Space-time descriptions of quantum fields interacting with optical cavities

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The purpose of this paper is to compare two unitary, deterministic approaches to the study of the interaction of quantized fields with atomic systems in optical cavities. In particular it is shown that the "modes of the universe" approach, in which each mode has both an intracavity and an outside part, is formally equivalent, in an appropriate limit, to the better known "input-output" formalism in its unitary (i.e., Hermitian-Hamiltonian based) form; the latter may be called a "quasimode" theory, since it treats the field inside the cavity as a separate degree of freedom. Differences that arise between the two approaches for numerical calculations involving a finite set of discrete modes are also pointed out. The formalism is illustrated with an in-depth discussion of the explicit solution for a single-photon pulse, of arbitrary shape and detuning, incident on a cavity containing a two-level atom.

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I. INTRODUCTION

The experimental demonstration of squeezed light in the 1980s prompted the development of a formalism appropriate to describe the coupling between a mode of the electromagnetic field inside an optical cavity and the modes of the field outside. The formal solution to the problem, known as the "input-output formalism," was presented by Collett and Gardiner in two classic papers [1,2], inspired by earlier work on the coupling of an oscillator to a "bath" of other oscillators (in this case, the cavity field and the outside modes, respectively).

Although it is possible to study this problem entirely in a closed, unitary framework (the basic Hamiltonian is already provided in Ref. [2]), many later studies have combined the basic input-output relations with a stochastic wave-function approach, in which probabilities for photon detection (and related quantities, such as correlation functions) are derived from a non-Hermitian Hamiltonian [3,4]. Nonetheless, the unitary approach (typically in the Heisenberg picture) has been used often over the years in order to answer such questions as the optimal shape for an incident single-photon pulse to drive an atomic system inside a cavity (see, for instance, Refs. [4,5]). Most recently, it has been adopted to provide a space-time description of the interaction of wave packets containing only a few photons with atoms in cavities [6,7].

Meanwhile, an alternative approach to the quantization of a cavity mode coupled to the outside world had been developed by Lang, Scully, and Lamb in the context of the quantum theory of the laser [8]. (See also Refs. [9–11] for other works along these lines.) Dubbed the "modes of the universe" approach, this was applied to the study of squeezed inputs and outputs in a couple of papers [12], where a connection was explicitly made to Gardiner and Collet's formalism, in particular pointing out that the latter's Langevin operator equations could be rederived from the modes of the universe approach. This same point was established in the work of Knoll *et al*. [13], who also showed the equivalence of a number of correlation functions obtained by both methods.

Since both approaches can lead, under the right circumstances, to the same Langevin equations (in the Heisenberg picture), it seems that it must be possible to show that the basic Hamiltonians used in the two approaches are also, in some sense, equivalent, and hence also the corresponding evolution equations in the Schrödinger or interaction pictures. The main goal of this paper is to provide a direct proof of this equivalence, which is otherwise not immediately obvious. This has been motivated by recent work on passive, single-photon optical gates, following a proposal by Koshino, Ishizaka and Nakamura [14]. The approach in Ref. [14] used a variant of what I will henceforth call the IO (for "input-output") Hamiltonian appropriate to the bad cavity limit; however, later, the results of Ref. [14] were extended to the good cavity limit using the MOU (or "modes of the universe") Hamiltonian by Pedrotti and the present author [15], and most recently [16] we have obtained a full analytical solution to the problem, also using the MOU approach. Hence a proof of the equivalence of both treatments seems to be called for.

Besides establishing this equivalence, this paper explores some of the differences, subtleties, and practical advantages or disadvantages of the two approaches (for example, in numerical calculations using only a finite number of modes, the two methods are *not* equivalent), and illustrates their usefulness for a space-time description of cavity quantum electrodynamics processes; in particular, a full analytical solution is provided to the problem of a single-photon pulse, of arbitrary shape, interacting with a cavity containing a single two-level atom (the solution is, of course, equivalent to the one obtained previously by Koshino and Ishihara in Ref. [7]; however, the derivation here is different, and explicit results for a number of interesting physical properties of the system are also presented and discussed).

II. THE "MODES OF THE UNIVERSE" APPROACH

A. The original formulation

In the "modes of the universe" approach presented in Ref. [12], one deals with the case of a one-sided cavity of length l (with a perfectly reflecting mirror and a mirror of amplitude transmission coefficient $\tilde{t} \ll 1$) by embedding it in a larger cavity (the "universe") of length $L + l$ (see Fig. 1).

By convention the outside cavity is taken to be bounded by a perfect mirror at $z = -L$, whereas the small cavity extends from $z = 0$ to $z = l$. One then calculates the electromagnetic

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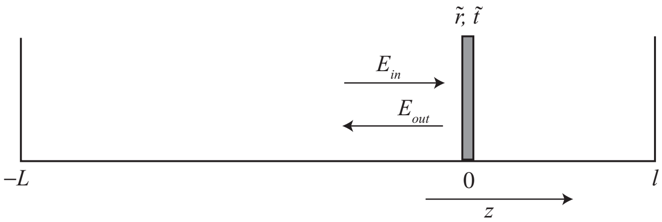

FIG. 1. The setup for the "modes of the universe" calculation.

field mode functions for this (one-dimensional) arrangement, for the whole space $-L \leqslant z \leqslant l$, which can be written as (Eq. (2.2) of Ref. [12])

$$U_k(z) = \begin{cases} \xi_k \sin k(z+L) & \text{for} \quad z < 0, \\ M_k \sin k(z-l) & \text{for} \quad z > 0. \end{cases} \tag{1}$$

The ξ_k are taken to be alternately $+1$ and -1, and the M_k then are approximately given by

$$M_k = \frac{(c\kappa/l)^{1/2}}{[(\Omega_k - \Omega_c)^2 + \kappa^2]^{1/2}}. \tag{2}$$

Here, as in Ref. [15], I have changed the notation of Ref. [12] slightly, so that κ, rather than Γ, is the amplitude decay rate for the small cavity; $\kappa = c\tilde{t}^2/4l$. The resonant frequency of the cavity is Ω_c; the frequency Ω_k of an individual mode is $\Omega_k = ck$ (later on, I will use the notation $\omega_k = \Omega_k - \Omega_c$). For finite L, the allowed values of the wave number k form a discrete set which can be obtained by solving a characteristic equation. If $L \gg l$, then one has $k_n \simeq n\pi/L$ (with n a positive integer) as a lowest-order approximation, but in general it is not correct to simply assume that $kL = n\pi$, as discussed below.

As shown in Ref. [12] [see, in particular, Eqs. (A4) and (A5)], Eq. (2) is, in fact, already an approximation, which assumes that the mirror transmission $\tilde{t}$ is small enough for the cavity resonances to be well separated, in which case each resonance can be well approximated by a Lorentzian. *The equivalence that is the object of this paper only holds in this limit*; the difference between IO and MOU approaches when the approximation (2) does not hold was considered by Barnett and Radmore [17].

