# Degree of polarization in quantum optics through the second generalization of intensity

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The classical definition of degree of polarization (DOP) is expressed in the quantum domain by replacing intensities through quantum mechanical average values of relevant number operators and is viewed as the first generalization of intensity. This definition assigns inaccurately the unpolarized status to some typical optical fields, e.g., amplitude-coherent phase-randomized and hidden-polarized light, which are not truly unpolarized light. The apparent paradoxical trait is circumvented by proposing a new definition of DOP in quantum optics through the second generalization of intensity. The correspondence of a new DOP to the usual DOP in quantum optics is established. It is seen that the two definitions disagree significantly for intense optical fields but coincide for weak light (thermal light) or for optical fields in which occupancy of photons in the orthogonal mode is very feeble. Our proposed definition of DOP, similar to other proposals in literature, reveals an interesting feature that states of polarization of optical quantum fields depend upon the average photons (intensity) present therein.

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Polarization of light, ensuring transversal character, is a centuries-old concept discovered by Bartholinus [1]. In classical optics, this trait of light is characterized by Stokes theory (parameters) [2] geometrically interpreted due to Poincare [3]. Remarkably, these Stokes parameters can also be applied to some optical quantum fields for inferring polarization nature, where they are defined to be quantum mechanical average values of the Stokes operators [4]. Although the polarization of the optical field has acquired indispensable candidacy for demonstrating fundamental issues of quantum mechanics as well as performing ingenious experiments in quantum optics [5], the basic understanding of optical polarization in terms of spatiotemporal variables of optical fields remains unexplored.

Although the studies on optical polarization may largely be classified in two extremes [perfect (complete) polarized state and unpolarized state], optical fields may exist in infinitely many states which are neither polarized nor unpolarized. In 1971, the unpolarized optical field is rigorously investigated wherein the structure of its density operator is discovered [6]. Other prominent works [7] on the state of unpolarized light have brought in new insights about its quantum nature in conjunction with its tomography. Also, in Ref. [6], it is emphasized that Stokes parameters prescribe insufficient conditions for characterizing the state of unpolarized electromagnetic radiation, especially when higher-order correlations between optical-field amplitudes are critical [8]. On the other hand, perfect polarized light is defined by requiring the disappearance of light (signal) in at least one transverse orthogonal mode [9], although this treatment does not provide a procedure for testing whether an arbitrary quantum state of light is perfect polarized or unpolarized.

Modern approaches for ascertaining the state of polarization witnessed two complementary methods: computable measures and operational measures. The former measure [10], based upon the “notion of distance” of optical quantum states from the state of unpolarized light, is applied to introduce expressions for degree of polarization (DOP). On the other hand, the latter approach is nothing but a Stokes-parametric approach. Notably, Klimov *et al.* [11] formulated a pragmatic and ingenious criterion for DOP in terms of minimal fluctuations in Stokes parameters on the Poincare sphere. Astute inspection of higher-order correlations in Stokes parameters and variables, where only equal numbers of bosonic creation and annihilation operators are involved [12], buttresses clinching evidence against a general propensity in favor of Stokes parameters because these Stokes-parametric correlation functions are, inherently, not synonymous to higher-order Glauber correlation functions [13]. Thus, not only the distance-based approach, being an abstract conception, lacks correspondence to classical description of the optical polarization and transcribes variant values of DOP for the same quantum state, but also skepticisms mount pertaining to operational measures due to unprecedented incisive analyses [14] highlighting the inadequacy of the Stokes theory. Moreover, Luis [15] contrived, by drawing analogy from SU(2) Lie algebra of Stokes operators to those for components of Jordon-Schwinger spin angular momenta [16], that the SU(2) $Q$ function is the most suitable distribution function for probabilistic description of optical polarization of quantum states on the Poincare sphere. The SU(2) $Q$ function of quantum states is, in turn, applied to cast a DOP as a “distance” from the uniform distribution possessed by unpolarized light. Later on, this SU(2) $Q$-function approach is generalized to characterize the states of polarization of the nonparaxial three-dimensional optical field [17], the description of which has witnessed various alternative approaches [18]. However, Karassiov [19] recognized that Stokes operators found a distinct sort of Hilbert space for their operation in contrast to those of spin angular momenta. This is why, recently in the spirit of a classical description of optical polarization, a quantum phase-space description of polarized optical field is carried out [20]. Nonetheless, some serious objections may be drawn to the Luis proposal: First, it does not ascribe the value unity for the coherent light (perfect polarized state), a multiphoton state; and second, the SU(2)

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$Q$ function does not connect manifolds with different photon excitation.

