Phase-measurement sensitivity beyond the standard quantum limit in an interferometer consisting of a parametric amplifier and a beam splitter

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We analyze a nonconventional interferometer that is formed with a parametric amplifier and a beam splitter for beam splitting and recombination. Because the outputs from a parametric amplifier are entangled and their quantum noise is correlated, the employment of the beam splitter will superimpose the two quantum fields and the destructive interference will lead to the subtraction of the quantum noise and to noise reduction in the output of the interferometer and hence an improvement of the signal-to-noise ratio (SNR) beyond the standard quantum limit or the shot noise limit. Furthermore, the injection of a squeezed state into the idler port of the parametric amplifier will lead to further improvement of the SNR. We will discuss the possibility of reaching the Heisenberg limit in such an interferometer. We find that the injection of a coherent state will degrade the performance in reaching the Heisenberg limit, whereas a squeezed state injection can improve it by a factor of 2 at best.

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I. INTRODUCTION

Optical interferometry [1] is an important method in precision measurement as the interference fringe is sensitive to phase shift of each arm. The fundamental technique of an optical interferometer is to split a light source into two arms by using an "element" and then recombine them with another element. So far, the basic elements in a traditional optical interferometer such as a Mach-Zehnder interferometer [2,3] are the beam splitters [4]. The sensitivity of these interferometers is limited by the vacuum quantum noise injected from the unused port [5]. Thus, it is natural that research efforts are mostly focused on finding better light sources such as squeezed states [5–7] for reducing the quantum noise in the interferometer. Such an approach has been applied to the Laser Interferometer Gravitational-Wave Observatory (LIGO) [8].

On the other hand, a nontraditional interferometer can be formed by replacing the linear beam splitters with some nonlinear elements. Such a nonlinear interferometer was first proposed by Yurke *et al.* [9] and the nonlinear elements there are four-wave mixers or parameter amplifiers. Because of the nature of an evolution operator as compared to a linear beam splitter, such an interferometer is often dubbed an SU(1,1) interferometer from a theoretician's point of view. But since nonlinear processes are involved in the beam splitting and recombination, we will call it a nonlinear interferometer. More nonlinear interferometers were proposed [10,11] with various types of unitary transformation serving as the beam splitting and recombination elements. The first nonlinear interferometer was realized in experiment in 2002 when Leibfried *et al.* [12] simulated the action of nth-order nonlinear optical beam splitters in a trapped ion system. But the quantum number is small as the efficiency exponentially decreases with the number. Recently, a new theoretical scheme was presented by Plick *et al.* [13] who proposed to inject a strong coherent beam to "boost" the photon number, which circumvents the low photon number problem. The experimental realization of such a nonlinear interferometer was reported by Jing *et al.* [14]. In this interferometer, the two elements are parametric amplifiers which increase the signal by amplification. Soon after this, Ou [15] presented a full quantum analysis of such kind of nonlinear interferometer and demonstrated the significant improvement of signal-to-noise ratio (SNR) beyond the standard quantum limit or the shot noise limit.

However, most of the nonlinear interferometer schemes mentioned above involve the same nonlinear process as the two elements for both beam splitting and recombination in the interferometer. In principle, these two elements can be different processes. In this paper, we study a nonlinear interferometer in which one of the elements is a parametric amplifier (PA) and the other is a beam splitter (BS). Because they are different, the order is important: We study both the "PA + BS" scheme and its reverse scheme of "BS + PA." What motivated our study is the fact that in the SU(1,1) interferometer scheme with two parametric amplifiers, quantum entanglement in the first PA plays an important role in canceling the amplified noise, while the amplification in the second PA leads to signal enhancement and ultimately to SNR improvement. So in this paper, we want to separate the roles of the two PAs. In the first scheme of PA + BS, we find that there is an improvement in SNR, which comes as a result of noise reduction of the interferometer because the quantum noise of the outputs of the parametric amplifier is entangled and destructive interference at the beam splitter leads to the quantum noise cancellation. Further improvement is possible with the injection of squeezed states into the idler mode of the parametric amplifier. In the reverse scheme of BS + PA, however, we find that although the signal is amplified, the noise is also amplified, resulting in no SNR improvement as compared to a linear interferometer.

The paper is organized as follows: In Sec. II, we first discuss in detail the scheme of PA + BS and consider the case when the idler mode of the PA is either vacuum or a squeezed state. We

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will discuss how the Heisenberg limit can be approached. Next in Sec. III, we discuss the scheme of BS + PA. We conclude in Sec. IV with a discussion.

