Quantum phase transitions of three-level atoms interacting with a one-mode electromagnetic field

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We apply the energy surface method to study a system of N_a three-level atoms interacting with a one-mode radiation field in the Ξ, Λ, and V configurations. We obtain an estimation of the ground-state energy, the expectation value of the total number of excitations, and the phase diagram of the model in the interaction parameter space and compare the results with the exact solutions. The phase diagram separates the behavior of, in general, a dominant single state without excitations ($M = 0$) from a collective state with excitations ($M \neq 0$). We have first- and second-order phase transitions, except for the V configuration, which only presents second-order phase transitions.

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I. INTRODUCTION

The Tavis-Cummings model [1], which describes the interaction of a collection of N two-level atoms with a quantized electromagnetic field in the dipolar and rotating-wave approximations, has an extensive use in quantum optics [2]. Recently this model has been physically realized using a QED cavity with Bose-Einstein condensates [3,4]. Particularly interesting has been the investigation of the phase transitions of the system in the thermodynamic limit [5], and at zero temperature [6,7].

The physics of three-level systems interacting with one or two quantized modes of the electromagnetic field is very rich and many special dynamical situations have been studied. In particular, a formalism to describe one three-level atom interacting with a one- or a two-mode field has been discussed together with the atomic level occupation probabilities, coherence properties, photon probability distribution, fluctuations, and the evolution of squeezing in a series of works [8]. For one three-level atom interacting with a one- or a two-mode field, it was found that the phase distribution properties of the field reflect the collapses and revivals of the level occupation probabilities. However, for the two-mode case, there are exceptions and the collapses and revivals are decorrelated from the phase field [9].

A comprehensive review of the dynamical interaction of an atom with radiation within the framework of the Jaynes-Cummings type has been done by Yoo and Eberly [10]. However, by means of the thermodynamic Green's function, the phase transitions of the Dicke model, including all modes of the radiation field, have been considered for a finite temperature in Ref. [11] for atoms confined in a cubic resonance wavelength. Recently, there has been a semiclassical treatment of the phase transitions in a system of three-level atoms in the Λ configuration interacting with a two-mode quantized electromagnetic field without the rotating-wave approximation, using, however, the generalized Holstein-Primakoff transformation to find the separatrix of the system in the thermodynamic limit [12].

The main contribution of this manuscript is the establishment of the separatrix of a system of three-level atoms interacting with a one-mode electromagnetic field for the three atomic configurations, Ξ, Λ, and V (cf. Fig. 1). Each separatrix determines in control parameter space (dipolar strengths) where the quantum phase transitions take place and which are of first and second order, depending on the values of the dipolar interactions. The presence of the quantum phase transitions can be clearly seen in the calculation of the ground-state energies and in the expectation value of the total number of excitations, as functions of the dipolar couplings μ_{12}, μ_{13}, and μ_{23}. The agreement with the corresponding exact quantum calculations is remarkable, in spite of considering a small number N_a of atoms ($N_a = 2$ and $N_a = 10$). Furthermore, by taking $\mu_{23} = 0$ in the Ξ and Λ configurations, and $\mu_{13} = 0$ in the V case, the corresponding Hamiltonian systems describe two-level problems in the manner of Tavis-Cummings, giving consistency to our results. In these cases it is straightforward to check that the separatrix coincides with the one established by Hepp and Lieb [5].

The organization of the paper is as follows. In Sec. II we establish the model and present the basic formulation of the problem. In Sec. III an analytic expression for the energy surface of the system is obtained, and we use it to obtain an estimation of the ground state of the system in the three basic configurations. In Sec. IV we solve the Hamiltonian for a finite number of particles and compare the results with those previously obtained. Finally, a summary of the general results and conclusions is outlined in Sec. V.