The total, quantized field is a superposition of the form $\sum_k \mathcal{E}_k U_k(z) a_k e^{-i\Omega_k t}$, with $\mathcal{E}_k = [\hbar\Omega_k/\epsilon_0 A(L+l)]^{1/2} \simeq (\hbar\Omega_k/\epsilon_0 AL)^{1/2}$ the "electric field per photon" (A is an appropriate cross-sectional area). The right- and left-propagating parts of this field in the region $-L \leqslant z < 0$ are identified with the incoming and outgoing fields:

$$E_{\rm in}^{(+)}(z,t) = \frac{1}{2i}\sum_k \left(\frac{\hbar\Omega_k}{\epsilon_0 AL}\right)^{1/2} \xi_k a_k e^{ik(z+L)-i\Omega_k t}, \tag{3}$$

$$E_{\rm out}^{(+)}(z,t) = -\frac{1}{2i}\sum_k \left(\frac{\hbar\Omega_k}{\epsilon_0 AL}\right)^{1/2} \xi_k a_k e^{-ik(z+L)-i\Omega_k t}, \tag{4}$$

whereas for the cavity field we have

$$\begin{aligned} E_{\rm cav}^{(+)}(z,t) &= \sum_k \left(\frac{\hbar\Omega_k}{\epsilon_0 AL}\right)^{1/2} M_k \sin[k(z-l)] a_k e^{-i\Omega_k t} \\ &\simeq \left(\frac{\hbar\Omega_c}{\epsilon_0 Al}\right)^{1/2} \sin[k_c(z-l)]\, a\, e^{-i\Omega_c t} \end{aligned} \tag{5}$$

with the cavity "single quasimode" operator

$$a(t) = \sqrt{\frac{l}{L}} \sum_k M_k a_k e^{-i(\Omega_k - \Omega_c)t} \tag{6}$$

(the above two equations correct some mistakes in Ref. [15]). The replacement, in the sine function, of the variable k by a single $k_c = \Omega_c/c$ is justified as follows: the range of k that characterizes a single cavity "quasimode" is $\Delta k \sim \kappa/c = \tilde{t}^2/4l$. For this range, $\Delta k\, l \sim \tilde{t}^2/4 \ll 1$ for a good mirror.

As defined in Eq. (6), the operator $a(t)$ and its Hermitian conjugate satisfy the commutation relations

$$\begin{aligned} [a(t), a^\dagger(t')] &= \sum_k \frac{c\kappa/L}{(ck - \Omega_c)^2 + \kappa^2} e^{-i(ck-\Omega_c)(t-t')} \\ &\simeq e^{-\kappa|t-t'|} \end{aligned} \tag{7}$$

(here and elsewhere sums over k are converted into integrals over frequency $\omega = ck$ via the replacement $\sum_k \to (L/\pi c)\int d\omega$; the integral over ω can be extended formally to $-\infty$ provided $\Omega_c \gg \kappa$).

B. A more convenient formulation

A problem with the above formalism is that the presence of L in the exponents in Eqs. (3) and (4) makes passage to the $L \to \infty$ limit nontrivial. As shown in Ref. [15], one cannot just replace e^{2ikL} by 1 in this limit. Rather, one has

$$e^{-2ikL} \simeq \frac{\kappa + i(\Omega_k - \Omega_c)}{\kappa - i(\Omega_k - \Omega_c)}. \tag{8}$$

Noting that $e^{ikL} \simeq e^{in\pi}$ to lowest order, we can write

$$\xi_k e^{ikL} \simeq e^{-i\phi_k}, \quad \phi_k = \tan^{-1}\left[\frac{\Omega_k - \Omega_c}{\kappa}\right] \tag{9}$$

and we can then absorb this phase in the definition of the operators a_k. As a result, Eqs. (3)–(5) are replaced by

$$E_{\rm in}^{(+)}(z,t) = \frac{1}{2i}\sum_k \left(\frac{\hbar\Omega_k}{\epsilon_0 AL}\right)^{1/2} a_k e^{ikz-i\Omega_k t}, \tag{10}$$

$$\begin{aligned} E_{\rm out}^{(+)}(z,t) = -\frac{1}{2i}\sum_k \left(\frac{\hbar\Omega_k}{\epsilon_0 AL}\right)^{1/2} \frac{\kappa + i(\Omega_k - \Omega_c)}{\kappa - i(\Omega_k - \Omega_c)} \\ \times a_k\, e^{-ikz-i\Omega_k t}, \end{aligned} \tag{11}$$

and

$$\begin{aligned} E_{\rm cav}^{(+)}(z,t) = \left(\frac{\hbar\Omega_c}{\epsilon_0 Al}\right)^{1/2} \sum_k \frac{\sqrt{c\kappa/L}}{\kappa - i(\Omega_k - \Omega_c)} a_k e^{-i\Omega_k t} \\ \times \sin[k_c(z-l)]. \end{aligned} \tag{12}$$

As we have shown in Ref. [16], in this form, the "modes of the universe" formalism can be simply derived by considering the "scattering modes of the cavity," in the following way. Consider an incoming, monochromatic field of amplitude $e^{ikz-i\Omega_k t}$ [as in Eq. (10)], incident on the small cavity from the left; standard methods, familiar from classical optics (boundary conditions, or multiple reflections and transmissions) can then be used to calculate the total intracavity and reflected field amplitudes. This leads directly, under the assumption of small transmission $\tilde{t}$, to the amplitude coefficients shown in

Eqs. (11) and (12); then, as in the previous subsection, one just considers all these fields (incident, intracavity, and transmitted) as parts of a single "mode of the universe" with creation and annihilation operators $a_k^\dagger$ and a_k. (As explained in Ref. [16], since this approach makes use of traveling waves instead of standing waves, there is a formal, but ultimately trivial, difference in the mode spacing to be used for discrete-mode calculations.)

The interaction of this quantized field with a two-level atom in the cavity would be described by a Hamiltonian like

$$\begin{aligned} H_{MOU} &= \hbar g (a\sigma^\dagger e^{-i(\Omega_c-\omega_a)t} + a^\dagger \sigma e^{i(\Omega_c-\omega_a)t}) \\ &= \hbar g \sum_k \frac{\sqrt{c\kappa/L}}{\kappa - i(\Omega_k-\Omega_c)} a_k \sigma^\dagger e^{-i(\Omega_k-\omega_a)t} + \text{H.c.} \end{aligned} \tag{13}$$

where σ and $\sigma^\dagger$ are atomic lowering and raising operators, and ω_a is the atomic resonant frequency. The quantities appearing in front of the summation sign in Eq. (12) are incorporated in the coupling constant g.

III. THE INPUT-OUTPUT APPROACH

A. The general case

In this section, I will consider the input-output Hamiltonian approach, as adapted by Koshino and coworkers [7] to the space-time description of fields interacting with atoms in cavities. For our geometry, their Hamiltonian can be written as

$$\begin{aligned} H_{IO} &= \hbar(\omega_a - \Omega_c)\sigma^\dagger\sigma + \int \hbar\omega_k b_k^\dagger b_k dk \\ &\quad + \hbar\sqrt{\frac{\kappa c}{\pi}} \int (a^\dagger b_k + b_k^\dagger a)\, dk + \hbar g(a\sigma^\dagger + \sigma a^\dagger). \end{aligned} \tag{14}$$

Here, the field is split into a continuum of outside modes b_k, and a cavity mode a, coupled by the third term in the Hamiltonian above. The frequencies ω_k are defined relative to the cavity frequency Ω_c, that is, $\omega_k = ck - \Omega_c$, and so is the zero of atomic energy. Regarding the latter, for consistency with previous work [15,16], I shall henceforth use $\delta_a \equiv -(\omega_a - \Omega_c)$ (note the sign!).

In this formalism, the input and output fields are calculated from the operators

$$b_{\text{in}}(z) = \frac{1}{\sqrt{2\pi}} \int b_k e^{i(k-k_c)z} dk \tag{15}$$

and

$$b_{\text{out}}(z) = \frac{1}{\sqrt{2\pi}} \int b_k e^{-i(k-k_c)z} dk = b_{\text{in}}(-z). \tag{16}$$

As in the previous section, it is understood that the integral runs only over positive values of k, centered around k_c. The notion of getting the output field operator from the input one by extending it from the $z < 0$ to the $z > 0$ region appears to have originated with Hofmann and Mahler [18].

As stated earlier, the equivalence between Eqs. (13) and (14) is not immediately apparent. Apart from some trivial differences (one is in an interaction picture and for discrete modes, the other in the Schrödinger picture and for a continuum), the Hamiltonian (14) contains what appears to be one more degree of freedom than Eq. (13), namely, the operators a and $a^\dagger$ for the intracavity field. The strategy will be to transform Eq. (14) into an interaction picture where a and $a^\dagger$ actually disappear.