II. THE SCHEME OF PA + BS

Nonlinear interferometers involving parametric amplifiers are depicted in Fig. 1, where (a) is the well-known SU(1,1) interferometer, (b) is the scheme of PA + BS, and (c) is its reverse of BS + PA. We first consider the scheme of PA + BS as shown in Fig. 1(b), where the first element to split an incoming coherent state is a PA and the beam recombining element is a beam splitter. The beam splitting effect of a PA can be seen in the input-output relations of the PA [16]

$$\hat{A} = G\hat{a}_{\rm in} + g\hat{b}^{\dagger}_{\rm in}, \quad \hat{B} = G\hat{b}_{\rm in} + g\hat{a}^{\dagger}_{\rm in}, \tag{1}$$

where we assume G and g are all positive and $G^2 - g^2 = 1$. With a coherent state $|\alpha\rangle$ input at port $\hat{a}_{\rm in}$ and vacuum at port $\hat{b}_{\rm in}$, the intensities at the outputs of the PA are simply $G^2|\alpha|^2 + g^2$ and $g^2|\alpha|^2 + g^2$, respectively. If the amplitude gain G is much larger than one, we have $G \approx g \gg 1$ and the outputs are nearly equal to each other. Furthermore, the two output fields are phase correlated so that the beam splitting is coherent.

To form an interferometer, we make the field $\hat{B}$ subject to a phase shift φ before we combine the fields $\hat{A}$ and $\hat{B}$ with a BS. The field intensity sensing this phase shift is $I_{\rm ps} \equiv \langle \hat{B}^{\dagger}\hat{B}\rangle = g^2|\alpha|^2 + g^2$. The input-output relations of the BS are

$$\hat{a}_{\rm out} = \sqrt{T}\hat{A} + \sqrt{R}\hat{B}e^{i\varphi}, \quad \hat{b}_{\rm out} = \sqrt{T}Be^{i\varphi} - \sqrt{R}\hat{A}. \tag{2}$$

Combining Eqs. (1) and (2), we obtain the input-output relations of the interferometer as

$$\begin{aligned}\hat{a}_{\rm out} &= \sqrt{T}G\hat{a}_{\rm in} + \sqrt{R}ge^{i\varphi}\hat{a}^{\dagger}_{\rm in} + \sqrt{T}g\hat{b}^{\dagger}_{\rm in} + \sqrt{R}Ge^{i\varphi}\hat{b}_{\rm in},\\ \hat{b}_{\rm out} &= \sqrt{T}Ge^{i\varphi}\hat{b}_{\rm in} - \sqrt{R}g\hat{b}^{\dagger}_{\rm in} + \sqrt{T}ge^{i\varphi}\hat{a}^{\dagger}_{\rm in} - \sqrt{R}G\hat{a}_{\rm in}.\end{aligned} \tag{3}$$

These are the operator relations. We will calculate the the fringe and noise of the interferometer for various input states at $\hat{a}_{\rm in}, \hat{b}_{\rm in}$.

A. Vacuum input at the idler mode of the PA

For a coherent state $|\alpha\rangle$ input at $\hat{a}_{\rm in}$ and vacuum at $\hat{b}_{\rm in}$, we find the output intensity as

$$\begin{aligned}\langle \hat{b}^{\dagger}_{\rm out}\hat{b}_{\rm out}\rangle &= g^2(1 + |\alpha|^2) + (R - 2\sqrt{TR}Gg\cos\varphi)|\alpha|^2\\ &= [g^2 + (g^2 + R)|\alpha|^2](1 - \mathcal{V}\cos\varphi)\\ &\approx I_{\rm ps}(1 - \mathcal{V}\cos\varphi) \quad \text{for } g^2 \gg R,\end{aligned} \tag{4}$$

which shows an interference fringe with a visibility of

$$\mathcal{V} \equiv \frac{2Gg\sqrt{TR}|\alpha|^2}{g^2 + (g^2 + R)|\alpha|^2} \approx \frac{2Gg\sqrt{TR}}{g^2 + R} \tag{5}$$

for $|\alpha| \gg 1$. The visibility will reach 100% when $R = g^2/(G^2 + g^2)$ for large α. Here $I_{\rm ps} \equiv \langle \hat{B}^{\dagger}\hat{B}\rangle = g^2(1 + |\alpha|^2) \approx g^2|\alpha|^2$ is the phase-sensing intensity.