II. MODEL HAMILTONIAN

We consider a quantum system of N_a three-level atoms, each atom being able to occupy one of three levels characterized by energies ω_1, ω_2, and ω_3 (where we have taken $\hbar = 1$), interacting dipolarly with a one-mode field of frequency Ω, and assume the rotating-wave approximation. We consider $\omega_1 \leqslant \omega_2 \leqslant \omega_3$. The Hamiltonian describing this system can be written as Ref. [10]

$$\hat{H} = \Omega\,\hat{a}^\dagger\hat{a} + \sum_{k=1}^{3}\omega_k\,\hat{A}_{kk} - \frac{\mu_{12}}{\sqrt{N_a}}(\hat{a}\,\hat{A}_{21} + \hat{a}^\dagger\,\hat{A}_{12}) - \frac{\mu_{13}}{\sqrt{N_a}}(\hat{a}\,\hat{A}_{31} + \hat{a}^\dagger\,\hat{A}_{13}) - \frac{\mu_{23}}{\sqrt{N_a}}(\hat{a}\,\hat{A}_{32} + \hat{a}^\dagger\,\hat{A}_{23}), \quad (1)$$

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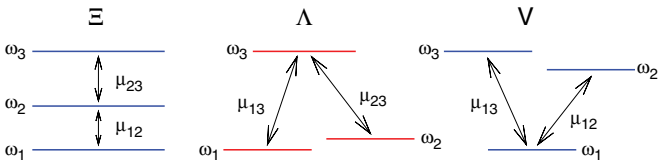

FIG. 1. (Color online) Schematic representation of the three atomic configurations. $\hbar\omega_i$ denotes the energy of the ith level, and μ_{ij} the dipolar coupling between levels i and j. We use $\hbar = 1$ throughout.

where the first two terms on the right-hand side are related to the free atomic and field parts. The $\hat{a}, \hat{a}^\dagger$ denote the annihilation and creation operators of the radiation field, and the operators $\hat{A}_{ij}$ are the generators of a U(3) algebra. The operators $\hat{a}$, $\hat{a}^\dagger$ satisfy the Heisenberg-Weyl algebra commutators

$$[\hat{a}, \hat{a}^\dagger] = \hat{1}, \quad [\hat{a}, \hat{a}] = 0,$$

while the atomic operators $\hat{A}_{ij}$ (weight for $i = j$, lowering for $i > j$ and raising with $i < j$) fulfill the U(3) algebra commutation relations

$$[\hat{A}_{ij}, \hat{A}_{kl}] = \delta_{jk}\hat{A}_{il} - \delta_{il}\hat{A}_{kj}.$$

The usual three-level atomic arrangements are obtained from Eq. (1), imposing $\mu_{13} = 0$ for the Ξ configuration, $\mu_{12} = 0$ for the Λ configuration, or $\mu_{23} = 0$ for the V configuration. In each case there are two constants of motion: the total number of atoms

$$\hat{N}_a = \hat{A}_{11} + \hat{A}_{22} + \hat{A}_{33}, \tag{2}$$

and the total number of excitations

$$\begin{aligned} &\hat{M}_\Xi = \hat{a}^\dagger\hat{a} + \hat{A}_{22} + 2\,\hat{A}_{33}, \quad \hat{M}_\Lambda = \hat{a}^\dagger\hat{a} + \hat{A}_{33}, \\ &\hat{M}_V = \hat{a}^\dagger\hat{a} + \hat{A}_{22} + \hat{A}_{33}. \end{aligned} \tag{3}$$

III. ENERGY SURFACE

In order to obtain an energy surface, we use as a trial state the direct product of coherent states in each subspace. The use of coherent states as trial states lets us determine in analytic form the expectation values of matter and field observables. Thus we use the tensor product of Heisenberg-Weyl HW(1) coherent states for the radiation part, $|\alpha\} = e^{\alpha\,\hat{a}^\dagger}\,|0\rangle$, and U(3) coherent states for the atomic part. The un-normalized U(3) coherent state can be constructed by taking the exponential of the lowering generators acting on the highest weight states of U(3),

$$|[h_1,h_2,h_3]\gamma_1,\gamma_2,\gamma_3\} = e^{\gamma_3\hat{A}_{21}}\,e^{\gamma_2\hat{A}_{31}}\,e^{\gamma_1\hat{A}_{32}}\,|\,[h_1,h_2,h_3]\rangle,$$

where we denote by

$$|\,[h_1,h_2,h_3]\rangle \equiv \left|\begin{matrix} h_1 & h_2 & h_3 \\ & h_1 & h_2 \\ & & h_1 \end{matrix}\right\rangle,$$

the highest weight state of the Gelfand-Tsetlin basis of the irreducible representation $[h_1, h_2, h_3]$ of U(3) [13]. The action of the raising operators $\hat{A}_{ij}$ on the highest weight state vanishes: $\hat{A}_{ij}\,|\,[h_1, h_2, h_3]\rangle = 0$.