For clarity, this will be done in two steps. We start by transforming the Hamiltonian (14) to a "standard" interaction picture, in which the first two terms disappear:

$$\begin{aligned} H'_{IO} &= \hbar\sqrt{\frac{\kappa c}{\pi}} \int (a^\dagger b_k e^{-i\omega_k t} + b_k^\dagger a e^{i\omega_k t})\, dk \\ &\quad + \hbar g(a\sigma^\dagger e^{-i\delta_a t} + \sigma a^\dagger e^{i\delta_a t}). \end{aligned} \tag{17}$$

The transformed input and output operators are then

$$b'_{\text{in}}(z,t) = \frac{1}{\sqrt{2\pi}} \int b_k e^{-i\omega_k(t-z/c)} dk \tag{18}$$

and

$$b'_{\text{out}}(z,t) = \frac{1}{\sqrt{2\pi}} \int b_k e^{-i\omega_k(t+z/c)} dk. \tag{19}$$

We now want to go to a second interaction picture in which the first term of H'_{IO} will disappear. If we call this term $A(t)$ we see that the transformed state vectors will obey the Schrödinger equation with a transformed Hamiltonian

$$\begin{aligned} H''_{IO} &= U(t,t_0)^\dagger H'_{IO} U(t,t_0) - A(t), \\ U(t,t_0) &= \mathcal{T} \exp\left[-(i/\hbar)\int_{t_0}^t A(t')dt'\right]. \end{aligned} \tag{20}$$

Here the lower limit of integration t_0 is an arbitrary time that we will soon formally extend to $-\infty$, and the $\mathcal{T}$ stands for time ordering.

We need to calculate the effect of the transformation on the operators a and b_k. Let $a(t)$ and $b_k(t)$ denote the transformed operators, whereas the absence of an explicit time argument will denote the original operators (or the transformed operators at the time t_0):

$$\begin{aligned} \frac{d}{dt} a(t) &= \frac{i}{\hbar}[A(t)a(t) - a(t)A(t)] \\ &= -i\sqrt{\frac{\kappa c}{\pi}} \int b_k(t) e^{-i\omega_k t} dk \end{aligned} \tag{21}$$

and

$$\frac{d}{dt} b_k(t) = \frac{i}{\hbar}[A(t)b_k(t) - b_k(t)A(t)] = -i\sqrt{\frac{\kappa c}{\pi}} e^{i\omega_k t} a(t), \tag{22}$$

assuming continuum commutation relations $[b_k, b_{k'}^\dagger] = \delta(k - k')$. The system (21) and (22) can be integrated as follows. First integrate Eq. (22) formally to get

$$b_k(t) = b_k - i\sqrt{\frac{\kappa c}{\pi}} \int_{t_0}^t e^{i\omega_k t'} a(t')\, dt', \tag{23}$$

then substitute in Eq. (21):

$$\begin{aligned} \frac{d}{dt} a(t) &= -i\sqrt{\frac{\kappa c}{\pi}} \int b_k e^{-i\omega_k t} dk \\ &\quad - \frac{\kappa c}{\pi} \int_{t_0}^t dt' \int dk\, e^{-i\omega_k(t-t')} a(t'). \end{aligned} \tag{24}$$

The integral over k in the last term can be approximated by $(2\pi/c)\delta(t-t')$ if Ω_c is sufficiently large, since in that case the lower limit of integration is effectively $-\infty$ (recall $\omega_k = ck - \Omega_c$, and $k>0$). Basically this amounts to requiring that Ω_c be large compared to all the other relevant time scales in the problem. The integral over t' then picks up half of the δ function, since the upper limit of integration is t. We end up with

$$\frac{d}{dt}a(t) = -\kappa a(t) - i\sqrt{\frac{\kappa c}{\pi}}\int b_k e^{-i\omega_k t}dk, \tag{25}$$

which integrates immediately to

$$a(t) = -i\int \frac{\sqrt{\kappa c/\pi}}{\kappa - i\omega_k} b_k\, e^{-i\omega_k t}dk \tag{26}$$

in the limit $t_0 \to -\infty$. This is directly comparable to the expressions that we found in the previous section. In particular, note that this $a(t)$ satisfies the same commutation relation (7) as its discrete-mode counterpart.

With this substitution, the Hamiltonian H''_{IO} becomes

$$H''_{IO} = -i\hbar g\int \frac{\sqrt{\kappa c/\pi}}{\kappa - i\omega_k} b_k\sigma^\dagger\, e^{-i(\omega_k+\delta_a)t}dk + \text{H.c.}, \tag{27}$$

which is directly comparable to Eq. (13), since $\omega_k + \delta_a = \Omega_k - \omega_a$. The phase factor represented by the $-i$ term can be absorbed in the definition of the atomic raising and lowering operators.

The only thing left to check is the transformation of the input and output operators (18) and (19). One must simply replace the original b_k by the transformed $b_k(t)$ given by Eq. (23), with the explicit expression (26) for $a(t)$. The result is

$$b''_{\text{in}}(z,t) = \frac{1}{\sqrt{2\pi}}\left[\int b_k e^{-i\omega_k(t-z/c)}dk - \frac{\kappa c}{\pi}\int dk\int_{t_0}^{t} dt'\int dk' \frac{e^{-i\omega_k(t-z/c)}e^{i(\omega_k-\omega_{k'})t'}}{\kappa - i\omega_{k'}} b_{k'}\right]. \tag{28}$$

With the same restrictions as above, the integral over k in the second term can be approximated by an integral over ω_k/c from minus infinity to infinity, which yields $(2\pi/c)\delta(t-z/c-t')$. For $z<0$ this always gives zero, since the point $t' = t - z/c > t$ is outside the interval of integration for t'. Hence

$$b''_{\text{in}}(z,t) = \frac{1}{\sqrt{2\pi}}\int b_k e^{-i\omega_k(t-z/c)}dk. \tag{29}$$

On the other hand, for $b''_{\text{out}}(z,t)$ one has the same formal expression with the opposite sign of z, which yields $(2\pi/c)\delta(t+z/c-t')$, and now the point $t' = t+z/c < t$ is inside the interval of integration for all $z<0$. With z understood to be negative, we get

$$\begin{aligned} b''_{\text{out}}(z,t) &= \frac{1}{\sqrt{2\pi}}\int b_k e^{-i\omega_k(t+z/c)}\left[1 - \frac{2\kappa}{\kappa - i\omega_k}\right]dk \\ &= -\frac{1}{\sqrt{2\pi}}\int \frac{\kappa + i\omega_k}{\kappa - i\omega_k} b_k e^{-i\omega_k(t+z/c)}dk, \end{aligned} \tag{30}$$

which is substantially the same as Eq. (11) in the previous section, only missing an overall factor of $e^{-i\Omega_c(t+z/c)}$, which can always be added by hand if necessary. Equation (29) for the input field also agrees with Eq. (10), again, up to an overall factor of $e^{-i\Omega_c(t-z/c)}$.

As noted above, the preferred approach of Hofmann, Koshino, and coworkers seems to be to derive both input and output fields from a single expression like the right-hand side of Eq. (15) above, with $z<0$ giving the input and $z>0$ the output field. Clearly, after going to the second interaction picture, one then ends up with an expression like the right-hand side of Eq. (28), which for $z<0$ yields Eq. (29), and for $z>0$ yields Eq. (30) with $t-z/c$ in place of $t+z/c$.

We see, therefore, that the input-output based approach, Eq. (14), with a separate degree of freedom for the cavity quasimode, is mathematically equivalent to the "modes of the universe" approach [Eq. (13)], up to a global unitary transformation, in the limit of a very narrow cavity resonance. The unitary transformation does not affect the initial state vector, since it is equal to 1 at $t=t_0$ (although we have formally let $t_0\to-\infty$, in practice it is sufficient to choose t_0 well before any pulse-cavity interaction takes place). Hence both methods should yield the same results for all physically relevant quantities.