Next, we examine the quantum noise in the output of the interferometer. As is well known, quantum noise is best measured by homodyne detection where the quadrature-phase amplitudes are the physical quantities. For the output port $\hat{b}_{\rm out}$, we have

$$\begin{aligned}\hat{X}_{b_{\rm out}} \equiv \hat{b}_{\rm out} + \hat{b}^{\dagger}_{\rm out} &= \sqrt{T}G\hat{X}_{b_{\rm in}}(\varphi) - \sqrt{R}g\hat{X}_{b_{\rm in}}\\ &\quad + \sqrt{T}g\hat{X}_{a_{\rm in}}(-\varphi) - \sqrt{R}G\hat{X}_{a_{\rm in}},\end{aligned} \tag{6}$$

where $\hat{X}_{b_{\rm in}}(\varphi) = e^{i\varphi}\hat{b}_{\rm in} + e^{-i\varphi}\hat{b}^{\dagger}_{\rm in}$, $\hat{X}_{b_{\rm in}} = \hat{b}_{\rm in} + \hat{b}^{\dagger}_{\rm in}$, $\hat{X}_{a_{\rm in}}(-\varphi) = e^{-i\varphi}\hat{a}_{\rm in} + e^{i\varphi}\hat{a}^{\dagger}_{\rm in}$, and $\hat{X}_{a_{\rm in}} = \hat{a}_{\rm in} + \hat{a}^{\dagger}_{\rm in}$. For a coherent state $|\alpha\rangle$ with $\alpha = i|\alpha|$ at $\hat{a}_{\rm in}$ and vacuum at $\hat{b}_{\rm in}$, we have

$$\begin{aligned}\langle \hat{X}^2_{b_{\rm out}}\rangle &= TG^2\langle \hat{X}^2_{b_{\rm in}}(\varphi)\rangle + Rg^2\langle \hat{X}^2_{b_{\rm in}}\rangle - \sqrt{TR}Gg\langle \hat{X}_{b_{\rm in}}(\varphi)\hat{X}_{b_{\rm in}}\\ &\quad + \hat{X}_{b_{\rm in}}\hat{X}_{b_{\rm in}}(\varphi)\rangle + Tg^2\langle \hat{X}^2_{a_{\rm in}}(-\varphi)\rangle + RG^2\langle \hat{X}^2_{a_{\rm in}}\rangle\\ &\quad - \sqrt{TR}Gg\langle \hat{X}_{a_{\rm in}}(-\varphi)\hat{X}_{a_{\rm in}} + \hat{X}_{a_{\rm in}}\hat{X}_{a_{\rm in}}(-\varphi)\rangle\\ &= G^2 + g^2 - 4\sqrt{TR}Gg + 8\sqrt{RT}Gg\sin^2(\varphi/2)\\ &\quad + 4Tg^2\sin^2\varphi|\alpha|^2.\end{aligned} \tag{7}$$

The interferometer usually works at the dark fringe with $\varphi \sim 0$. So with a small phase shift $\delta(\ll 1)$ for φ and a strong coherent state injection ($|\alpha|^2 \gg 1$), we have

$$\begin{aligned}\langle \hat{X}^2_{b_{\rm out}}\rangle &\approx G^2 + g^2 - 4\sqrt{RT}Gg + 4Tg^2\delta^2|\alpha|^2\\ &= \langle \hat{X}^2_{b_{\rm out}}\rangle_N + 4T\delta^2 I_{\rm ps}\end{aligned} \tag{8}$$

with the noise part $\langle \hat{X}^2_{b_{\rm out}}\rangle_N \equiv G^2 + g^2 - 4\sqrt{RT}Gg$ and the phase-sensing intensity $I_{\rm ps} \approx g^2|\alpha|^2$ for $|\alpha| \gg 1$. Hence the

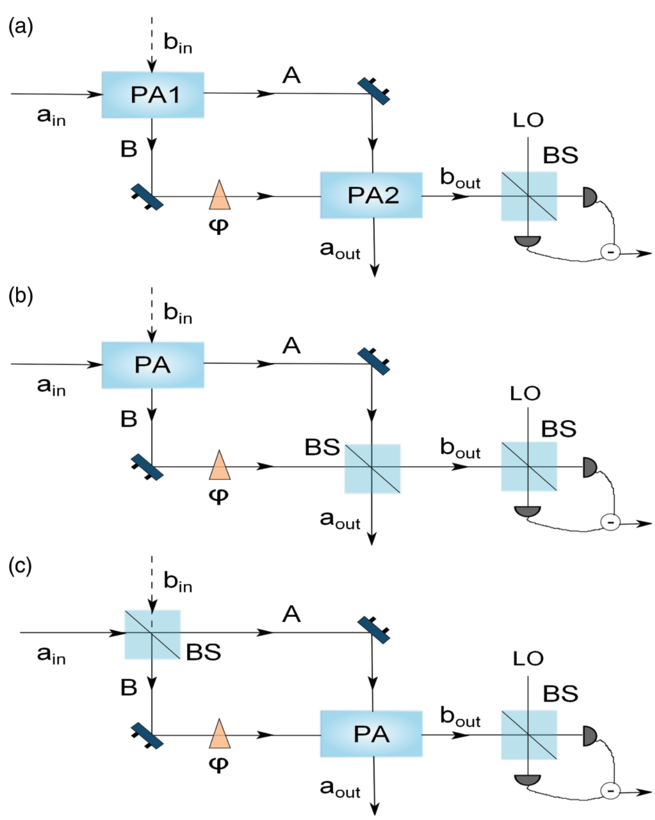

FIG. 1. (Color online) Schematic diagram for (a) an SU(1,1) nonlinear interferometer with two parametric amplifiers (PA); (b) an interferometer with a PA as the beam splitting element and a beam splitter (BS) as the beam recombination element; and (c) reversed order of PA and BS from (b).