In this contribution we only consider the completely symmetric representation $[N_a, 0, 0]$ of U(3), where N_a denotes the number of atoms. Therefore the trial state is the tensor product

$$|N_a,\alpha,\gamma_2,\gamma_3\} = e^{\alpha\,\hat{a}^\dagger}|0\rangle \otimes e^{\gamma_3\hat{A}_{21}}\,e^{\gamma_2\hat{A}_{31}}|[N_a,0,0]\rangle.$$

The parameter γ_1 does not appear in the coherent state because $\hat{A}_{32}\,|\,[N_a, 0, 0]\rangle = 0$.

The expectation value with respect to this state of the model Hamiltonian (1) gives the energy surface

$$\begin{aligned} &\mathcal{H}(\varrho,\varrho_2,\varrho_3,\vartheta_1,\vartheta_2,\vartheta_3) \\ &\quad = \Omega\,\varrho^2 + N_a\frac{\omega_1 + \omega_2\,\varrho_3^2 + \omega_3\,\varrho_2^2}{1+\varrho_2^2+\varrho_3^2} - 2\sqrt{N_a}\varrho \\ &\qquad \times \frac{\mu_{12}\varrho_3\cos\vartheta_3 + \mu_{13}\varrho_2\cos\vartheta_2 + \mu_{23}\varrho_2\varrho_3\cos\vartheta_1}{1+\varrho_2^2+\varrho_3^2}, \end{aligned}$$

where we have used the identifications $\gamma_k = \varrho_k\,\exp(i\,\varphi_k)$, with $k = 2, 3$, and $\alpha = \varrho\,\exp(i\,\varphi)$. Additionally, the angles $\vartheta_1 = \varphi - \varphi_2 + \varphi_3$, $\vartheta_2 = \varphi - \varphi_2$, and $\vartheta_3 = \varphi - \varphi_3$ were defined.

The minima for the energy surface are obtained when $\vartheta_i = 0, \pi$ for $i = 1,2,3$, and

$$\mu_{12}\,\cos\vartheta_3 > 0, \quad \mu_{13}\,\cos\vartheta_2 > 0, \quad \mu_{23}\,\cos\vartheta_1 > 0.$$

To have an intensive quantity we divide the energy surface $\mathcal{H}(\varrho,\varrho_2,\varrho_3,\vartheta_1,\vartheta_2,\vartheta_3)$ by N_a, and use the conditions of a