Recent trends in quantum optics spearhead new physical effects such as quantum Darwinism, quantum imaging, ghost imaging, and spatiotemporal multipartite entanglement [21] in which spatiotemporal features of the optical field, its quantum state engineering, and bases-based quantum measurements are harnessed. Vociferously, none of the preceding definitions of DOP, whether they may be a computable (distance based) or operational (Stokes parametric) or SU(2) $Q$ function, explore innate relationships possessed by spatiotemporal properties of the optical quantum fields.

Our viewpoint on optical polarization stems from its classical description, i.e., by the superposition of two transverse orthogonal harmonic oscillators of synchronized frequency emulating two transverse orthogonal components of a harmonic electromagnetic plane wave in any basis of description preserving nonrandom values of "ratio of real amplitudes" and "difference in phases" or nonrandom values of the "ratio of their complex amplitudes" which define the "index of polarization" [22] for perfect polarized light. In a quantum regime, a quantum criterion is established [see Eq. (7)] by invoking the fact that is due to Mehta and Sharma [9]. This criterion prescribes a recipe for characterizing whether a light in any arbitrary quantum state is perfectly polarized and picks out simultaneously the characteristic parameters [23].

We urge that the vacuum state of the optical field (virtual photons) must find a paramount position in the theory of optical polarization. Therefore, in this paper, we introduce an alternative judicious expression of DOP by employing the second generalization of intensity in which virtual vacuum photons enter through the projection operation. Our definition, contrary to other prevalent proposals for DOP, meets the very basic requirements of the term DOP in verbatim furnishing a unit value for the perfect polarized state (coherent state) and a vanishing value for the unpolarized state of light.

First, we shall describe our criterion to characterize perfect optical polarization to establish consistency. A plane monochromatic optical field propagating along the $z$ direction in free space can, in general, be described by a vector potential $\vec{\mathrm{A}}$ in the form

$$\vec{\mathrm{A}} = \hat{\mathrm{e}}_x A_{0x}\cos(\psi - \phi_x) + \hat{\mathrm{e}}_y A_{0y}\cos(\psi - \phi_y),$$
$$\psi = \omega t - kz,$$

or in analytic-signal representation

$$\vec{\mathcal{A}} = (\hat{\mathrm{e}}_x \underline{A}_x + \hat{\mathrm{e}}_y \underline{A}_y)e^{-i\psi}, \quad \underline{A}_{x,y} = A_{0x,0y}e^{i\phi_{x,y}}, \tag{1}$$

where $\underline{A}_{x,y}$ are classical complex amplitudes; $A_{0x,0y}$, real amplitudes and phase parameters, $\phi_{x,y}$ $(0 \leqslant \phi_{x,y} < 2\pi)$ possess, in general, random spatiotemporal variation with angular frequency, $\omega$ in linear polarization basis $(\hat{\mathrm{e}}_x,\hat{\mathrm{e}}_y)$ of transverse plane to $\vec{\mathrm{k}}(=k\hat{\mathrm{e}}_z)$ which is the propagation vector of magnitude, $k$ and $\hat{\mathrm{e}}_{x,y,z}$ are unit vectors along the respective $x$, $y$, and $z$ axes forming a right-handed triad.

We consider a pellucid property of perfect polarized optical field, namely, the nonrandom ratio of complex amplitudes of transverse orthogonally polarized modes

$$\underline{A}_y/\underline{A}_x = p, \tag{2}$$

as a definition. Here, $p$ is a nonrandom complex parameter in the linear-polarization basis ($\hat{\mathrm{e}}_x$, $\hat{\mathrm{e}}_y$) and is termed as the index of polarization [22] which renders characteristic polarization parameters (ratio of real amplitudes and difference in phases). Evidently, one may note that the polarized optical field (through nonrandom $p$) is effectively a monomodal optical field since only one random complex amplitude suffices for its complete statistical description.

Additionally, if one introduces new parameters $A_0$ (real random amplitude defining global intensity), $\chi_0$ (polar angle), $\Delta_0$ (azimuth angle), $\phi$ (random global phase) on a Poincare sphere, satisfying inequalities $0 \leqslant A_0$, $0 \leqslant \chi_0 \leqslant \pi$, $-\pi < \Delta_0 \leqslant \pi$, $0 \leqslant \phi < 2\pi$, respectively, involving transforming equations in terms of old parameters $A_0 = (A_{0x}^2 + A_{0y}^2)^{1/2}$, $\chi_0 = 2\tan^{-1}(A_{0y}/A_{0x})$, and $\Delta_0 = \phi_y - \phi_x$; $\phi = (\phi_x + \phi_y)/2$, the analytical signal $\vec{\mathcal{A}}$ in Eq. (1) yields a self-instructive form

$$\vec{\mathcal{A}} = \hat{\varepsilon}_0\mathcal{A}; \quad \mathcal{A} = \underline{A}e^{-i\Psi}; \quad \underline{A} = A_0 e^{i\phi},$$
$$\hat{\varepsilon}_0 = \hat{\mathrm{e}}_x \cos\frac{\chi_0}{2}e^{-\Delta_0/2} + \hat{\mathrm{e}}_y \sin\frac{\chi_0}{2}e^{\Delta_0/2}. \tag{3}$$

