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Bethe-logarithm calculation using the B-spline method

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A method based on the B-spline basis set is developed that significantly simplifies the Bethe-logarithm calculations for the atomic hydrogen. Without any auxiliary optimization and extrapolation, this method not only can calculate the Bethe logarithms of low-lying states to high precision using relatively small basis sets, but can also calculate high-lying Rydberg states efficiently. For the ground state, the Bethe logarithm is calculated to 24 significant figures using 450 B-spline functions. For high-lying Rydberg states, we can reproduce all the results of Jentschura and Mohr [Phys. Rev. A **72**, 012110 (2005)].

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I. INTRODUCTION

In 1947, Lamb and Retherford [1] discovered that there was a small difference in energy between the $2s_{1/2}$ and $2p_{1/2}$ states of atomic hydrogen using a microwave method, which was in disagreement with the prediction of Dirac theory stating that the $2s_{1/2}$ and $2p_{1/2}$ states are degenerate. In the same year, a satisfactory explanation for this shift was given by Bethe using a radiation field theory [2]. In general, the lowest-order quantum electrodynamic (QED) correction for a hydrogen energy level can be written as [3] (in atomic units throughout unless otherwise stated)

$$\triangle E_L(nlj) = \frac{4}{3}\frac{\alpha^3}{\pi n^3}\left\{\delta_{l0}\left[\ln(\alpha^{-2}) + \frac{11}{24} - \frac{1}{5}\right] - \ln\left[k_0(nl)/R_\infty\right] + \frac{3}{8}\frac{\frac{\delta_{j,l+\frac{1}{2}}}{l+1} - \frac{\delta_{j,l-\frac{1}{2}}}{l}}{(2l+1)}\right\}, \quad (1)$$

where $\ln[k_0(nl)/R_\infty]$ is called the Bethe logarithm and is labeled $\beta(nl)$. The Bethe logarithm is an essential part of the Lamb shift and is the most difficult to calculate as well. For the atomic hydrogen, the Bethe logarithms of the ground state and low-lying states have been calculated by many researchers [4–10], including the group theoretical method of Lieber and Huff [4,5], the iteration-variation method of Goldman [6], the discrete technique with a higher-order Neville-Richardon extrapolation method of Haywood and Morgan [7], the momentum-space finite basis-set approach of Mallampalli and Sapirstein [8], and the finite basis-set method of Drake and Goldman involving a large range of nonlinear parameters [9,10]. The most noteworthy is that the Drake-Goldman method has been also successfully applied to the calculations of the Bethe logarithms for helium [10,11] and lithium [12,13].

In recent years, attention has been drawn to Rydberg states of one-electron systems. One reason is that the theory of Rydberg states with a long natural lifetime can be formulated more precisely than for low-lying states. Combining precision frequency measurements with QED calculations for Rydberg states of hydrogenlike ions, one can search for information on physical constants of nature [14]. Jentschura *et al.* proposed the application of this idea to investigate the Rydberg constant [15] and nuclear masses [16]. Experiments are underway on Rydberg states in hydrogenlike systems. For example, Tan *et al.* developed an experimental method for one-electron ions in circular Rydberg states at the National Institute of Standards and Technology (NIST) [17]. Thus the investigation of hydrogenic Bethe logarithms of Rydberg states is of great importance. Jentschura and Mohr recently calculated the Bethe logarithms for states with high n and l ($n \leqslant 200$, $l \leqslant 199$) using an integral representation method and a spectral representation method [18]. Both methods are adapted to different ranges of principal and angular momentum quantum numbers. They noted that their methods are relatively complex and rely partly on multiprecision arithmetic libraries due to the numerical difficulties. To our knowledge, there are no other methods that have been applied to the calculations of hydrogenic Bethe logarithms for such Rydberg states. It is necessary and important to perform an independent verification of Jentschura and Mohr's calculations using an independent, alternative method [18].