SNR for measuring a small phase shift of δ is

$$R_{\rm nl} = \frac{4T\delta^2 I_{\rm ps}}{G^2 + g^2 - 4\sqrt{RT}Gg}. \quad (9)$$

For a given gain of the PA, we find the SNR has a maximum value of

$$R_{\rm nl_{max}} = 4\delta^2 I_{\rm ps}(G^2 + g^2) \quad (10)$$

when $T = T_{\rm max} = (G^2 + g^2)^2/(8G^2g^2 + 1)$. Under this condition, the output noise becomes

$$\langle \hat{X}^2_{b_{\rm out}} \rangle_N = \frac{G^2 + g^2}{8G^2g^2 + 1} \approx \frac{1}{2(G^2 + g^2)} \quad (11)$$

for $G \gg 1$. This noise level is reduced below the vacuum noise level by $2(G^2 + g^2)$.

Compared to a traditional linear interferometer, which has a SNR of $R_l = 2\delta^2 I_{\rm ps}$ [15], we have a SNR improvement by

$$\frac{R_{\rm nl_{max}}}{R_l} = 2(G^2 + g^2). \quad (12)$$

This improvement factor is the same as the noise reduction factor in Eq. (11), indicating that the improvement is all from noise reduction. It is interesting to compare this result with that from the scheme of PA + PA [the SU(1,1) interferometer]. From Ref. [15], we find the SNR improvement for PA + PA is $2G^2$, which is about two times (when $G \sim g$) smaller than that in Eq. (12). This difference is a result of a different mechanism for the SNR improvement: For the scheme of PA + PA, the SNR improvement is due to the signal increase, whereas here for PA + BS, the improvement is from noise reduction.

For $G \gg 1$, we have $G \approx g$ and when the SNR $R_{\rm nl} = 1$, we reach the minimum measurable phase shift

$$\delta_m = \sqrt{\frac{1}{4I_{\rm ps}(G^2 + g^2)}} \approx \sqrt{\frac{1}{8I_{\rm ps}G^2}} = \frac{\delta_{\rm SQL}}{2G}, \quad (13)$$

where $\delta_{\rm SQL} \equiv 1/\sqrt{2I_{\rm ps}}$ is the standard quantum limit or the shot noise limit of phase measurement that can be achieved in a traditional linear interferometer. So our scheme of PA + BS improves upon the SQL by a factor of $2G$.

B. Squeezed state at the idler mode of the PA

As we can see, the improvement in the SNR in the phase measurement in the previous part is due to quantum noise reduction. On the other hand, the input to the unused idler mode ($\hat{b}_{\rm in}$) of the PA is in vacuum in our previous calculation. So, can we reduce the noise further by placing $\hat{b}_{\rm in}$ in a squeezed state? We will investigate it next. In this case, the input port $\hat{a}_{in}$ is in a coherent state $|\alpha\rangle$ while the input port $\hat{b}_{\rm in}$ is in a squeezed vacuum state, i.e.,

$$\begin{aligned} \xi^\dagger \hat{b}_{\rm in}\xi &= \hat{b}_{\rm in}\cosh r - \hat{b}^\dagger_{\rm in} e^{2i\theta}\sinh r, \\ \xi^\dagger \hat{b}^\dagger_{\rm in}\xi &= \hat{b}^\dagger_{\rm in}\cosh r - \hat{b}_{\rm in} e^{-2i\theta}\sinh r, \end{aligned} \quad (14)$$

where ξ is the squeezing operator so that the input state is $\xi|{\rm vac}\rangle$ and r is the squeezing parameter [17]. Setting $\theta = 0$, we find straightforwardly the output intensity of the interferometer as

$$\begin{aligned} \langle \hat{b}^{s\dagger}_{\rm out}\hat{b}^s_{\rm out}\rangle &= g^2 + (TG^2 + Rg^2)(\sinh^2 r + |\alpha|^2) \\ &\quad - 4\sqrt{TR}Gg\cos\varphi\,|\alpha|^2, \end{aligned} \quad (15)$$

and the quadrature-phase amplitude noise as

$$\begin{aligned} \langle (\hat{X}^s_{b_{\rm out}})^2 \rangle &= TG^2[\sinh^2 r + \cosh^2 r - 2\sinh r\cosh \\ &\quad \times r(1 - 2\sin^2\varphi)] + Tg^2 + 4Tg^2|\alpha|^2\sin^2\varphi \\ &\quad + RG^2 - 2\sqrt{RT}Gg\left(1 - 2\sin^2\frac{\varphi}{2}\right) + Rg^2e^{-2r} \\ &\quad - 2\sqrt{TR}Gg\left(1 - 2\sin^2\frac{\varphi}{2}\right)e^{-2r}. \end{aligned} \quad (16)$$