Interpretatively, this form of vector potential $\vec{\mathcal{A}}$ in Eq. (3) may be construed as a single-mode polarized optical field, statistically explicable by a single complex amplitude $\underline{A}$ polarized in the fixed direction $\hat{\varepsilon}_0$ specifying the polarization mode $(\hat{\varepsilon}_0,\vec{\mathrm{k}})$. Here, the complex unit vector $\hat{\varepsilon}_0$ $(\hat{\varepsilon}_0^* \cdot \hat{\varepsilon}_0 = 1)$ assigns the parametrized expression of the index of polarization on the Poincare sphere $p = \underline{A}_y/\underline{A}_x = \tan\frac{\chi_0}{2}e^{i\Delta_0}$. Visibly, the state of optical polarization is specified by the nonrandom values of $p$, which, in turn, is fixed by the nonrandom values of $\chi_0$ and $\Delta_0$ defining a point $(\hat{\varepsilon}_0)$ on the unit Poincare sphere analogous to its counterpart Stokes parameters.

One may develop a quantum theory for perfect optical polarization on a similar classical lineage. In quantum optics, the optical field [Eq. (1)] is described by the vector potential operator

$$\vec{\hat{\mathcal{A}}} = \left(\frac{2\pi}{\omega V}\right)^{1/2}[(\hat{\mathrm{e}}_x\hat{\mathrm{a}}_x + \hat{\mathrm{e}}_y\hat{\mathrm{a}}_y)e^{-i\psi} + \mathrm{H.c.}]$$
$$= \left(\frac{2\pi}{\omega V}\right)^{1/2}[(\hat{\varepsilon}\,\hat{\mathrm{a}}_{\hat{\varepsilon}} + \hat{\varepsilon}_\perp\hat{\mathrm{a}}_{\hat{\varepsilon}_\perp})e^{-i\psi} + \mathrm{H.c.}],$$

in the linear-polarization basis $(\hat{\mathrm{e}}_x,\hat{\mathrm{e}}_y)$ or in the elliptic-polarization basis $(\hat{\varepsilon},\hat{\varepsilon}_\perp)$ [24], respectively, where $\omega$ is the angular frequency of the optical field and $V$ is the quantization volume of the cavity, and H.c. stands for Hermitian conjugate. Orthonormal properties of $\hat{\varepsilon}(=\varepsilon_x\hat{\mathrm{e}}_x + \varepsilon_y\hat{\mathrm{e}}_y)$ and $\hat{\varepsilon}_\perp(=\varepsilon_{\perp x}\hat{\mathrm{e}}_x + \varepsilon_{\perp y}\hat{\mathrm{e}}_y)$ provide the relationships between bosonic-annihilation operators $\hat{\mathrm{a}}_{\hat{\varepsilon}}$ $(\hat{\mathrm{a}}_{\hat{\varepsilon}_\perp})$ with those in the linear-polarization basis $(\hat{\mathrm{e}}_x,\hat{\mathrm{e}}_y)$ as

$$\hat{\mathrm{a}}_{\hat{\varepsilon}} = \varepsilon_x^*\hat{\mathrm{a}}_x + \varepsilon_y^*\hat{\mathrm{a}}_y, \quad \hat{\mathrm{a}}_{\hat{\varepsilon}_\perp} = \varepsilon_{\perp x}^*\hat{\mathrm{a}}_x + \varepsilon_{\perp y}^*\hat{\mathrm{a}}_y. \tag{4}$$

The pure (mixed) dynamical state of a monochromatic optical beam, propagating along the $z$ axis and polarized in the mode $(\hat{\varepsilon}_0, \vec{\mathrm{k}})$, may be specified by a state vector $|\psi\rangle(\rho)$ in its Hilbert space. Obviously, such light does not have signal (photons) in the orthogonal mode $(\hat{\varepsilon}_{0\perp},\vec{\mathrm{k}})$, i.e.,

$$\hat{\mathrm{a}}_{\hat{\varepsilon}_{0\perp}}|\psi\rangle = 0, \tag{5}$$

or $\hat{a}_{\hat{\varepsilon}_{0\perp}}\rho = 0$, which yields, on applying Eq. (4), $(\varepsilon^*_{0\perp x}\hat{a}_x + \varepsilon^*_{0\perp y}\hat{a}_y)|\psi\rangle = 0$. Refurbishing it by the orthogonality relation between $\hat{\varepsilon}_0$ and $\hat{\varepsilon}_{0\perp}$, one obtains the defining equation (criterion) for perfect optical polarization