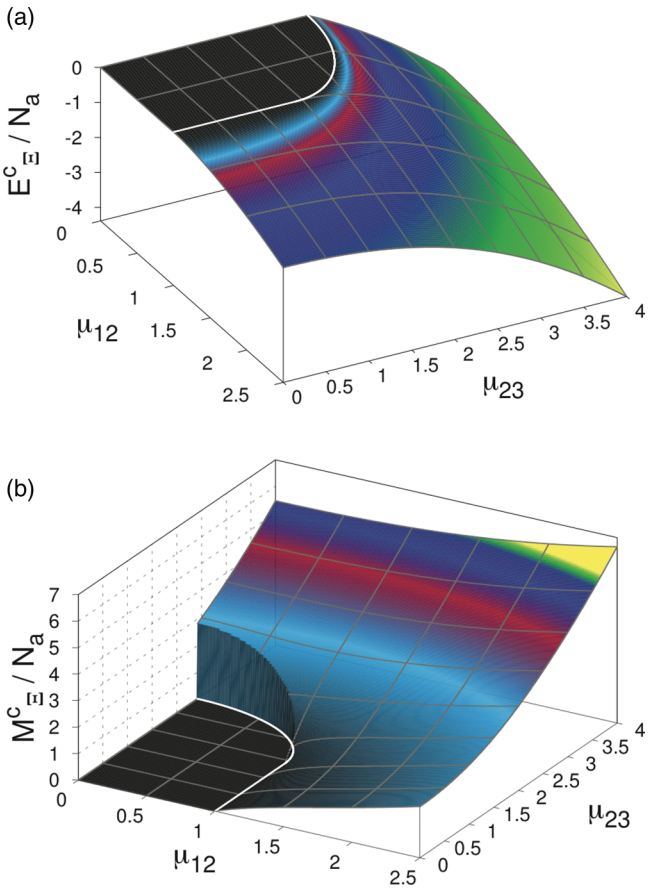

FIG. 2. (Color online) Numerical results for the lowest value of the energy surface for the Ξ configuration (a) and its corresponding estimation for the constant of motion M_Ξ (b), as functions of μ_{12} and μ_{23}. We consider the resonant case $\omega_{32} = \omega_{21} = 1$, with $\omega_1 = 0$.

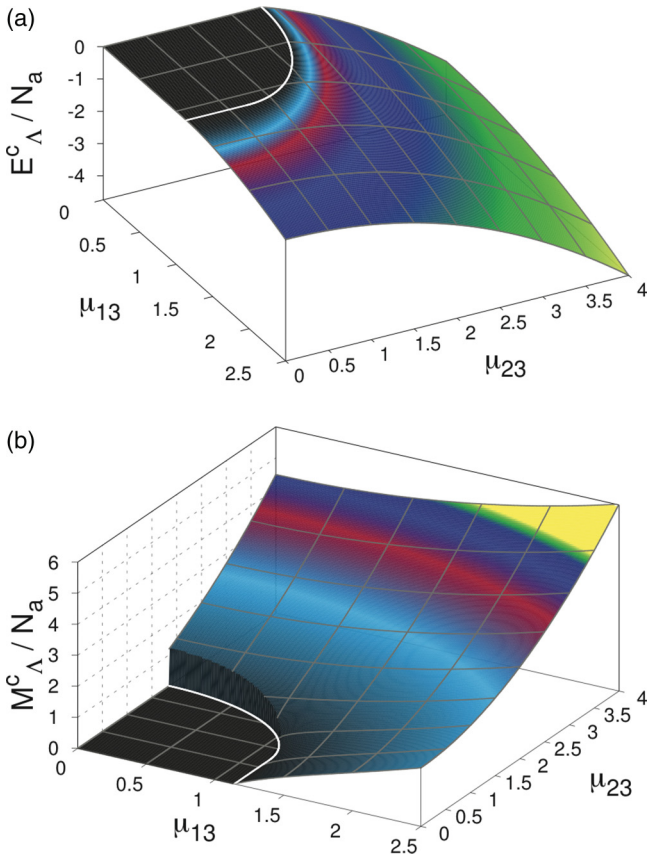

FIG. 3. (Color online) Numerical results for the lowest value of the energy surface for the Λ configuration (a) and its corresponding estimation for the constant of motion M_Λ (b), as functions of μ_{13} and μ_{23}. We consider the nonresonant case $\omega_{31} = 1.3$, $\omega_{32} = 0.8$, and $\omega_1 = 0$.