B. The very bad cavity limit

A limit that is often of interest is when the cavity bandwidth κ is much larger than any other frequencies involved in the interaction: this includes the pulse bandwidth, and the coupling rate g. In that case, and on resonance ($|\Omega_k - \Omega_c| \ll \kappa$ for all k), one can just neglect all the $\Omega_k - \Omega_c$ terms in Eqs. (10)–(13) to get

$$E^{(+)}_{\text{in}}(z,t) = \frac{1}{2i}\sum_k \left(\frac{\hbar\Omega_k}{\epsilon_0 AL}\right)^{1/2} a_k e^{ikz - i\Omega_k t}, \tag{31}$$

$$E^{(+)}_{\text{out}}(z,t) = -\frac{1}{2i}\sum_k \left(\frac{\hbar\Omega_k}{\epsilon_0 AL}\right)^{1/2} a_k\, e^{-ikz - i\Omega_k t}, \tag{32}$$

and an interaction Hamiltonian

$$H = \hbar g\sqrt{\frac{c}{\kappa L}}\sum_k a_k\sigma^\dagger e^{-i(\omega_k+\delta_a)t} + \text{H.c.} \tag{33}$$

This is clearly the discrete-mode, interaction-picture version of the Hamiltonian

$$\begin{aligned} H &= -\hbar\delta_a\sigma^\dagger\sigma + \int \hbar\omega_k b_k^\dagger b_k dk \\ &\quad + \hbar g\sqrt{\frac{c}{\kappa\pi}}\int (\sigma^\dagger b_k + b_k^\dagger\sigma)\, dk, \end{aligned} \tag{34}$$

which looks a lot like Eq. (14), only the last term is gone and the "cavity mode" a has been replaced, in the third term, by $g^2\sigma/\kappa$ (a replacement often justified by "adiabatic elimination" methods [19]). This Hamiltonian (34) is sometimes referred to as the "one-dimensional atom limit" (see, e.g., Ref. [6]). Instead of depending separately on g and κ, it involves only the ratio g^2/κ; one often writes $\Gamma = 2g^2/\kappa$. Equation (34) is essentially the Hamiltonian used in the study of cavity-mediated single-photon quantum logical gates in Ref. [14]. Space-time descriptions of quantum fields interacting with atom-cavity systems based on this Hamiltonian can also be found in Ref. [6] and references therein. Clearly, no special effort is necessary to derive this Hamiltonian from the MOU formalism; it suffices to neglect ω_k versus κ in all the resonant denominators.

IV. PRACTICAL CONSIDERATIONS

A. Numerical calculations with a finite, discrete set of modes

While analytical results based on the Hamiltonians (13) or (27) are possible in some cases (namely, for one- and two-photon incident pulses: see, e.g., Refs. [6,7,16]), sometimes one may have to resort to numerical calculations, which necessarily must be performed with a finite, discrete set of modes. In this case, the equivalence between the two Hamiltonians established in Sec. III A (which explicitly involved the continuum limit of an infinite number of modes) does *not* exactly hold.

This point is illustrated in Fig. 2, which refers to the following situation: a Gaussian pulse containing exactly one photon is incident from the left on a cavity containing a two-level atom, initially in the ground state. The atom-cavity parameters are $g = 5/T$ and $\kappa = 10/T$, where T is the pulse duration [see Eq. (35) below]. The initial state of the field is described, in terms of the MOU operators a_k of Sec. II, by the sum $\sum_k c_k(0)a_k^\dagger|\text{vac}\rangle$, where $|\text{vac}\rangle$ is the vacuum state and the coefficients c_k are given by

$$c_k(0) = \left(\frac{\pi}{2}\right)^{1/4}\sqrt{\frac{cT}{L}}\, e^{-ikz_0}\, e^{-(cT(k-k_0)/2)^2} \tag{35}$$

with $k = n\pi/L$, and $ck_0 = n_0 c\pi/L = \Omega_c$, the cavity resonant frequency; in practice, we just set $k - k_0 = n\pi/L$ with $n = -(N-1)/2,\ldots,0,\ldots,(N-1)/2$, where N is the total number of modes (an overall phase factor $e^{ik_0z_0}$ may be ignored). For sufficiently large N, the expression (35) is normalized to a very good approximation. The packet is initially centered at z_0, which we take to be a negative number much larger than the pulse width cT, so the interaction between the pulse and cavity is initially negligible.

As time passes, the state $|\Psi(t)\rangle$ of the overall (field + atom/cavity) system changes. The probability to detect a photon at a space-time point goes as $I(t,z) \sim \langle\Psi(t)|E^{(-)}(t,z)E^{(+)}(t,z)|\Psi(t)\rangle$. This can be specialized to the probability to detect a photon in either the input or the output field, by using either the operator (10) or (11). To be precise, we can define

$$\begin{aligned} I_{\text{in}}(t,z) &= \frac{1}{2L}\left\| \sum_k a_k(t)e^{-i(\omega_k t - kz)}|\Psi(t)\rangle \right\|^2, \\ I_{\text{out}}(t,z) &= \frac{1}{2L}\left\| \sum_k \frac{\kappa + i\omega_k}{\kappa - i\omega_k}\, a_k(t)e^{-i(\omega_k t + kz)}|\Psi(t)\rangle \right\|^2. \end{aligned} \tag{36}$$

The normalization in Eqs. (36) is such that, for an initial one-photon state, $I_{\text{in}}(0,z)$ integrates to unity over the region $-L \leqslant z \leqslant 0$.

Figure 2(a) shows the function $I_{\text{out}}(t,z)$, which may be thought of as the outgoing pulse, at two different instants, $t = 4T$ and $t = 6T$; the incoming pulse at $t = 0$ was centered at $z_0 = -\pi cT$. The corresponding calculation in the IO formalism, that is, with an appropriately discretized version of the Hamiltonian (14), is shown in Fig. 2(b). The exact result from the infinite, continuous mode approach (which can be obtained easily, as will be shown in the next section) is also plotted in both figures for reference.

It is apparent that for any finite number of modes there are differences in detail between the two approaches and between each and the exact result. Both numerical approximations show wiggles near $z = 0$ in the still-interacting ($t = 4T$) pulse; these are unphysical and can be discounted, as will be shown in the subsection immediately following. The postinteraction ($t = 6T$) pulse is reproduced more accurately, but the IO formalism for small N shows it shifted from its true position by a small distance.

In general, the MOU approach is more economical (since it has one degree of freedom fewer than the IO model) and, for the same number of modes, more accurate for numerical calculations, perhaps because it has the interaction with the empty cavity "built in" from the start. Mathematically, the resonant denominators of Eq. (13) may be helpful in reducing the influence of large-frequency modes, which in the IO approach rely solely on interference for their cancellation.

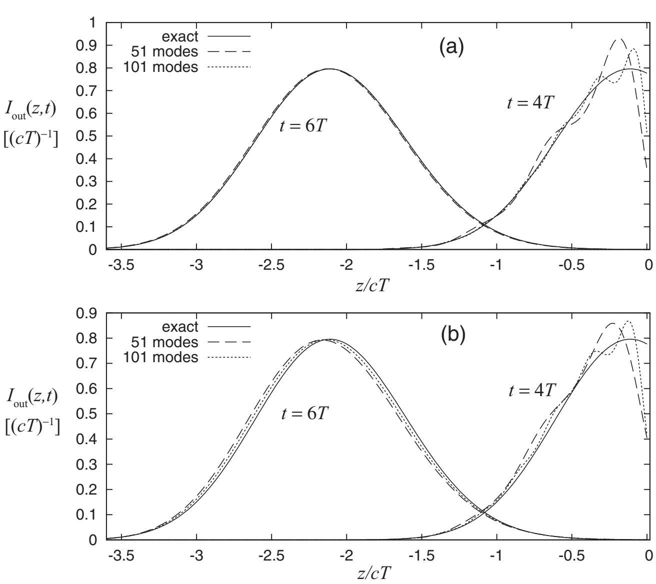

FIG. 2. The outgoing pulse, as described by the photon probability distribution function [in units of $(cT)^{-1}$], for times $t = 4T$ and $t = 6T$, calculated by (a) the MOU approach and (b) the IO approach. The number of modes used in the numerical integration of the Schrödinger equation is indicated; the exact result is also shown in both graphs for reference.