$$\hat{a}_y|\psi\rangle = p\hat{a}_x|\psi\rangle \tag{6}$$

or $\hat{a}_y\rho = p\hat{a}_x\rho$, the quantum analog to the classical perfect optical-polarization criterion [Eq. (2)], giving $p = \varepsilon_{0y}/\varepsilon_{0x}$ $[= \tan\frac{\chi_0}{2}e^{i\Delta_0}$, from Eq. (3)]. Multiplying Eq. (5) from the left by the inverse annihilation operator $\hat{a}_x^{-1} = \hat{a}_x^\dagger(1 + \hat{a}_x^\dagger\hat{a}_x)^{-1}$ [25], we obtain

$$\hat{P}|\psi\rangle = p(1 - \hat{V}_x)|\psi\rangle \tag{7}$$

or $\hat{P}\rho = p(1 - \hat{V}_x)\rho$, where $\hat{P}(\equiv\hat{a}_x^{-1}\hat{a}_y)$ is recognized as polarization operator and $\hat{V}_x$ is the vacuum projection operator for $x$-polarized virtual photons $\hat{V}_x \equiv \sum_{n_y=0}^{\infty}|0,n_y\rangle\langle n_y,0|$, where $n_y$ is the number of $y$ polarized photons. As a demonstration of the criterion [Eq. (7)], one may consider the biphotonic qutrit state $|\psi\rangle = \frac{1}{3}|2,0\rangle + \frac{2}{3}|1,1\rangle + \frac{2}{3}|0,2\rangle$, which provides $p = \sqrt{2}$ showing that light is plane polarized in a direction making an angle of $2\tan^{-1}(\sqrt{2})$ with the $x$ axis having the characteristic parameter ratio of real amplitudes and difference in phases equal to $\sqrt{2}$ and 0, respectively.

Second, the *first generalization of intensity and inadequacy of DOP* is pointed out. A simple experiment may be accomplished to record the maximum and minimum intensities of a light falling on a polarizer whose fast axis is set along a unit polarization vector $\hat{\varepsilon}_0$. Obviously, for a light of arbitrary state of polarization, one obtains the extremum intensities $(I_{\hat{\varepsilon}_0})_{\max} = I_{\text{pol}} + \frac{1}{2}I_{\text{unpol}}$ and $(I_{\hat{\varepsilon}_0})_{\min} = \frac{1}{2}I_{\text{unpol}}$, where $I_{\text{pol}}, I_{\text{unpol}}$ stand for intensities of polarized and unpolarized light, respectively. DOP, in classical optics, is expressed as

$$P = \frac{I_{\text{pol}}}{I_{\text{total}}} = \frac{(I_{\hat{\varepsilon}_0})_{\max} - (I_{\hat{\varepsilon}_0})_{\min}}{(I_{\hat{\varepsilon}_0})_{\max} + (I_{\hat{\varepsilon}_0})_{\min}}. \tag{8}$$

For polarized light, $(I_{\hat{\varepsilon}_0})_{\min} = 0$ implying DOP, $P = 1$, and for unpolarized state, DOP, $P = 0$ because $(I_{\hat{\varepsilon}_0})_{\max} = (I_{\hat{\varepsilon}_0})_{\min}$.

A natural generalization of Eq. (8) to the quantum domain can be affected by replacing the intensity $I_{\hat{\varepsilon}_0}$ by the quantum mechanical average value of the photon number operator $\hat{N}_{\hat{\varepsilon}_0}$, i.e., $I_{\hat{\varepsilon}_0} \to n_{\hat{\varepsilon}_0} = \text{Tr}[\hat{\rho}\hat{N}_{\hat{\varepsilon}_0}]$, where Tr stands for trace, $\hat{\rho}$ is density operator for the optical field, $\hat{N}_{\hat{\varepsilon}_0} = \hat{a}^\dagger_{\hat{\varepsilon}_0}\hat{a}_{\hat{\varepsilon}_0}$, and $\hat{a}^\dagger_{\hat{\varepsilon}_0}$ ($\hat{a}_{\hat{\varepsilon}_0}$) is the creation (annihilation) operator for the optical-field mode polarized along $\hat{\varepsilon}_0$. Hence, in the quantum domain, Eq. (8) takes the form

$$P^{(I)} = \frac{(n_{\hat{\varepsilon}_0})_{\max} - (n_{\hat{\varepsilon}_0})_{\min}}{(n_{\hat{\varepsilon}_0})_{\max} + (n_{\hat{\varepsilon}_0})_{\min}}. \tag{9}$$

Evidently, Eq. (9) is a quantum version of the definition for DOP in classical optics [Eq. (8)] and may be regarded as the first generalization of intensity [26]. Let us verify whether Eq. (9) meets basic requirements, viz., the DOP, $P^{(I)}$ attains zero value for the unpolarized state and unit value for the perfectly polarized state.