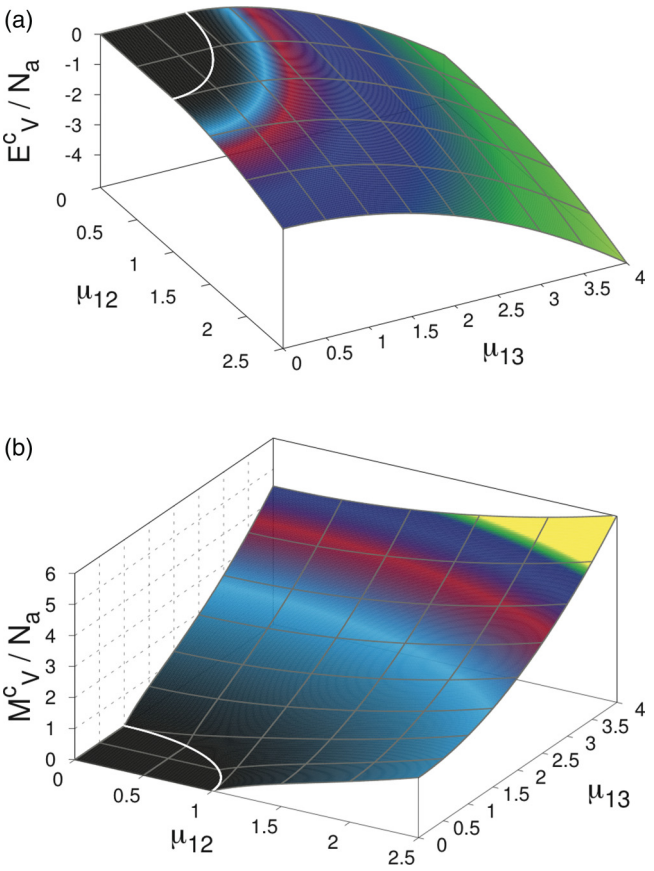

FIG. 4. (Color online) Numerical results for the lowest value of the energy surface for the V configuration (a) and its corresponding estimation for the constant of motion M_V (b), as functions of μ_{12} and μ_{13}. We consider $\omega_{31} = \omega_{21} = 1$ with $\omega_1 = 0$.

minimum for the angles ϑ_i to obtain

$$\mathcal{E}(\bar{\varrho}, \varrho_2, \varrho_3) = \bar{\varrho}^2 + \frac{\omega_1 + \omega_2\,\varrho_3^2 + \omega_3\,\varrho_2^2}{1 + \varrho_2^2 + \varrho_3^2} - 2\,\bar{\varrho}\,\frac{|\mu_{12}|\,\varrho_3 + |\mu_{13}|\,\varrho_2 + |\mu_{23}|\,\varrho_2\,\varrho_3}{1 + \varrho_2^2 + \varrho_3^2}, \tag{4}$$

where $\bar{\rho} = \rho/\sqrt{N_a}$, and ω_k and μ_{kl} are now measured in units of $[\hbar\Omega]$. $E = \mathcal{E}\,N_a\,\Omega$ is given in units of $[\hbar\Omega]$, and in what follows we will set $\Omega = 1$.

Using the Ritz variational principle, one finds the best variational approximation to the ground-state energy of the system and its corresponding eigenstate for the energy surface (4). The values of $\bar{\varrho}_c$ are always given in terms of ϱ_{2c} and ϱ_{3c}. These last two, independent of N_a, are found numerically. For the configuration Ξ the estimation of the ground energy is plotted in Fig. 2(a) as a function of μ_{12} and μ_{23}. In Fig. 2(b) we show also the corresponding value for the constant of motion M_Ξ. In both figures we used the resonant case $\omega_{32} = \omega_{21} = 1$, and $\omega_1 = 0$, with the notation $\omega_{ij} \equiv \omega_i - \omega_j$. The white lines indicate the separatrix of the configuration.

For the configuration Λ the result for the semiclassical ground energy is plotted in Fig. 3(a) as a function of μ_{13} and μ_{23}. In Fig. 3(b) we show the corresponding value of the constant of motion M_Λ. In both figures we used $\omega_{31} = 1.3$, $\omega_{32} = 0.8$, and $\omega_1 = 0$. Note that, in this case, we chose for illustrative purposes to work away from resonance. Again, the white lines denote the separatrix of the system.

Finally, considering the V configuration we obtain the semiclassical ground energy shown in Fig. 4(a) as a function of μ_{12} and μ_{13}, and in Fig. 4(b) the corresponding value of the constant of motion M_V is displayed. In both figures we used again the resonant case $\omega_{31} = \omega_{21} = 1$, with $\omega_1 = 0$. The white line is the corresponding separatrix for the configuration.

The separatrix of the system is obtained by analyzing the stability and equilibrium properties of the energy surface by means of the catastrophe formalism. Thus the locus of the points where the thermodynamic phase transition $M = 0 \rightarrow M \neq 0$ occurs is given in each case by the following:

For the Ξ configuration,

$$\mu_{12}^2 + [|\mu_{23}| - \sqrt{\omega_{31}}]^2\,\Theta(|\mu_{23}| - \sqrt{\omega_{31}}) = \omega_{21}; \tag{5}$$

for the Λ configuration,

$$\mu_{13}^2 + [|\mu_{23}| - \sqrt{\omega_{21}}]^2\,\Theta(|\mu_{23}| - \sqrt{\omega_{21}}) = \omega_{31}; \tag{6}$$