B. Space and time dependance of input and output fields in the presence of interaction

It was shown at the end of Sec. III A that the input and output fields in the second interaction picture, b''_{in} and b''_{out}, depend on t and z only through the combinations $t - z/c$ and $t + z/c$, respectively. This is what one expects for right- and left-traveling fields outside the interaction region. It is, however, not immediately clear that the same result still holds when dealing with expressions such as Eq. (36), when the state vector $|\Psi(t)\rangle$ changes as a result of the interaction with the atomic field inside the cavity. In fact, the numerical results just presented show that this is not the case for a finite-mode calculation, since the interacting pulses calculated in Fig. 2 are not just shifted, truncated copies of the postinteraction ones.

Nonetheless, in the physical limit of an infinite continuum of modes, the expectation values involving the input (output) field do depend only on $t - z/c$ ($t + z/c$), as shown by the lines marked "exact" in Fig. 2. The simplest way to prove this result in general is to go fully to the Heisenberg picture appropriate to the IO Hamiltonian [Eq. (27), for a continuous set of modes], in which, of course, the state vector no longer changes. In this picture we have, for the input field operator,

$$\begin{aligned} b_{\rm in}^{(H)}(z,t) &= \mathcal{T}\exp\left[\frac{i}{\hbar}\int_{t_0}^{t} H''_{IO}(t')dt'\right]\frac{1}{\sqrt{2\pi}}\int b_k e^{-i\omega_k(t-z/c)}dk \\ &\quad\times \mathcal{T}\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} H''_{IO}(t')dt'\right], \end{aligned} \tag{37}$$

using Eq. (29) as the starting point. To show that $b_{\rm in}^{(H)}(z,t)$ depends only on $t - z/c$ it suffices to show that $(\partial_t + c\,\partial_z)b_{\rm in}^{(H)} = 0$. Applying this to Eq. (37) we get

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial z}\right)b_{\rm in}^{(H)} = \frac{i}{\hbar\sqrt{2\pi}}\int [H''_{IO},b_k]^{(H)}e^{-i\omega_k(t-z/c)}dk. \tag{38}$$

Again, it suffices to show that the right-hand side of Eq. (38) vanishes in the original picture, in which H''_{IO} is given by Eq. (27) and the b_k are time-independent, since Eq. (38) is just related to that picture by a unitary transformation. We get

$$[H''_{IO}(t),b_k] = -i\hbar g\frac{\sqrt{\kappa c/\pi}}{\kappa + i\omega_k}\sigma e^{i(\omega_k-\omega_a)t}. \tag{39}$$

Substituting in Eq. (38), we end up with something proportional to $\int e^{i\omega z/c}/(\kappa + i\omega)\,d\omega$, which vanishes for $z < 0$, which proves the desired result. Similarly, for the outgoing field, which is given by Eq. (30) in the second interaction picture, the factor $(\kappa + i\omega_k)/(\kappa - i\omega_k)$ ensures that $(\partial_t - c\,\partial_z)b_{\rm out}^{(H)}$ ends up proportional to $\int e^{-i\omega z/c}/(\kappa - i\omega)\,d\omega$, which also vanishes for $z < 0$.

We conclude that, even in the presence of interaction, expectation values of quantities involving the input field will depend only on $t - z/c$, and expectation values of quantities involving the output field will depend only on $t + z/c$; that is, the parts of the pulse in the $z < 0$ region do not change either as they go into the cavity or after leaving it. This means, as already anticipated, that the oscillations seen in the interacting ($t = 4T$) pulses in Fig. 2 are unphysical. On the other hand, note that the analytical result just proved required the formal addition of an infinite, continuous set of modes, so naturally one should not expect to see it hold for calculations involving a finite, discrete set.

This result is very useful because it means, in essence, that the asymptotic, postinteraction wave function contains all the relevant physics. This final state, in turn, can be calculated analytically with relative ease for a number of cases, as illustrated in the next section.

V. EXAMPLE: SINGLE-PHOTON, TWO-LEVEL ATOM

To illustrate the formalism, in this section I will solve without (any further) approximations the problem of a single-photon pulse incident on a cavity that contains a single two-level atom. This has been done before in Ref. [7], but it may be interesting, nonetheless, to present a unified view of the various regimes that are possible for this system. As always, spontaneous emission from the excited state (out of the sides of the cavity) will be neglected, although it can actually be included in the model with some additional effort [7].

A. Output pulse for atom initially in ground state

With only one excitation in the system, the total state can be written at any time as

$$|\Psi(t)\rangle = \left[\int C_g(k,t)b_k^\dagger\,dk\,|g\rangle + C_e(t)|e\rangle\right]|{\rm vac}\rangle. \tag{40}$$

Assuming that the atom starts in the ground state, the coefficients $C_g(k,t_0)$ represent the initial pulse (or "wave function of the photon") in momentum space. Since there is a one-to-one correspondence between the wave numbers k and the frequencies ω_k, in what follows it will be useful sometimes to work with coefficients $C_g(\omega,t)$ which depend on ω rather than k. The precise correspondence, which ensures that $\int |C(\omega,t)|^2 d\omega = \int |C(k,t)|^2 dk$, is $C_g(\omega,t) = C_g(k,t)/\sqrt{c}$, where c is the speed of light and $\omega = ck - \Omega_c$.

With the Hamiltonian (27) the equations of motion for the coefficients of the state vector are

$$\dot{C}_g(k,t) = -ig\frac{\sqrt{c\kappa/\pi}}{\kappa + i\omega_k}\,e^{i(\omega_k+\delta_a)t}C_e(t), \tag{41a}$$

$$\dot{C}_e(t) = -ig\int\frac{\sqrt{c\kappa/\pi}}{\kappa - i\omega_k}\,C_g(k,t)e^{-i(\omega_k+\delta_a)t}\,dk. \tag{41b}$$

We can integrate formally Eq. (41a), substitute the result in Eq. (41b), and carry out the integral over k with $\omega_k = ck - \Omega_c$. The result is the integral equation

$$\begin{aligned} \dot{C}_e &= -g^2\int_{t_0}^{t} e^{-(\kappa+i\delta_a)(t-t')}C_e(t')dt' \\ &\quad - ig\int dk\frac{\sqrt{c\kappa/\pi}}{\kappa - i\omega_k}\,C_g(k,t_0)e^{-i(\omega_k+\delta_a)t}. \end{aligned} \tag{42}$$

Introduce now the Fourier transform of $C_e(t)$

$$C_e(t) = \frac{1}{2\pi}\int \tilde{C}_e(\omega)e^{-i\omega t}d\omega \tag{43}$$

and substitute Eq. (43) in Eq. (42), formally letting the lower limit of the time integral go to minus infinity. One can then solve for $\tilde{C}_e(\omega)$, with the result

$$\tilde{C}_e(\omega) = -2ig\sqrt{\pi\kappa}\frac{C_g(\omega-\delta_a,-\infty)}{g^2 - i\omega[\kappa - i(\omega-\delta_a)]}. \tag{44}$$

Also, the time integral of the right-hand side of Eq. (41a), from (formally) minus infinity to infinity, is just proportional to $\tilde{C}_e(\omega+\delta_a)$. Putting this together with Eq. (44) immediately yields the state coefficients after the interaction is over:

$$C_g(\omega,\infty) = C_g(\omega,-\infty) - \frac{2g^2\kappa}{\kappa+i\omega}\,\frac{C_g(\omega,-\infty)}{g^2-i(\omega+\delta_a)(\kappa-i\omega)} = \frac{\omega+i\kappa}{\omega-i\kappa}\,\frac{(\omega-i\kappa)(\omega+\delta_a)-g^2}{(\omega+i\kappa)(\omega+\delta_a)-g^2}C_g(\omega,-\infty). \tag{45}$$