Let us consider an amplitude-coherent phase-randomized (multiphoton) optical field propagating along the positive $\hat{e}_z$ direction, the quantum state of which, in the transverse linear polarization basis ($\hat{e}_x$, $\hat{e}_y$), may be specified by the density operator

$$\hat{\rho} = \frac{1}{(2\pi A_0)^2}\iint d^2\alpha_x d^2\alpha_y \delta(|\alpha_x| - A_0) \times \delta(|\alpha_y| - A_0)|\alpha_x,\alpha_y\rangle\langle\alpha_x,\alpha_y|, \tag{10}$$

where $\delta(-)$ is a Dirac-$\delta$ function, $A_0$ is a real amplitude, $|\alpha_x,\alpha_y\rangle$ are bimodal quadrature coherent states, $(\hat{a}_x,\hat{a}_y)|\alpha_x,\alpha_y\rangle = (\alpha_x,\alpha_y)|\alpha_x,\alpha_y\rangle$, and $(\hat{a}_x,\hat{a}_y)$ are annihilation operators for $x$-and $y$-polarized photons, respectively. Noting $\hat{a}_{\hat{\varepsilon}_0} = \varepsilon^*_{0x}\hat{a}_x + \varepsilon^*_{0y}\hat{a}_y$ [Eq. (4)], one obtains the intensity along $\hat{\varepsilon}_0$ as

$$\begin{aligned} n_{\hat{\varepsilon}_0} &= \text{Tr}\big[\hat{\rho}\hat{N}_{\hat{\varepsilon}_0}\big] \\ &= \text{Tr}[\hat{\rho}|\varepsilon_{0x}|^2\,\hat{a}_x^\dagger\hat{a}_x + |\varepsilon_{0y}|^2\hat{a}_y^\dagger\hat{a}_y \\ &\quad + \varepsilon_{0x}\varepsilon^*_{0y}\hat{a}_x^\dagger\hat{a}_y + \varepsilon^*_{0x}\varepsilon_{0y}\hat{a}_y^\dagger\hat{a}_x] \\ &= (|\varepsilon_{0x}|^2 + |\varepsilon_{0y}|^2)A_0^2 = A_0^2, \end{aligned} \tag{11}$$

independent of the unit polarization vector $\hat{\varepsilon}_0(=\varepsilon_{0x}\hat{e}_x + \varepsilon_{0y}\hat{e}_y$ and $|\varepsilon_{0x}|^2 + |\varepsilon_{0y}|^2 = 1)$. Clearly, Eq. (11) demonstrates that for this multiphoton (amplitude-coherent phase-randomized) optical field the DOP, $P^{(I)}$ is zero, suggesting it, unequivocally, to the status of unpolarized state. But, this is not true because all its quantum statistical properties are not symmetric about the direction of propagation $\hat{e}_z$ [8]. This opposite instance breeds doubts about the definition (9) obtained through the first generalization of intensity in quantum optics, which, in turn, necessitates another generalization (the second generalization of intensity).

Third, the *DOP through the second generalization of intensity* is introduced by considering those measurement events in which one of the exit channels of the polarization analyzer registers no photons and takes the average intensity in the other channel. It is, therefore, proposed that instead of replacing $I_{\hat{\varepsilon}_0}$ in Eq. (8) by $n_{\hat{\varepsilon}_0} = \text{Tr}[\hat{\rho}\hat{N}_{\hat{\varepsilon}_0}]$ for accomplishing the quantum version $P^{(I)}$ [Eq. (9)] of DOP in quantum optics, we must replace $I_{\hat{\varepsilon}_0}$ by

$$n_{\hat{\varepsilon}_0} = \text{Tr}\left[\hat{\rho}\hat{N}_{\hat{\varepsilon}_0}\hat{\mathcal{V}}_{\hat{\varepsilon}_{0\perp}}\right], \tag{12}$$

where $\hat{\mathcal{V}}_{\hat{\varepsilon}_{0\perp}}$ is $\hat{\varepsilon}_{0\perp}$ mode's vacuum projection operator, i.e., $\hat{\mathcal{V}}_{\hat{\varepsilon}_{0\perp}} = |0\rangle_{\hat{\varepsilon}_{0\perp}\hat{\varepsilon}_{0\perp}}\langle 0|$. Equation (12) may be regarded as the second generalization of intensity which leads to the second modification in DOP in quantum optics as

$$P^{(\text{II})} = \frac{(n_{\hat{\varepsilon}_0})_{\max} - (n_{\hat{\varepsilon}_0})_{\min}}{(n_{\hat{\varepsilon}_0})_{\max} + (n_{\hat{\varepsilon}_0})_{\min}}. \tag{13}$$