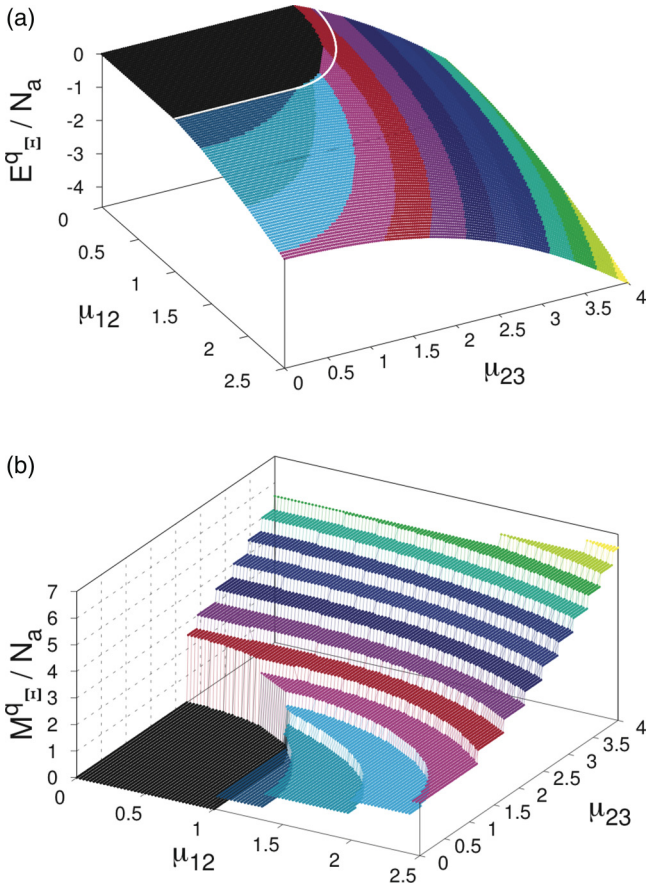

FIG. 5. (Color online) Numerical results for the ground-state energy of the model Hamiltonian for the Ξ configuration (a) and the corresponding value of the constant of motion M_Ξ (b), as functions of μ_{12} and μ_{23}. We consider the case $N_a = 2$, $\omega_{32} = \omega_{21} = 1$, and $\omega_1 = 0$. The white line corresponds to the semiclassical separatrix.

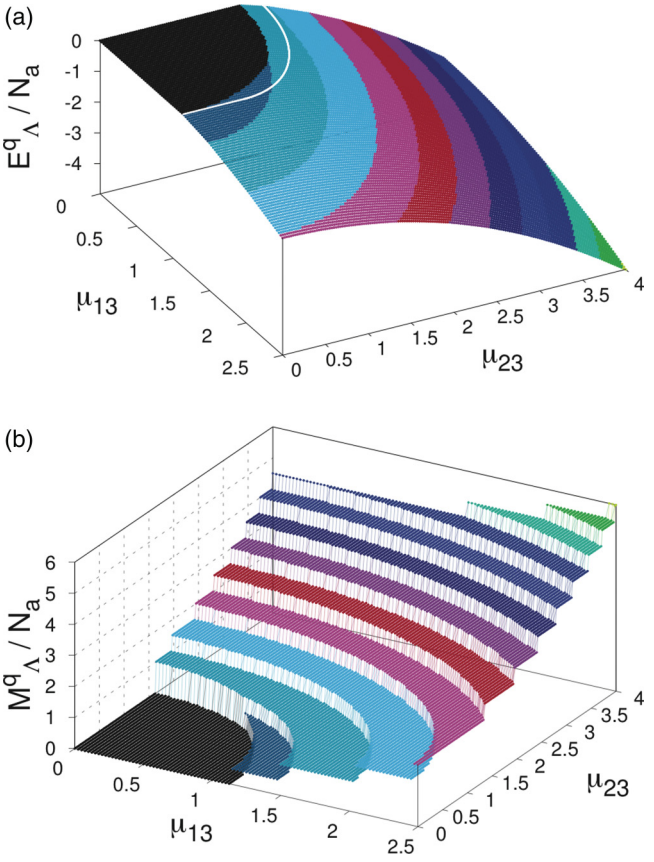

FIG. 6. (Color online) Numerical results for the ground-state energy of the model Hamiltonian for the Λ configuration (a) and the corresponding value of the constant of motion M_Λ (b), as functions of μ_{13} and μ_{23}. We consider the case $N_a = 2$, $\omega_{31} - \Omega = 0.3$, $\omega_{32} - \Omega = -0.2$, and $\omega_1 = 0$. The white line corresponds to the semiclassical separatrix.