With a small phase shift δ ($\delta \ll 1$) for φ and the approximation of strong coherent state injection, that is, $|\alpha| \gg e^r, 1$, we have

$$\begin{aligned} \langle (\hat{X}^s_{b_{\rm out}})^2 \rangle &= (\sqrt{T}G - \sqrt{R}g)^2e^{-2r} + (\sqrt{T}g - \sqrt{R}G)^2 \\ &\quad + 4Tg^2|\alpha|^2\delta^2. \end{aligned} \quad (17)$$

This leads to the SNR for phase measurement as

$$R^s_{\rm nl} = \frac{4Tg^2|\alpha|^2\delta^2}{(\sqrt{T}G - \sqrt{R}g)^2e^{-2r} + (\sqrt{T}g - \sqrt{R}G)^2}. \quad (18)$$

For a fixed G and r, we find that when

$$T = T_{\rm max} = \frac{(G^2e^r + g^2e^{-r})^2}{4G^2g^2\cosh^2 r + (G^2e^r + g^2e^{-r})^2}, \quad (19)$$

$R^s_{\rm nl}$ reaches the maximum value:

$$\begin{aligned} R^s_{\rm nl_{max}} &= 4g^2|\alpha|^2\delta^2(G^2e^{2r} + g^2) \approx 4I_{\rm ps}\delta^2(G^2e^{2r} + g^2) \\ &\approx 4I_{\rm ps}\delta^2G^2e^{2r} \qquad \text{for } e^r \gg 1, \end{aligned} \quad (20)$$

where $I_{\rm ps} \approx g^2|\alpha|^2$ for $|\alpha| \gg e^r, 1$. Comparing to Eq. (10), we find that the squeezed state can improve the SNR further by about e^{2r}, similar to the scheme of PA + PA. With $R^s_{\rm nl_{max}} = 1$, we obtain the minimum measurable phase shift

$$\delta_m = \sqrt{\frac{1}{4I_{\rm ps}(G^2e^{2r} + g^2)}} \approx \frac{\delta_{\rm SQL}}{\sqrt{2}Ge^r} \quad \text{for } r \gg 1. \quad (21)$$

For a strong squeezing, we cannot make the approximation of $|\alpha| \gg e^r$. So the intensity of the phase-sensing field is

$$\begin{aligned} I^s_{\rm ps} &= \langle \hat{B}^\dagger\hat{B} \rangle = G^2\sinh^2 r + g^2(1 + |\alpha|^2) \approx G^2\sinh^2 r \\ &\quad + g^2|\alpha|^2 \quad (|\alpha|^2, e^r \gg 1), \end{aligned} \quad (22)$$

which we will set as a constant. With $|\alpha|^2, e^r \gg 1$, Eq. (18) is changed to

$$R^{s'}_{\rm nl} = \frac{4T(g^2|\alpha|^2 + G^2\sinh r\cosh r)\delta^2}{(\sqrt{T}G - \sqrt{R}g)^2e^{-2r} + (\sqrt{T}g - \sqrt{R}G)^2}. \quad (23)$$

The above expression is maximum when T takes the value in Eq. (19) and the optimized value is

$$R^{s'}_{\rm nl_{max}} = 4(g^2|\alpha|^2 + G^2\sinh r\cosh r)G^2e^{2r}\delta^2 \approx 4I_{\rm ps}\delta^2G^2e^{2r}. \quad (24)$$

Notice that the above expression is exactly the same as Eq. (20) but without the approximation of $|\alpha|^2 \gg e^r$. With $R^{s'}_{\rm nl_{max}}$ set to 1, we obtain the minimum measurable phase shift as

$$\delta'_m = \sqrt{\frac{1}{4I_{\rm ps}G^2e^{2r}}}. \quad (25)$$

C. Approaching the Heisenberg limit

From Eq. (25), it seems that we can improve the sensitivity indefinitely as we increase the squeezing parameter. However, when e^r is large so that $G^2e^{2r} \approx 4G^2\sinh^2 r \gg g^2|\alpha|^2$, we have $G^2e^{2r} \approx 4I_{ps}$ and Eq. (25) becomes

$$\delta_m' \approx \frac{1}{4I_{ps}}, \tag{26}$$

which is the Heisenberg limit. So, with strong squeezing, we can approach the ultimate limit of phase-measurement sensitivity.

