}(21t + i) = 16t + i - 5 \quad \text{for} \quad t \geqslant 1, \quad 10 \leqslant i \leqslant 14,$$
$$d_{ea}(21t + i) = 16t + i - 6 \quad \text{for} \quad t \geqslant 1, \quad 15 \leqslant i \leqslant 19.$$

It is trivial to verify that the EAQECCs [[14,3,9;9]], [[18,3,12;13]], and [[19,3,13;14]] in (1) and the EAQECCs $[[21t+6,3,16t+3;21t+1]]$, $[[21t+14,3,16t+9;21t+9]]$, $[[21t+18,3,16t+12;21t+13]]$, and $[[21t+19,3,16t+13;21t+14]]$ for $t \geqslant 1$ in (3)–(6) are optimal codes and saturate the linear EA-Plotkin bound; the others in (1)–(6) are near optimal codes. Summarizing the above discussions, the theorem follows.

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