Under the assumption that the initial state has the photon entirely outside the cavity, the spectrum of the incoming pulse, $f_{\rm in}(\omega)$, is simply proportional to $C_g(\omega,-\infty)$. On the other hand, from Eq. (36) [see also Eq. (11)] we see that the spectrum of the outgoing pulse is given by the product of the $C_g(\omega,\infty)$ coefficients by a factor $(\kappa+i\omega_k)/(\kappa-i\omega_k)$. We therefore have

$$f_{\rm out}(\omega) = -\frac{(\omega-i\kappa)(\omega+\delta_a)-g^2}{(\omega+i\kappa)(\omega+\delta_a)-g^2}\,f_{\rm in}(\omega), \tag{46}$$

which shows that the incoming spectrum simply gets multiplied by a (frequency-dependent) phase shift, $-e^{-2i\phi(\omega)}$, with

$$\phi = \tan^{-1}\left[\frac{\kappa}{\omega-g^2/(\omega+\delta_a)}\right]. \tag{47}$$

If the central frequency of the pulse is Ω_0, one may define the field-cavity detuning Δ as $\Delta=\Omega_0-\Omega_c$ and write ω in the above expression as $\omega=\Delta+\omega'$, where ω' is a new variable that captures the pulse's shape. The pulse's bandwidth may be written as $\Delta\omega=\Delta\omega'$.

One may now consider several limits of interest. In the adiabatic limit, the pulse is very long and $\Delta\omega \ll g,\kappa$, so the pulse is just multiplied by the constant phase factor $e^{-2i\phi_0}$, with ϕ_0 given by Eq. (47) with $\omega=\Delta$.

Another case is when $\Delta\omega\ll\kappa$, but g^2/κ may be smaller than, or of the same order as, $\Delta\omega$. This is basically the bad cavity regime, since it requires $\kappa\gg g$. The general expression in this limit is not much simpler than Eq. (46), but when the detunings Δ and δ_a vanish one has approximately

$$f_{\rm out}(\omega) \simeq \frac{i\omega+\Gamma/2}{i\omega-\Gamma/2}\,f_{\rm in}(\omega) \quad \text{(bad cavity, } \Delta=\delta_a=0), \tag{48}$$

where the parameter $\Gamma=2g^2/\kappa$ is often used to characterize the bad cavity limit; see Sec. III B above. This limit looks essentially (up to an overall minus sign) like an empty cavity [compare with Eq. (46) for $g=0$], only with a decay rate $\Gamma/2$ instead of κ.

The spatial or temporal profile of the outgoing pulse can be easily obtained, in practice, by Fourier transforming Eq. (46) [as in Eq. (36)]. We may define a "photon wave function" $\psi(\tau)=1/\sqrt{2\pi}\int f(\omega)e^{-i\omega\tau}\,d\omega$, so that $I(\tau)=|\psi(\tau)|^2$ gives the normalized probability distribution function to find the photon, as a function of τ, where (by the results of Sec. IV B) $\tau=t-z/c$ for the incoming pulse and $t+z/c$ for the outgoing one. A partial fraction decomposition of Eq. (46) yields

$$f_{\rm out}(\omega) = \left[-1-\frac{\kappa}{s}\left(\frac{i\delta_a-\mu_1}{i\omega+\mu_1}-\frac{i\delta_a-\mu_2}{i\omega+\mu_2}\right)\right]f_{\rm in}(\omega), \tag{49}$$

where $s=\frac{1}{2}\sqrt{(\kappa+i\delta_a)^2-4g^2}$ and $\mu_{1,2}=(-\kappa+i\delta_a)/2\mp s$; the latter represents the resonances of the atom-cavity system. Because the real part of $\mu_{1,2}$ is always negative, one can always write $(i\omega+\mu_i)^{-1}=-\int_{-\infty}^0 e^{-(i\omega+\mu_i)\tau'}d\tau'$, so the Fourier transform can be written

$$\psi_{\rm out}(\tau) = -\psi_{\rm in}(\tau)+\frac{\kappa}{s}\int_{-\infty}^{\tau}[(i\delta_a-\mu_1)e^{\mu_1(\tau-\tau')} - (i\delta_a-\mu_2)e^{\mu_2(\tau-\tau')}]\psi_{\rm in}(\tau')d\tau'. \tag{50}$$

The first term on the right-hand side of Eq. (50) is the only one present in the $\kappa\to 0$ limit, where the cavity is replaced by a perfectly reflecting mirror (the minus sign is, of course, arbitrary, depending on the type of mirror). The two other terms are convolutions of the input pulse with the response functions associated with the two resonances of the system.

For reference, for an input Gaussian pulse with spectrum $f_{\rm in}(\omega)=(T/\sqrt{2\pi})^{1/2}e^{-(\omega-\Delta)^2T^2/4}$, the explicit result (valid in both the good and bad cavity regimes) is

$$\psi_{\rm out}(\tau) = \frac{T^{1/2}}{(2\pi)^{3/4}}\Bigg[-\frac{2\sqrt{\pi}}{T}\,e^{-(\tau/T)^2-i\tau\Delta} + \frac{\pi\kappa}{s}\sum_{j=1,2}(-1)^j(i\delta_a-\mu_j)e^{(\mu_j+i\Delta)^2T^2/4+\mu_j\tau} \times\left(1+\mathrm{erf}\left[\frac{\tau}{T}+\frac{T(\mu_j+i\Delta)}{2}\right]\right)\Bigg]. \tag{51}$$

Figures 3–6 below show examples of outgoing pulses for the same incoming, Gaussian pulse, and different cavity parameters. Figure 3 is close to the adiabatic regime, Fig. 4 is in the bad cavity regime, and Fig. 5 in the good cavity regime. According to the discussion in Sec. IV B one may get a spacetime description from these pictures in the following way: imagine a vertical line drawn through the figure and moving to the right as time passes. At any given time, the part of the input pulse to the *right* of the line represents the part of the pulse that has not gone yet into the cavity (one has to imagine this part reflected about the vertical line, that is, traveling to the right; this is because for the input pulse $\tau=t-z/c$, so τ and z increase in opposite directions), whereas the part of the output pulse to the *left* of the vertical line represents the part of the pulse that has already come out of the cavity, at that same instant.

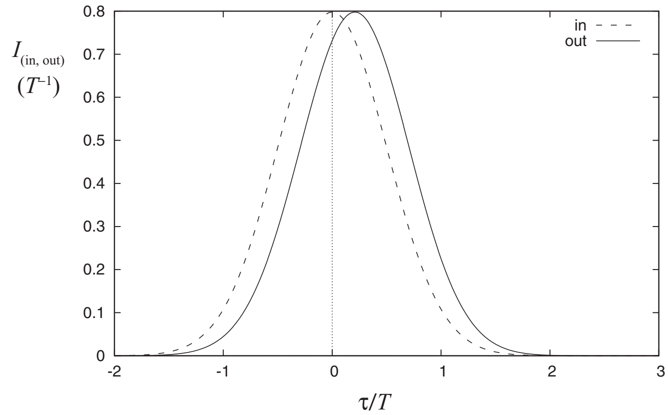

FIG. 3. Input and output probabilities to find the photon as functions of τ (in units of T^{-1}), for $\kappa T=gT=10$ (close to the adiabatic regime).