We first show that the definition (13) meets the basic requirements, i.e., the DOP, $P^{(\text{II})}$ picks zero value for the unpolarized state and unity value for the perfectly polarized state. For a beam polarized along $\hat{\varepsilon}_0$, propagating in the $\hat{e}_z$ axis, we note $(n_{\hat{\varepsilon}_0})_{\max} = n_{\hat{\varepsilon}_0}$ and $(n_{\hat{\varepsilon}_0})_{\min} = n_{\hat{\varepsilon}_{0\perp}} = 0$, which provides the unit value for DOP, $P^{(\text{II})}$. Next, for unpolarized light having density operator [6] in the basis $(\hat{\varepsilon}_0,\hat{\varepsilon}_{0\perp})$, $\rho = \sum_{n=0}^{\infty}B_n\sum_{r=0}^{\infty}|r,n-r\rangle_{(\hat{\varepsilon}_0,\hat{\varepsilon}_{0\perp})(\hat{\varepsilon}_0,\hat{\varepsilon}_{0\perp})}\langle r,n-r|$, gives the number of photons $n_{\hat{\varepsilon}_0} = \sum nB_n = B_1 + 2B_2 + 3B_3 + \cdots$ from Eq. (12), showing independence on polarization vector $\hat{\varepsilon}_0$, which clearly gives the value zero for DOP, $P^{(\text{II})}$ from Eq. (13).

Moreover, correspondence of Eq. (13) can be seen through the vacuum projection operator $\mathcal{V}_{\hat{\varepsilon}_{0\perp}}$ in Weyl representation [27], i.e., $\hat{\mathcal{V}}_{\hat{\varepsilon}_{0\perp}} = (\mathbb{1} - :\hat{\mathrm{N}}_{\hat{\varepsilon}_{0\perp}}: + \frac{1}{2!}:\hat{\mathrm{N}}^2_{\hat{\varepsilon}_{0\perp}}: - \frac{1}{3!}:\hat{\mathrm{N}}^3_{\hat{\varepsilon}_{0\perp}}: + \cdots)$, where $\hat{\mathrm{N}}_{\hat{\varepsilon}_{0\perp}} = \hat{\mathrm{a}}^\dagger_{\hat{\varepsilon}_{0\perp}}\hat{\mathrm{a}}_{\hat{\varepsilon}_{0\perp}}$ is the number operator of virtual photons and the symbol : : denotes the normal ordering of creation and annihilation operators. Evidently, it demonstrates that if occupancy in the $\hat{\varepsilon}_{0\perp}$ mode is extremely feeble, i.e., $n_{\hat{\varepsilon}_{0\perp}} \ll 1$, $\hat{\mathcal{V}}_{\hat{\varepsilon}_{0\perp}} \approx \mathbb{1}$, an identity operator, which, in turn, ensures the equality of photons in two definitions $n_{\hat{\varepsilon}_0} \approx n_{\hat{\varepsilon}_0}$ and, hence, the two definitions coincide. Also, if the beam is very weak, i.e., $\mathrm{Tr}[\rho(\hat{\mathrm{N}}_{\hat{\varepsilon}_0} + \hat{\mathrm{N}}_{\hat{\varepsilon}_{0\perp}})] = n_{\hat{\varepsilon}_0} + n_{\hat{\varepsilon}_{0\perp}} \ll 1$, which is the case of a thermal light, the two definitions agree.

The foregoing discussion demonstrates that if the photon numbers in the orthogonal mode become significant, the two definitions of DOP will substantially be dissimilar. Furthermore, nonlinear dependence of photons $n_{\hat{\varepsilon}_0}$ on the orthogonally polarized photons $n_{\hat{\varepsilon}_{0\perp}}$ leads to an interesting feature that the DOP may not only depend on the nature of the optical field, but also on the average photon numbers (intensity). That is, if we find DOP for the field $\rho = \int d^2\alpha d^2\beta P(\alpha,\beta)|\alpha,\beta\rangle\langle\alpha,\beta|$, and then increase the average photon numbers by a factor of $m$ without affecting the nature of the field, i.e., replacing its density operator by $\rho = \int d^2\alpha d^2\beta P(\alpha,\beta)|\sqrt{m}\alpha, \sqrt{m}\beta\rangle\langle\sqrt{m}\alpha,\sqrt{m}\beta|$ or by $\rho = \int d^2\alpha d^2\beta\, P'(\alpha,\beta)|\alpha,\beta\rangle\langle\alpha,\beta|$, with $P'(\alpha,\beta) = (1/m)\,P(\alpha/\sqrt{m},\,\beta/\sqrt{m})$, the degree of polarization changes. We shall explore this peculiar aspect for two multiphoton optical fields (amplitude-coherent phase-randomized and hidden optical-polarized field) in the following discussion. Such an intensity-dependent feature of various DOPs has been intensively surveyed in Ref. [10].