With this motivation, we present here a simple and efficient method for calculating the hydrogen Bethe logarithm. The method is based on the B-spline basis set with an exponential distribution of knots. It can be applied not only to calculate low-lying states with high precision using relatively small basis sets, but also to calculate highly-excited Rydberg states with the principal quantum number $n > 200$. This paper is organized as follows. An outline of the Bethe logarithm is given in Sec. II, together with the details of our computational method. Some selected results are presented in Sec. III. Finally, a summary is given in Sec. IV.

II. COMPUTATIONAL METHOD

A. Bethe logarithm

In the acceleration gauge, the Bethe logarithm of the hydrogen atom can be represented in the form [9]

$$\beta(nl) = \frac{B}{C}, \quad (2)$$

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where

$$B = \sum_{n'} \int \frac{\left|\langle \psi_{nl}|\frac{\vec{r}}{r^3}|\psi_{n'l'}\rangle\right|^2 \ln|E_{n'} - E_n|}{E_{n'} - E_n} \tag{3}$$

and

$$C = \begin{cases} \sum_{n'} \int \frac{|\langle \psi_{nl}|\frac{\vec{r}}{r^3}|\psi_{n'l'}\rangle|^2}{E_{n'} - E_n} & \text{for } s \text{ state} \\ \frac{2}{n^3} & \text{otherwise.} \end{cases} \tag{4}$$

Here, E_n is the energy of the state of interest $|\psi_{nl}\rangle$, $E_{n'}$ is one of the intermediate states with the corresponding eigenfunction $|\psi_{n'l'}\rangle$, and the summation-integration symbol represents a summation over all bound states and an integration over the continuum. The wave function of hydrogen can be written as

$$\psi_{nlm}(\vec{r}) = f_n(r) Y_{lm}(\hat{r}), \tag{5}$$

where $Y_{lm}(\hat{r})$ is the spherical harmonics, and $f_n(r)$ is the radial wave function normalized according to

$$\int_0^\infty |f_n(r)|^2 r^2 dr = 1. \tag{6}$$

The matrix element $\langle \psi_{nl}|\frac{\vec{r}}{r^3}|\psi_{n'l'}\rangle$ can be evaluated explicitly by

$$\begin{aligned} &\langle \psi_{nl}|\frac{\vec{r}}{r^3}|\psi_{n'l'}\rangle \\ &= \sum_{\mu} (-1)^m \begin{pmatrix} l & 1 & l' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & 1 & l' \\ m & \mu & m' \end{pmatrix} \\ &\quad \times \left[\frac{3(2l+1)(2l'+1)}{4\pi}\right]^{\frac{1}{2}} \int_0^\infty f_n(r) \frac{1}{r^2} f_{n'}(r) r^2 dr. \end{aligned} \tag{7}$$

B. Structure of basis set

When calculating the Bethe logarithm using a finite basis set, the structure of the basis set is essential. One has to consider carefully two special requirements for constructing a basis set [10]. First, the basis functions used in the intermediate states should include the initial state, which can improve the rate of convergence. Second, the basis set can generate intermediate states of high-enough energies, which is critical in the calculation of the Bethe logarithm; in other words, the major contribution to the $\beta(nl)$ lies in the continuum states of high energy. More detailed information on the calculation of the Bethe logarithm can be found in [10].

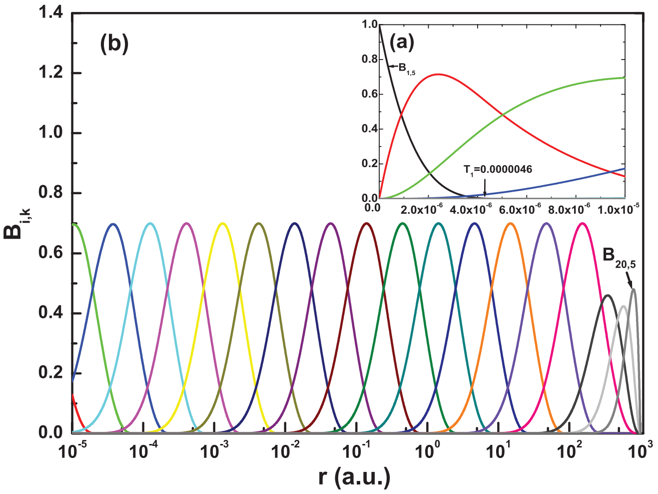

FIG. 1. (Color online) Twenty B splines of order $k = 5$ with an exponential distributing knot sequence. (a) The B splines in the range of 0 to 1×10^{-5}. (b) The B splines in the range of 1×10^{-5} to 1000. T_1 represents the first internal knot point, i.e., $T_1 = t_{k+1}$, and the knot parameter $\alpha = 0.021$.