In practice, however, it is difficult to obtain a large squeezing. So can we still approach the Heisenberg limit without squeezed state injection? From Eq. (13) we find that although the sensitivity in the PA + BS scheme is better than the standard quantum limit, it is still scaled as $1/\sqrt{I_{ps}}$ for a fixed G of the PA. However, if we fix the strength of the injected coherent state, that is, the quantity α with $|\alpha|^2 \gg 1$ and change I_{ps} by G of the PA, Eq. (13) is changed to

$$\delta_m = \sqrt{\frac{1}{4I_{ps}(G^2+g^2)}} \approx \sqrt{\frac{1}{8I_{ps}g^2}} \quad \text{for} \quad g \gg 1$$
$$= \frac{|\alpha|}{2\sqrt{2}I_{ps}}, \tag{27}$$

where we replace g^2 with $I_{ps}/|\alpha|^2$. For large I_{ps}, it scales as $1/I_{ps}$ for fixed α but with a large coefficient $|\alpha|(\gg 1)$.

If $|\alpha| \sim 1$, we need to keep those terms that were otherwise dropped in Eq. (8). The phase-sensing intensity is then

$$I_{ps} = g^2(1+|\alpha|^2) \tag{28}$$

and Eq. (8) is changed to

$$\langle \hat{X}^2_{b_{out}} \rangle = G^2 + g^2 - 4\sqrt{TR}Gg + (2\sqrt{TR}Gg + 4Tg^2|\alpha|^2)\delta^2. \tag{29}$$

The SNR for phase measurement is then

$$R_{nl} = \frac{(2\sqrt{TR}Gg + 4Tg^2|\alpha|^2)\delta^2}{G^2+g^2-4\sqrt{TR}Gg}. \tag{30}$$

The above expression, though it looks simple, has a very complicated formula at optimum value for T. Before we go to the general case, let us look at a simple case.

In the PA + PA scheme, the Heisenberg limit is reached without the coherent state injection [9]. So, let us consider the same situation here.

When all the inputs to the parametric amplifier are in vacuum, we have

$$\langle \hat{b}^\dagger_{out}\hat{b}_{out} \rangle = g^2 = I_{ps}. \tag{31}$$

So there is no interference in the output intensity. This is because the two output fields (A and B) from the PA have phase anticorrelation, that is, $\varphi_A + \varphi_B$ = const. and $\varphi_A - \varphi_B$ is completely random. On the other hand, the phase anticorrelation can be revealed in higher-order measurement such as homodyne measurement of quadrature-phase amplitudes. In this case, Eq. (30) gives

$$\langle \hat{X}^2_{b_{out}} \rangle \approx G^2 + g^2 - 4\sqrt{TR}Gg + 2\sqrt{TR}Gg\delta^2. \tag{32}$$

Hence the SNR is

$$R_{nl} = \frac{2\sqrt{TR}Gg\delta^2}{G^2+g^2-4\sqrt{TR}Gg}. \tag{33}$$

When $T = T_{max} = 1/2$, we have the maximum R_{nl}:

$$R_{nl} = Gg\delta^2(G+g)^2 = \sqrt{I_{ps}(I_{ps}+1)}\delta^2(\sqrt{I_{ps}+1}+\sqrt{I_{ps}})^2$$
$$\approx 4\delta^2 I_{ps}^2 \quad \text{for} \quad I_{ps} \gg 1. \tag{34}$$

Thus, the minimum measurable phase shift is

$$\delta_m = \frac{1}{2I_{ps}}. \tag{35}$$

This is the Heisenberg limit. Because there is no coherent state injection to boost the phase-sensing intensity I_{ps}, this scheme encounters the same problem of low I_{ps} as the scheme of PA + PA without a coherent state input.

It is interesting to compare the results in Eqs. (26) and (35). Equation (26) is obtained when the squeezing parameter is so large that $e^r \gg |\alpha|^2$, or the coherent state can be dropped. This corresponds to the situation when there is no coherent state input but squeezed state input at the input port of the PA. So injection of a squeezed state in the unused port will further improve the sensitivity by a factor of 2 as compared to vacuum input.