and for the V configuration,

$$\frac{\mu_{12}^2}{\omega_{21}} + \frac{\mu_{13}^2}{\omega_{31}} = 1. \tag{7}$$

Each separatrix is indicating the minimum values of the dipole transition strengths between the levels where the total number of excitations can be different from zero, that is, establishes the regions where the ground state has a collective character formed by a linear combination of states with excited atoms and/or field excitations, as opposed to those in which it has, in general, a single-state configuration.

The Ehrenfest procedure [14] is used to determine numerically the order of the phase transitions of the system. We want to emphasize that for the cases of equal detuning or in resonance for the V and Λ configurations the order can be obtained analytically. In summary, the three-level atoms present first- and second-order phase transitions; the first ones are associated to discontinuities in the first derivative of the energy surface while the second ones to discontinuities of the second derivative of the energy surface [14]. The first-order phase transitions imply level crossings in the ground state of the system, while for the second-order ones there is an absence of level crossings in the ground state, when the control parameters are changed [15]. For the Ξ configuration, the phase transition across $\mu_{12} = \sqrt{\omega_{21}}$ in the separatrix is of second order, and the one that takes place along the segment of the circumference is of first order. For the Λ configuration something similar happens: across $\mu_{13} = \sqrt{\omega_{31}}$ in the separatrix the phase transition is of second order, and along the segment of circumference in the separatrix is of first order. For the V configuration all transitions are always of second order.

IV. QUANTUM CASE

In the quantum case we must construct the proper basis for each configuration taking into account (3). Thus the basis is characterized by N_a, the number of atoms, which also defines the U(3) highest weight state, and M, the total excitation number for the configuration considered.

The value of the matrix elements of the U(3) generators in the Hamiltonian (1) in the Gelfand-Tsetlin basis are given in Ref. [16]. Using the matrix elements, we calculate the matrix Hamiltonian and through a diagonalization procedure obtain the ground-state energies, and the expectation value of the total number of excitations. In some cases for all the configurations

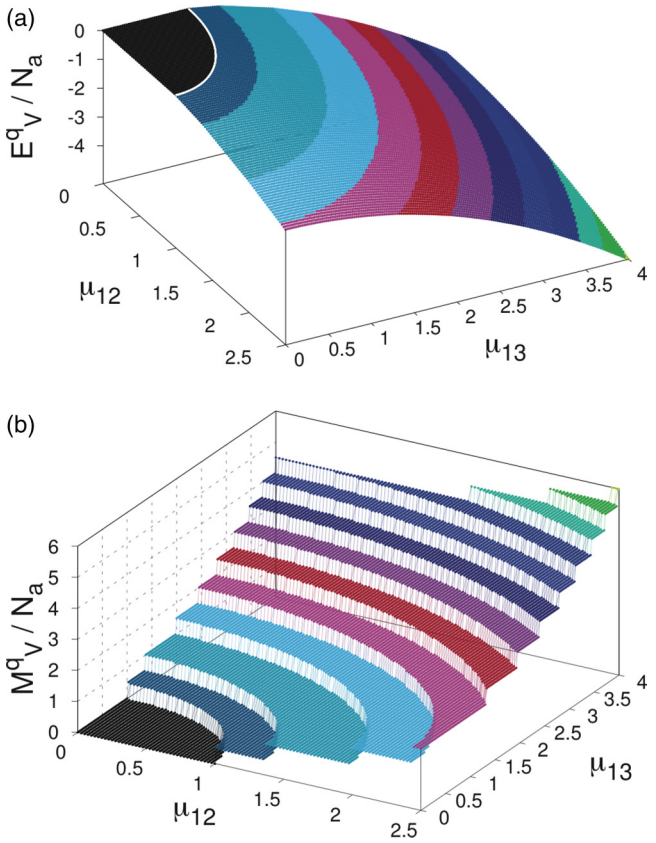

FIG. 7. (Color online) Numerical results for the ground-state energy of the model Hamiltonian for the V configuration (a) and the corresponding value of the constant of motion M_V (b), as functions of μ_{12} and μ_{13}. We consider the case $N_a = 2$, $\omega_{31} = \omega_{21} = 1$, and $\omega_1 = 0$. The white line corresponds to the semiclassical separatrix.