Finally, we shall test the efficacy of Eq. (13) to assess the polarization states of some typical multiphoton optical fields for which the conventional definition (9) fails as it assigns the status of the unpolarized state. Applying Eq. (12) for evaluation of photon numbers (intensity) in the mode $(\hat{\varepsilon}_0,\vec{\mathrm{k}} = \mathrm{k}\hat{\mathrm{e}}_z)$, one obtains

$$\begin{aligned} n_{\hat{\varepsilon}_0} &= \mathrm{Tr}\left[\hat{\rho}\hat{\mathrm{N}}_{\hat{\varepsilon}_0}\hat{\mathcal{V}}_{\hat{\varepsilon}_{0\perp}}\right] \\ &= \mathrm{Tr}\left[\hat{\rho}\sum_{n=0}^{\infty} n|n,0\rangle_{(\hat{\varepsilon}_0,\hat{\varepsilon}_{0\perp})(\hat{\varepsilon}_0,\hat{\varepsilon}_{0\perp})}\langle n,0|\right] \\ &= \sum_{n=0}^{\infty} n\langle n,0|\hat{\rho}|n,0\rangle_{(\hat{\varepsilon}_0,\hat{\varepsilon}_{0\perp})}, \end{aligned} \tag{14}$$

where $\hat{\mathcal{V}}_{\hat{\varepsilon}_{0\perp}} = \sum_{n=0}^{\infty}|n,0\rangle_{(\hat{\varepsilon}_0,\hat{\varepsilon}_{0\perp})(\hat{\varepsilon}_0,\hat{\varepsilon}_{0\perp})}\langle n,0|$ has been inserted. Insertion of Eq. (10) in Eq. (14), after parametrizing the basis vectors

$$\hat{\varepsilon}_0 = \cos\frac{\chi}{2}e^{-i\Delta/2}\hat{\mathrm{e}}_x + \sin\frac{\chi}{2}e^{-i\Delta/2}\hat{\mathrm{e}}_y,$$
$$\hat{\varepsilon}_{0\perp} = \sin\frac{\chi}{2}e^{-i\Delta/2}\hat{\mathrm{e}}_x - \cos\frac{\chi}{2}e^{-i\Delta/2}\hat{\mathrm{e}}_y,$$

and expressing the complex amplitudes $\alpha_{x,y} = |\alpha_{x,y}|e^{i\varphi_{x,y}}$ in polar form, yields

$$n_{\hat{\varepsilon}_0} = (2\pi)^{-2}\int_{\varphi_x=0}^{2\pi} d\varphi_x \int_{\varphi_y=0}^{2\pi} d\varphi_y |\alpha_{\hat{\varepsilon}_0}|^2 \exp\left(-|\alpha_{\hat{\varepsilon}_{0\perp}}|^2\right), \tag{15}$$

with

$$\begin{aligned} \alpha_{\hat{\varepsilon}_0} &= a\left\{\cos\frac{\chi}{2}\exp\left[\mathrm{i}\left(\varphi_x + \frac{1}{2}\Delta\right)\right]\right. \\ &\quad \left. + \sin\frac{\chi}{2}\exp\left[\mathrm{i}\left(\varphi_y - \frac{1}{2}\Delta\right)\right]\right\}, \\ \alpha_{\hat{\varepsilon}_{0\perp}} &= a\left\{\sin\frac{\chi}{2}\exp\left[\mathrm{i}\left(\varphi_x + \frac{1}{2}\Delta\right)\right]\right. \\ &\quad \left. - \cos\frac{\chi}{2}\exp\left[\mathrm{i}\left(\varphi_y - \frac{1}{2}\Delta\right)\right]\right\}. \end{aligned}$$

Since the integrand in Eq. (15) involves only the difference $\theta = \varphi_x - \varphi_y$ and not $\varphi_x$ and $\varphi_y$ independently, one may simplify Eq. (15) to yield

$$\begin{aligned} n_{\hat{\varepsilon}_0} &= \left(A_0^2\right)^2 e^{-A_0^2}\int_{\theta=0}^{2\pi}\frac{d\theta}{2\pi}\left[1 + \sin\frac{\chi}{2}\cos(\theta+\Delta)\right] \\ &\quad \times \exp\left[A_0^2\sin\chi\cos(\theta+\Delta)\right]. \end{aligned} \tag{16}$$