In this work, the radial wave function of the initial and intermediate states are constructed by the same finite B-spline basis set,

$$f_n(r) = \sum_{i=1}^{N} C_i^n B_i(r), \tag{8}$$

where N is the number of B splines [19]. It is obvious that this type of basis set satisfies the first requirement mentioned above. A B-spline function is a set of piecewise polynomials of certain order defined in a finite region $[0,R]$. This finite region is designed as a set of monotonically increasing points called the knot sequence $\{t_i\}$. We denote $B_{i,1}$ to be a B-spline function of order one, defined by

$$B_{i,1}(r) = \begin{cases} 1, & t_i \leqslant r < t_{i+1} \\ 0, & \text{others.} \end{cases} \tag{9}$$

TABLE I. Convergence study of the Bethe logarithm for the 1s state of hydrogen using k = 15th-order B-spline basis set, confined in a box of size 100 a.u. Here, N is the number of kth-order B splines, $\beta(1s)$ is the 1s-state Bethe logarithm, and C is the denominator defined in Eq. (4).

N	$\beta(1s)$	$C(1s)$
50	2.290 980 4	1.999 999 87
100	2.290 981 372 6	1.999 999 999 999 977
200	2.290 981 375 205 552 09	2.000 000 000 000 000 02
300	2.290 981 375 205 552 301 336	2.000 000 000 000 000 000 006 3
400	2.290 981 375 205 552 301 341 2	1.999 999 999 999 999 999 999 98
450	2.290 981 375 205 552 301 342 529 8	1.999 999 999 999 999 999 999 999 76
Huff [5]	2.290 981 375 205 552 301 342 544 96	
Drake and Goldman [10]	2.290 981 375 205 552 301 33	

TABLE II. Values of the Bethe logarithm for the ground state of hydrogen at different knot sequences. Here, α is the parameter in Eq. (11), T_1 is the first interior knot point, $E_{\max}$ is the highest energy, and $\delta\beta(1s)$ is the difference from the exact value. All calculations are carried out using the same set of parameters: $N = 300$, $k = 15$, and $R = 200$. Digits in italics are not converged.

α	T_1	$E_{\max}$	$\beta(1s)$	$\delta\beta(1s)$
0.005	4.10×10^{-1}	5.76×10^{3}	2.2*58*	3.30×10^{-2}
0.025	2.41×10^{-2}	1.59×10^{6}	2.28*9 0*	1.98×10^{-3}
0.035	4.59×10^{-3}	4.32×10^{7}	2.290 *61*	3.71×10^{-4}
0.045	8.06×10^{-4}	1.38×10^{9}	2.290 9*15*	6.64×10^{-5}
0.065	2.17×10^{-5}	1.83×10^{12}	2.290 97*9 6*	1.83×10^{-6}
0.075	3.43×10^{-6}	7.23×10^{13}	2.290 981 *09*	2.90×10^{-7}
0.085	5.32×10^{-7}	2.96×10^{15}	2.290 981 3*30*	4.53×10^{-8}
0.105	1.23×10^{-8}	5.38×10^{18}	2.290 981 37*4 1*	1.07×10^{-9}
0.115	1.84×10^{-9}	2.36×10^{20}	2.290 981 375 *05*	1.61×10^{-10}
0.125	2.74×10^{-10}	1.05×10^{22}	2.290 981 375 *18*	2.56×10^{-11}
0.135	4.04×10^{-11}	4.75×10^{23}	2.290 981 375 202 *0*	3.57×10^{-12}
0.145	5.94×10^{-12}	2.16×10^{25}	2.290 981 375 205 *02*	5.28×10^{-13}
0.165	1.26×10^{-13}	4.65×10^{28}	2.290 981 375 205 5*4 1*	1.14×10^{-14}
0.175	1.83×10^{-14}	2.17×10^{30}	2.290 981 375 205 55*0 6*	1.67×10^{-15}
0.185	2.65×10^{-15}	1.03×10^{32}	2.290 981 375 205 552 *06*	2.42×10^{-16}
0.195	3.82×10^{-16}	4.86×10^{33}	2.290 981 375 205 552 *27*	3.53×10^{-17}
0.205	5.49×10^{-17}	2.32×10^{35}	2.290 981 375 205 552 2*96*	5.08×10^{-18}
0.225	1.13×10^{-18}	5.35×10^{38}	2.290 981 375 205 552 301 *24*	1.05×10^{-19}
0.235	1.60×10^{-19}	6.31×10^{39}	2.290 981 375 205 552 301 3*35 5*	7.04×10^{-21}
Huff [5]			2.290 981 375 205 552 301 342 544 96	