one can get analytic expressions for the lowest energy states for a few number of atoms. For example, for the Ξ configuration and one atom we obtain

$$E_{M_\Xi} = M_\Xi - \sqrt{M_\Xi\,\mu_{12}^2 + (M_\Xi - 1)\,\mu_{23}^2},$$

which coincides with the results given in Ref. [10] after an identification of the parameters used here.

In Figs. 5–7, we show the ground-state energy and the value of the constant of motion M for the Ξ, Λ, and V configurations as functions of the interaction parameters for $N_a = 2$ atoms and for the same parameter values as in the semiclassical case, for comparison purposes. We can observe that the resemblance between the semiclassical and quantum results is excellent. In the quantum case we also take notice of phase crossovers in the ground-state energy which occur every time the value of the constant of motion changes [17]. This may be seen as different shades in the energy surface and is better seen in the plot for M where there is a jump when a phase crossover takes place. For the Ξ configuration we see in Fig. 5 that there are phase transitions from the region $M_\Xi = 0$ directly to regions $M_\Xi = 1, 2, \ldots, 5$ for $N_a = 2$ atoms. When the number of atoms increases, greater values of M_Ξ are pulled toward the region with $M_\Xi = 0$, in such a way that there will be phase transitions from this value to greater M_Ξ's. For the other configurations, one has a similar effect when the number of atoms increases.

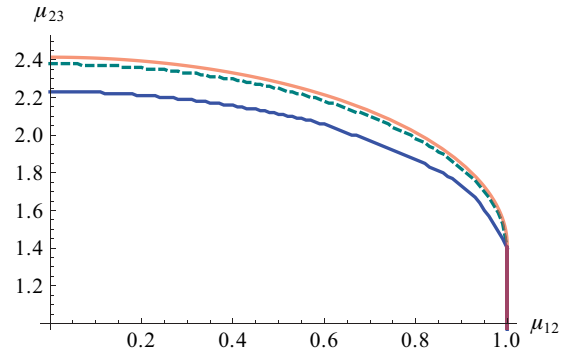

FIG. 8. (Color online) Comparison of the equipotential curves of the ground-state energy of the model Hamiltonian in the Ξ configuration for $N_a = 2, 10$, lower continuous and middle broken lines, respectively, with $\omega_{32} = \omega_{21} = 1$ and $\omega_1 = 0$. The upper line corresponds to the semiclassical separatrix, to which all tend as $N_a \to \infty$.

When the number of atoms N_a increases, all the curves where the constant of motion changes tend to the classical separatrix. This behavior is shown in Fig. 8, where we plot the equipotential curves for the ground-state energy of the model in the Ξ configuration, with $N_a = 2, 10$ atoms for $M_\Xi = 0$, and compare with the semiclassical separatrix.

V. CONCLUSIONS

We studied the three main configurations of a system of N_a three-level atoms interacting with a one-mode radiation field, which is the simplest generalization of the Tavis-Cummings model. Using HW(1) and U(3) coherent states, we established the energy surface for the system and obtained an approximate expression for the ground state of the system, as well as the separatrix in the parameters of the model, which defines the locus of the quantum phase transitions. Although the phase transitions in all the configurations are similar, in the thermodynamic limit there are only two zones, and the precise form for the V configuration is qualitatively different to the Ξ and Λ configurations. For the Ξ and Λ configurations, the phase transition from one region to the other can be of first or second order, depending on the zone where the separatrix is crossed; for the V configuration all phase transitions are of second order. The comparison with the exact ground state of the model shows that our approximation is excellent, at least for the energy and the expectation value of the constant of motion of each configuration. We were also able to verify numerically that when the number of atoms goes to infinity, the multitude of quantum phase crossovers tend to the thermodynamic limit.

ACKNOWLEDGMENTS

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