Use of the standard formula for the modified Bessel function of order $m$ [28], $I_m(z) = \frac{1}{\pi}\int_0^{\pi} d\theta\,\cos(m\theta)\exp[z\cos\theta]$ in the above expressions results as

$$n_{\hat{\varepsilon}_0} = n_0 e^{-n_0}\left[I_0(n_0\sin\chi) + \sin\chi\, I_1(n_0\sin\chi)\right], \tag{17}$$

with intensity $n_0 \equiv A_0^2$. Since the modified Bessel function $I_m(x)$ is a monotonically increasing function of $x$, for given $m$, the maximum and minimum intensities in $(\hat{\varepsilon}_0,\vec{\mathrm{k}} = \mathrm{k}\hat{\mathrm{e}}_z)$ modes will be

$$\left(n_{\hat{\varepsilon}_0}\right)_{\max} = n_0 e^{-n_0}[I_0(n_0) + I_1(n_0)] \quad \text{for} \quad \chi = \frac{\pi}{2}, \tag{18}$$

$$\left(n_{\hat{\varepsilon}_0}\right)_{\min} = n_0 e^{-n_0} \quad \text{for} \quad \chi = 0 \text{ or } \pi, \tag{19}$$

and the value of $\Delta$ does not matter. Substituting Eqs. (18) and (19) into Eq. (13), one gets

$$P^{(\mathrm{II})} = \frac{\left(n_{\hat{\varepsilon}_0}\right)_{\max} - \left(n_{\hat{\varepsilon}_0}\right)_{\min}}{\left(n_{\hat{\varepsilon}_0}\right)_{\max} + \left(n_{\hat{\varepsilon}_0}\right)_{\min}} = \frac{I(n_0) - 1}{I(n_0) + 1}, \tag{20}$$

where $I(n_0) \equiv I_0(n_0) + I_1(n_0)$. The DOP [Eq. (20)] obtained through the second generalization of intensity evidently demonstrates the dependence on average photons $(n_0)$. On limiting cases for average photons, when $n_0 \to 0$ (few photonic regime), $I(n_0) \to 1$ and $P^{(\mathrm{II})} \to 0$, which is palpable because for small $n_0$ only the second-order correlation functions (Stokes theory) prevail, ensuring the unpolarized state. But, for intense multiphoton optical fields, i.e., $n_0 \to \infty, I(n_0) \to \infty$ which signifies the typical nature $P^{(\mathrm{II})} \to 1$ ascertain perfect polarization.

Next, let us apply Eq. (13) for the unimodular hidden optical-polarized field [29]. It is an optical field whose characteristic polarization parameters ratio of real amplitudes and sum of phases are unity and zero, respectively, possessing nonrandom nature in contrast to the usual polarized optical field in which the ratio of real amplitudes and difference of phases are nonrandom characteristic polarization parameters. The state of such an optical field is specified by the density operator

$$\begin{aligned} \hat{\rho} &= \frac{1}{(2\pi A_0)^2}\iint d^2\alpha_x d^2\alpha_y \delta(|\alpha_x| - A_0)\delta(|\alpha_y| - A_0) \\ &\quad \times \left||\alpha_x|e^{i\theta_x}, |\alpha_y|e^{i\theta_y}\right\rangle\left\langle|\alpha_x|e^{i\theta_x}, |\alpha_y|e^{i\theta_y}\right|, \end{aligned} \tag{21}$$

with the condition $\theta_x = -\theta_y = \theta'$. It may be noted that the calculations for various terms in Eq. (13) proceed in similar fashions yielding the same results. Equation (17) would be the same in both cases, which leads to an equivalent expression for DOP [Eq. (20)] and a similar interpretational tenet as that in the earlier case.

*Concluding,* an expression for degree of polarization (DOP) in quantum optics is proposed by inserting a vacuum-mode projection operator in the definition of intensity of optical field. Its correspondence with the usual definition of DOP in the quantum domain, derived by replacing intensity in the classical definition of DOP by quantum mechanical average values of number operators, is sought. The efficacy of the proposed definition is demonstrated for typical multiphoton optical fields where the usual definition fails to predict the true polarization nature. Precisely, the proposed definition of DOP uses a pragmatic approach through modifying the very definition of intensity rather than to rely on the abstract notion of distance of quantum states from the unpolarized state as well as to incomplete correlation functions of Stokes variables (parameters). Therefore, the polarization of the optical quantum field is either characterized by the criterion [Eq. (7)] for perfect polarization or by the definition [Eq. (13)] for the partial (mixed) polarization. Moreover, for multimodal multiphoton optical quantum fields, generalization of DOP is straightforward [30].

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