The B-spline functions of order k are built up from $B_{i,1}$ using the recursion relation

$$B_{i,k}(r) = \frac{r - t_i}{t_{i+k-1} - t_i} B_{i,k-1}(r) + \frac{t_{i+k} - r}{t_{i+k} - t_{i+1}} B_{i+1,k-1}(r). \tag{10}$$

To obtain precise bound states, as well as continuum states of high-enough energies, we employ a knot sequence according to

$$\begin{aligned} t_i &= 0, \quad i = 1,2,\ldots,k, \\ t_i &= R\frac{e^{(\lambda \frac{i-1}{n-1})} - 1}{e^{\lambda} - 1}, \quad i = k+1, k+2, \ldots, N, \\ t_i &= R, \quad i = N+1, N+2, \ldots, N+k-1, \end{aligned} \tag{11}$$

where $\lambda = \alpha R$, with α being an adjustable parameter. The knot sequence consists of knots of multiplicity k at $r = 0$, knots of multiplicity $k - 1$ at $r = R$, and simple knots at interior points. In Fig. 1, 20 B splines of order 5 are given. These B splines are defined in the finite region [0, 1000] with the knot sequence as defined above. Figure 1(a) shows the B splines in the range of 0 to 1×10^{-5}, and Fig. 1(b) shows the B splines in the range of 1×10^{-5} to 1000. Also in the figure, T_1 is the first internal knot point. This choice makes the last B splines to be zero at $r = R$, so that the basis set satisfies the boundary condition of the wave function at $r = R$. There are different asymptotic behaviors between s and non-s states at origin $r = 0$. The wave

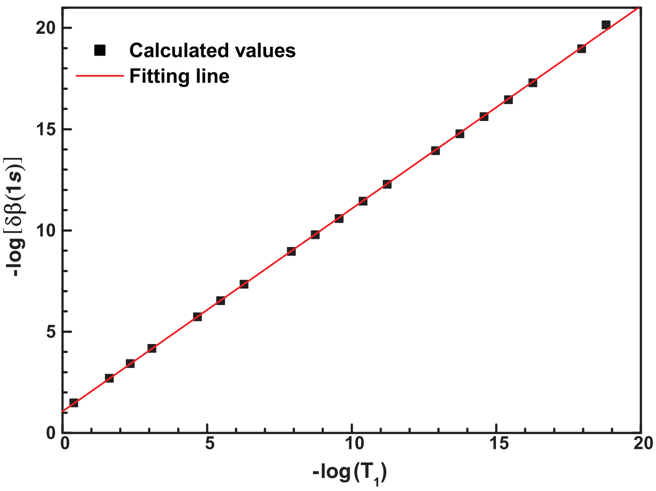

FIG. 2. (Color online) A relation between the first interior knot T_1 and the precision $\delta\beta(1s)$.