The two extreme cases of $|\alpha|^2 \gg 1$ and $|\alpha|^2 \ll 1$ both approach the Heisenberg limit at large g for the PA, although they have quite different proportional coefficients. Next, we go back to Eqs. (28) and (30) to look at the case with a fixed intermediate value of $|\alpha|^2 \sim 1$. We first need to optimize T for a maximum R_{nl}. The formula for T_{max} is quite complicated. But in the limit of $g^2 \gg 1$ or $I_{ps} \gg 1$, we have

$$T_{max} \approx \frac{1}{2} + \frac{|\alpha|^2}{8g^4(1+2|\alpha|^2)}. \tag{36}$$

The maximum SNR is then

$$R_{nl_{max}} \approx (Gg + 2g^2|\alpha|^2)(G+g)^2\delta^2 \approx \frac{4(1+2|\alpha|^2)\delta^2 I_{ps}^2}{(1+|\alpha|^2)^2} \tag{37}$$

for $I_{ps} \gg 1$. Setting $R_{nl_{max}} = 1$, we obtain

$$\delta_m = \frac{\eta}{2I_{ps}} \tag{38}$$

with

$$\eta = \sqrt{\frac{(1+|\alpha|^2)^2}{1+2|\alpha|^2}}. \tag{39}$$

Equation (38) recovers Eqs. (27) and (35) for $|\alpha|^2 \gg 1$ and $\ll 1$, respectively.

Notice that η in Eq. (39) has a minimum value of 1 for $|\alpha|^2 = 0$. This indicates that a coherent state boost in the nonlinear interferometer does not help in approaching the Heisenberg limit. We always require the gain $g \gg 1$ for $I_{ps} \gg 1$ to approach the Heisenberg limit.

For arbitrary I_{ps}, we obtain δ_m by setting $R_{nl} = 1$ in Eq. (30) after optimizing T. In Fig. 2, we plot δ_m so obtained versus I_{ps} for various values of $|\alpha|^2$. Notice that when $I_{ps} \sim 1$, we have a $1/\sqrt{I_{ps}}$ dependence, although all are below the standard

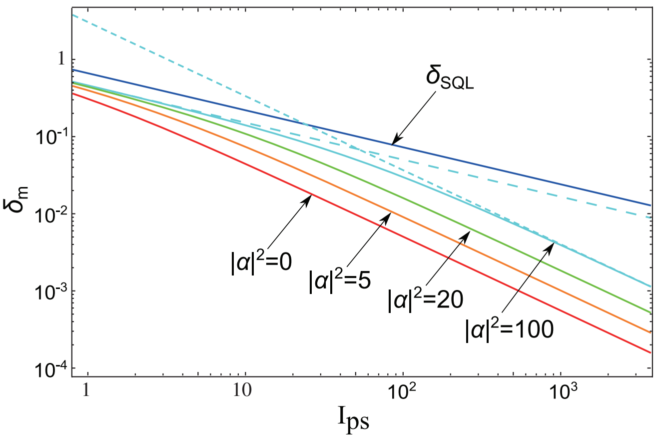

FIG. 2. (Color online) The minimum measurable phase δ_m as a function of phase-sensing intensity I_{ps} in log-log scale for various values of coherent state injection: $|\alpha|^2 = 0,5,20,100$. The standard quantum limit of $\delta_{SQL} = 1/\sqrt{2I_{ps}}$ is also shown. The dotted line has a slope of -1, whereas the slope of the dashed line is $-1/2$.

quantum limit or shot noise limit of $\delta_{SQL} = 1/\sqrt{2I_{ps}}$. It is interesting to note that although coherent state injection can boost the phase-sensing intensity, Fig. 2 shows that it will always degrade the sensitivity for phase measurement.

III. THE SCHEME OF BS + PA

Now for completeness, let us consider the case when the PA and the BS are reversed in order, that is, the scheme of BS + PA as shown in Fig. 1(c). As before, we will have a coherent state input at one input port and vacuum in the other port of the BS. The input-output relations of BS and PA are

$$\hat{A} = \sqrt{T}\hat{a}_{in} - \sqrt{R}\hat{b}_{in}, \quad \hat{B} = \sqrt{T}\hat{b}_{in} + \sqrt{R}\hat{a}_{in}, \\ \hat{a}_{out} = G\hat{A} + g\hat{B}^\dagger e^{-i\varphi}, \quad \hat{b}_{out} = G\hat{B}e^{i\varphi} + g\hat{A}^\dagger. \tag{40}$$

The total input-output relations can be expressed as

$$\hat{a}_{out} = G\sqrt{T}\hat{a}_{in} + g\sqrt{R}e^{-i\varphi}\hat{a}^\dagger_{in} - G\sqrt{R}\hat{b}_{in} + g\sqrt{T}\hat{b}^\dagger_{in}e^{-i\varphi}, \\ \hat{b}_{out} = G\sqrt{T}e^{i\varphi}\hat{b}_{in} - g\sqrt{R}\hat{b}^\dagger_{in} + G\sqrt{R}\hat{a}_{in}e^{i\varphi} + g\sqrt{T}\hat{a}^\dagger_{in}. \tag{41}$$