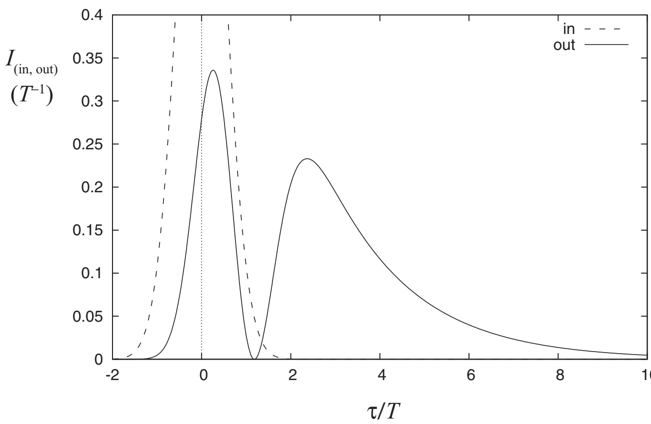

FIG. 4. Input and output probabilities to find the photon as functions of τ, for $\kappa T = 4$, $gT = 1$ (bad cavity regime).

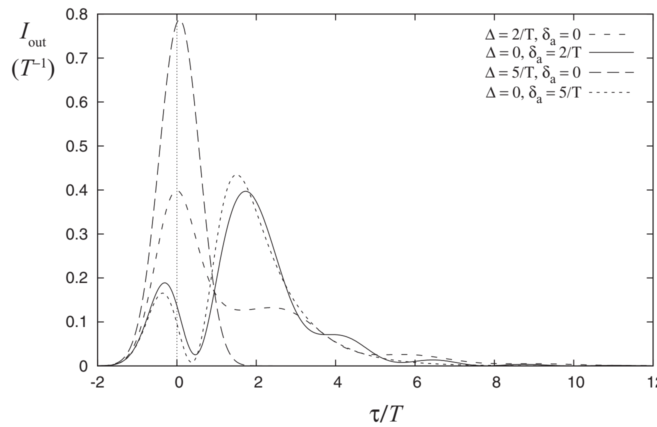

FIG. 6. Output pulses for $\kappa T = 0.5$, $gT = 1$, and $\Delta = 2/T, 5/T$ (with $\delta_a = 0$) and $\delta_a = 2/T, 5/T$ (with $\Delta = 0$).

Although the parameters of Fig. 3 are close to the adiabatic regime, the figure shows a small difference between the incoming and outgoing pulses, in the form of a shift in time (or space). This is in fact what one would expect from an expansion of the phase Eq. (47) to first order in ω': multiplying the incident spectrum by such a linearly growing phase factor just results in a displacement in the τ domain. For $\Delta = \delta_a$, this displacement, or delay, is equal to $2\kappa/g^2$, or $0.2T$ for the pulse in the figure.

Figure 2 illustrates the apparition, in the bad cavity regime, of a fairly slow decay rate, associated, in our notation, with the parameter μ_2; for large κ, and $\delta_a = 0$, this slow rate goes as g^2/κ ($\sim 1/4T$ for the parameters in the figure). The zero in the output pulse results from a cancellation between all three terms in Eq. (50) which typically survives in the empty cavity limit ($g \to 0$); in that limit, however, the μ_2 term in Eq. (50) goes to zero.

Figure 5 shows additional zeros and oscillations that appear in the good cavity limit, as the characteristic decay rates $\mu_{1,2}$ develop an imaginary part. It is tempting to associate these features with Rabi oscillations; mathematically, of course, they have the same origin, but it would be wrong to conclude, for instance, that when the probability of finding the photon in the outgoing pulse is exactly zero the atom must be in the excited state, because for these pulses the probability to excite the atom never reaches very high values, even on resonance, as will be explained below. Indeed, for any time τ when $I_{\text{out}}(\tau)$ is zero there is a nonvanishing probability that the photon may have left the cavity *before* then, and this alone implies that the excitation probability cannot be equal to 1 at that time.

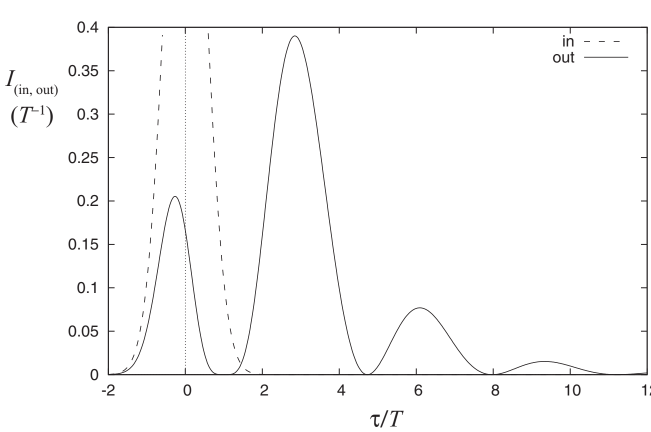

FIG. 5. Input and output probabilities to find the photon as functions of τ, for $\kappa T = 0.5$, $gT = 1$ (good cavity regime).

Finally, Fig. 6 shows the effect, on the output pulse, of varying the detunings, for the same cavity parameters as in Fig. 5. Clearly, for sufficiently large detuning Δ between the cavity and the pulse, the cavity appears as a perfectly reflecting mirror and the output pulse approaches the input one, whereas for sufficiently large detuning δ_a between the atom and the cavity the output pulse approaches the limit appropriate for an empty cavity, mentioned above.

B. Excitation probability, and decay of initially excited atom

For an atom initially (formally, at $t = -\infty$) in the ground state, the excitation probability as a function of time is given by $P_e(t) = |C_e(t)|^2$ [Eq. (43), with $\tilde{C}_e(\omega)$ given by Eq. (44)]. We can write this as

$$P_e(t) = \left| \int_{-\infty}^{\infty} \frac{g\sqrt{\kappa/\pi}}{(\omega + i\kappa)(\omega + \delta_a) - g^2} \, C_g(\omega, -\infty) \, e^{-i\omega t} \, d\omega \right|^2$$
$$\equiv \left| \int_{-\infty}^{\infty} S(\omega) \, C_g(\omega, -\infty) \, e^{-i\omega t} \, d\omega \right|^2 \qquad (52)$$

with the function $S(\omega)$ defined by this equation. It is not hard to see that in fact $\int |S(\omega)|^2 d\omega = 1$, just as $\int |C_g(\omega, -\infty)|^2 d\omega = 1$, and therefore, by the Cauchy-Schwartz inequality, the integral in Eq. (52) will always be strictly smaller than 1 unless $C(\omega, -\infty) = S^*(\omega) e^{i\omega t_e}$, in which case the probability will be equal to 1 at $t = t_e$.

The input pulse corresponding to a spectrum $C(\omega, -\infty) = S^*(\omega)$ is

$$\psi_{\text{in,opt}}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{g\sqrt{\kappa/\pi}}{(\omega - i\kappa)(\omega + \delta_a) - g^2} e^{-i\omega t} \, d\omega$$
$$= \frac{g\sqrt{2\kappa}}{\sqrt{(\kappa - i\delta_a)^2 - 4g^2}} (e^{-\mu_1^* t} - e^{-\mu_2^* t}), \quad \text{for } t < 0 \qquad (53)$$

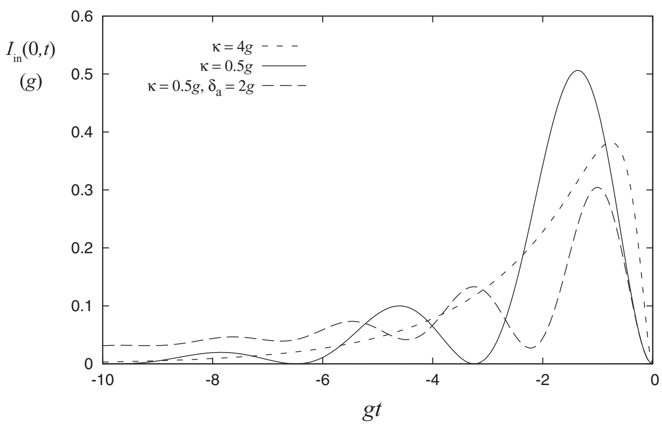

FIG. 7. Optimal driving pulses for $\kappa = 4g$ (bad cavity regime), $\kappa = 0.5g$ (good cavity), and $\kappa = 0.5g, \delta_a = 2g$. Time is in units of g^{-1}; the vertical axis is the probability of finding the photon per unit time (in units of g). Reflecting these pulses around the $t = 0$ vertical axis gives the decay pulses for an initially excited atom, for the same parameters.