For a coherent state $|\alpha\rangle$ input at $\hat{a}_{in}$ and vacuum at $\hat{b}_{in}$, the output intensity is

$$\langle\hat{b}^\dagger_{out}\hat{b}_{out}\rangle = 2I_{ps}(G^2R + g^2T - 2Gg\sqrt{RT}\cos\varphi) + g^2, \tag{42}$$

where $I_{ps} \equiv \langle\hat{B}^\dagger\hat{B}\rangle = |\alpha|^2/2$. For strong coherent state input $|\alpha|^2 \gg 1$, we have

$$\langle\hat{b}^\dagger_{out}\hat{b}_{out}\rangle \approx 2(G^2R + g^2T)I_{ps}(1 - \mathcal{V}\cos\varphi), \tag{43}$$

with the visibility as $\mathcal{V} = 2Gg\sqrt{RT}/(G^2R + g^2T)$, which is 100% when $T = G^2/(G^2 + g^2)$, and Eq. (43) becomes

$$\langle\hat{b}^\dagger_{out}\hat{b}_{out}\rangle = \frac{4G^2g^2I_{ps}}{G^2+g^2}(1 - \mathcal{V}\cos\varphi). \tag{44}$$

So, the fringe size is increased by a factor of $2G^2g^2/(G^2 + g^2) \approx 2G^2$ for large G, similar to the scheme of PA + PA.

Now let us look at the output noise. The quadrature-phase amplitude of the output of the interferometer is

$$\hat{X}_{b_{out}} = G\sqrt{T}\hat{X}_{b_{in}}(\varphi) - g\sqrt{R}\hat{X}_{b_{in}} + G\sqrt{R}\hat{X}_{a_{in}}(\varphi) + g\sqrt{T}\hat{X}_{a_{in}}. \tag{45}$$

For the given input states, we have

$$\langle\hat{X}^2_{b_{out}}\rangle = G^2 + g^2 + 4G^2R|\alpha|^2\sin^2\varphi. \tag{46}$$

With a small phase shift δ for φ, we can get

$$\langle\hat{X}^2_{b_{out}}\rangle \approx G^2 + g^2 + 4G^2R|\alpha|^2\delta^2 = G^2 + g^2 + 8G^2RI_{ps}\delta^2. \tag{47}$$

So, the output noise is $G^2 + g^2 \approx 2G^2$ for large G. Hence the SNR is

$$R_{nl} = \frac{8G^2RI_{ps}\delta^2}{G^2+g^2}. \tag{48}$$

Then we have

$$\frac{R_{nl}}{R_l} = \frac{4G^2R}{G^2+g^2} \approx 2R \sim 1 \quad \text{for large } G, \tag{49}$$

where $R_l = 2\delta^2I_{ps}$. With $R_{nl} = 1$, we obtain the minimum measurable phase shift as

$$\delta_m = \sqrt{\frac{G^2+g^2}{8G^2RI_{ps}}} \approx \sqrt{\frac{1}{4RI_{ps}}} \quad \text{for large } G. \tag{50}$$

This result is the same as that for a linear interferometer. Notice that output noise is increased by a factor of $2G^2$ as compared to the vacuum noise level. So, even though the interference fringe is increased by a factor of $2G^2$, the quantum noise is also increased by a similar amount compared to a linear interferometer, leading to no increase in the SNR, as shown in Eq. (49). So, there is no advantage in the scheme of BS + PA as compared to a linear interferometer.

The amplified noise in the output of the interferometer is because the two fields from the beam splitter are not correlated and their noise will be amplified independently when fed to the two ports of the amplified. On the other hand, in the scheme of PA + PA, the input fields to the second amplifier are correlated quantum mechanically so that their noise can be canceled when superimposed at the second amplifier.

IV. CONCLUSION AND DISCUSSION

In summary, we investigated two schemes of nonlinear interferometers involving a PA and a BS with vacuum and squeezed state input at the unused port. We found that similar to the SU(1,1) interferometer with a coherent state boost [PA + PA scheme in Fig. 1(a)], the scheme of PA + BS [Fig. 1(b)] can also beat the standard quantum limit of phase-measurement sensitivity by a similar amount. But the reversed scheme of BS + PA [Fig. 1(c)] works at the SQL without any advantage over a traditional linear interferometer. The employment of the squeezed state in the unused port of the PA can increase the SNR and the sensitivity further.

The Heisenberg limit of $1/I_{ps}$ dependence can be reached when the gain of the PA is large with or without a coherent state boost. However, a coherent state injection will always decrease the sensitivity. On the other hand, a squeezed state input instead of the coherent state will improve the sensitivity by a factor of 2 at best.

It should be noted that the scheme of PA + BS requires the two outputs of the PA be frequency degenerate. Otherwise, a linear frequency converter [18] has to be used to replace the BS in order to mix the two fields of different frequencies [19].

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