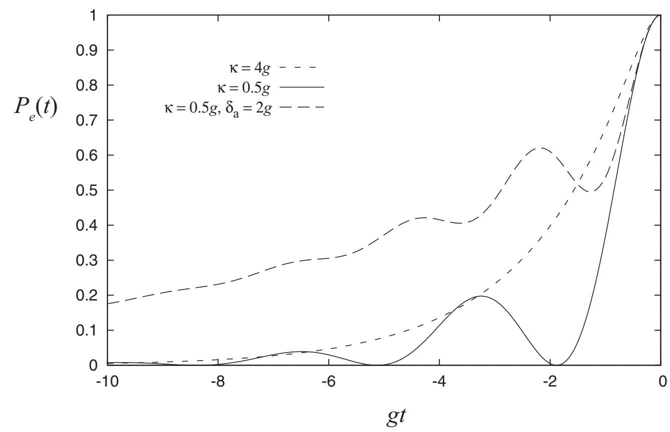

FIG. 8. Excitation probabilities for the optimal pulses shown in Fig. 7.

and 0 for $t > 0$. When such a pulse is incident on the atom-cavity system, it will drive the atom to the excited state at the time $t = 0$. The corresponding output pulse can be obtained by Fourier-transforming Eq. (46), with $f_{\text{in}}(\omega) = S^*(\omega)$, which immediately shows that the following relation holds:

$$\psi_{\text{out}}(t) = -\psi^*_{\text{in,opt}}(-t). \tag{54}$$

This result vanishes for $t < 0$, i.e., for this particular input pulse there is no reflected (outgoing) pulse at all, as long as the input pulse lasts. Clearly, for $t > 0$, when $\psi_{\text{in,opt}}$ "switches off" and ψ_{out} "switches on," the latter just gives us the decay of an initially excited atom in the cavity, a problem that can also be solved starting directly from Eq. (42) above, in its homogeneous form (that is, without the second, driving term on the right-hand side).

Equation (53) represents the optimal way to drive the cavity-atom system: at the end of the $\psi_{\text{in,opt}}$ pulse the photon is certain to have gone into the cavity (zero reflection probability) and to have excited the atom ($|C_e|^2 = 1$). The observation that the pulse that accomplishes this is just the same as the pulse given off by an initially excited atom, only time-reversed, was made a long time ago by Cirac *et al.* [4]. Interestingly, full excitation is possible, in principle, even for a finite atom-cavity detuning δ_a.

Figure 7 shows several of these optimal pulses, as functions of time (evaluated, for instance, at the entrance to the cavity, $z = 0$). The parameters correspond to Figs. 4 and 5, as well as one of the detuned situations considered in Fig. 6.

Turning Fig. 7 around also shows that the pulse emitted by an initially excited atom in the good cavity limit has oscillations ("bright and dark fringes") similar to the ones we found in the previous subsection. Their physical origin, in this case, is actually quite clear. In the good cavity limit, the cavity-atom system single-excitation spectrum consists of two distinct lines centered at $\text{Im}(\mu_1)$ and $\text{Im}(\mu_2)$. The state in which the atom is initially excited is an equally weighted superposition of these two modes, so a subsequent state in which the photon is outside the cavity and the atom in the ground state could be the result of two indistinguishable (as long as the frequency of the photon is not observed) decay processes, whose relative phases change with the observation time [see Eq. (53)]; the fringes are the result of the interference between these two alternative paths. One may also think of them as an example of "quantum beats."

Finally, the expression (52) for the excitation probability can be evaluated for an arbitrary input pulse as described by the initial spectrum $C_g(\omega, -\infty)$. The result can be trivially written in terms of a convolution of $\psi^*_{\text{in,opt}}$ with ψ_{in}:

$$\begin{aligned} P_e(t) &= \left| \int_{-\infty}^{\infty} \psi^*_{\text{in,opt}}(t')\psi_{\text{in}}(t+t')\,dt' \right|^2 \\ &= \frac{2\kappa g^2}{\sqrt{\left(\kappa^2 - 4g^2 - \delta_a^2\right)^2 + 4\kappa^2\delta_a^2}} \\ &\quad \times \left| \int_{-\infty}^{\infty} (e^{\mu_1(t-t')} - e^{\mu_2(t-t')})\psi_{\text{in}}(t')\,dt' \right|^2. \end{aligned} \tag{55}$$

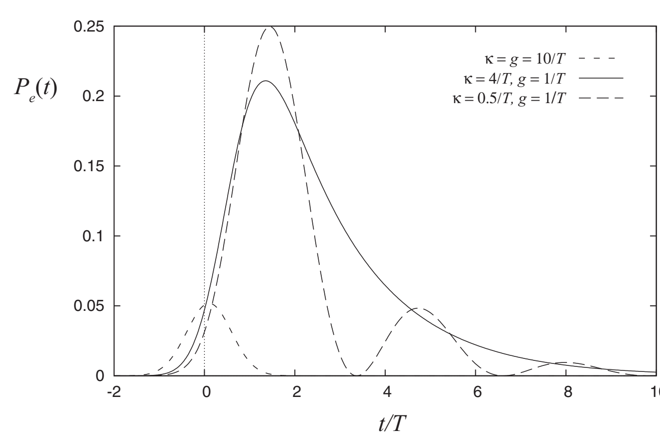

FIG. 9. Excitation probabilities for the Gaussian pulse and the cavity parameters used in Figs. 1–3. Solid line: bad cavity regime. Long-dashed line: good cavity regime. Short-dashed line: near-adiabatic regime.

For the Gaussian input pulse considered above the explicit result is

$$P_e(t) = \frac{T}{\sqrt{2\pi}} \frac{\kappa g^2}{|2s|^2} \Bigg| \sum_{j=1,2} (-1)^j e^{(\mu_j + i\Delta)^2 T^2/4 + \mu_j t} \left(1 + \mathrm{erf}\left[\frac{t}{T} + \frac{T(\mu_j + i\Delta)}{2}\right]\right)\Bigg|^2. \tag{56}$$

Figure 8 shows the excitation probabilities calculated for the optimal pulses in Fig. 7, whereas Fig. 9 shows the result (56) for an incident Gaussian pulse, for the parameters used in Figs. 1–3. Figure 8 shows the interesting result that the excitation probability also displays oscillations in the good cavity limit, even for an optimal pulse. Figure 8 shows that the Gaussian pulse is not very good at exciting the atom, in general. As expected, the excitation probability is smallest for the $\kappa = g = 10/T$ case, since this is very close to the adiabatic regime.

VI. CONCLUSIONS

The complete formal equivalence of the input-output formalism and the modes of the universe approach to deal with the problem of the coupling between an atomic system in a cavity and the external quantized field has been established in a particular limit (namely, when the cavity transmission losses are small enough to allow for a Lorenzian approximation to the cavity resonance line). The MOU approach has a number of potential advantages for numerical calculations: it involves intrinsically fewer equations, and fewer modes are typically required to obtain reliable results. It is also conceptually more satisfying, since it can be derived directly from the optical boundary conditions, and hence easily adapted to any type of cavity (see, for instance, the treatment in Ref. [16] for a double-sided cavity). Although its original formulation was formally cumbersome, the most recent one in terms of scattering modes is conceptually quite straightforward (again, see Ref. [16]).

The formalism has been illustrated for a very simple case of a two-level atom in a cavity and a single-photon pulse. Of course, the real usefulness of the formalism will have to be shown by addressing open problems; these may include new photonic gates, or ways to combine few-photon pulses.

ACKNOWLEDGMENT

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