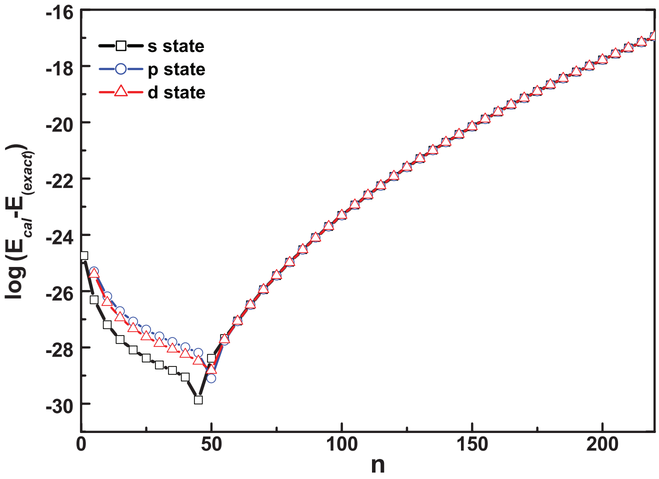

FIG. 3. (Color online) The difference between our calculated results and the exact energy levels for Rydberg states with the principal quantum number $n \leqslant 220$ and the orbital angular momentum quantum number $l \leqslant 2$.

TABLE III. Bethe logarithms for s, p, and d states using 3000 B spline with $k = 11$. All decimal digits shown are significant.

n	$\beta(ns)$	$\beta(np) \times 10^2$	$\beta(nd) \times 10^3$	$\beta(nf) \times 10^3$
2	2.811 769 894 (1)	– 3.001 670 863 (1)		
3	2.767 663 612 (1)	– 3.819 022 939 (1)	– 5.232 148 141 (1)	
4	2.749 811 840(1)	– 4.195 489 460 (1)	– 6.740 938 877 (1)	– 1.733 661 482 (1)
5	2.740 823 728 (1)	– 4.403 469 559 (1)	– 7.600 751 258 (1)	– 2.202 168 381 (1)
6	2.735 664 207 (1)	– 4.531 219 769 (1)	– 8.147 203 962 (1)	– 2.502 179 760 (1)
7	2.732 429 129 (1)	– 4.615 517 726 (1)	– 8.519 223 294 (1)	– 2.709 095 727 (1)
8	2.730 267 261 (1)	– 4.674 135 200 (1)	– 8.785 042 984 (1)	– 2.859 114 559 (1)
9	2.728 751 166 (1)	– 4.716 569 995 (1)	– 8.982 032 294 (1)	– 2.971 901 488 (1)
10	2.727 646 939 (1)	– 4.748 289 336 (1)	– 9.132 272 249 (1)	– 3.059 094 279 (1)
20	2.723 967 084 (1)	– 4.860 818 451 (1)	– 9.694 501 704 (1)	– 3.399 111 574 (1)
30	2.723 247 984 (1)	– 4.884 701 661 (1)	– 9.823 005 803 (1)	– 3.481 789 018 (1)
40	2.722 991 205 (1)	– 4.893 508 814 (1)	– 9.871 887 451 (1)	– 3.514 138 599 (1)
50	2.722 871 077 (1)	– 4.897 701 457 (1)	– 9.895 562 973 (1)	– 3.530 062 203 (1)
60	2.722 805 384 (1)	– 4.900 019 544 (1)	– 9.908 798 484 (1)	– 3.539 058 061 (1)
70	2.722 765 592 (1)	– 4.901 434 390 (1)	– 9.916 939 183 (1)	– 3.544 632 085 (1)
80	2.722 739 679 (1)	– 4.902 360 892 (1)	– 9.922 300 373 (1)	– 3.548 323 104 (1)
90	2.722 721 868 (1)	– 4.903 000 433 (1)	– 9.926 017 221 (1)	– 3.550 892 879 (1)
100	2.722 709 102 (1)	– 4.903 460 354 (1)	– 9.928 699 409 (1)	– 3.552 753 548 (1)
110	2.722 699 641 (1)	– 4.903 802 125 (1)	– 9.930 698 160 (1)	– 3.554 143 907 (1)
120	2.722 692 436 (1)	– 4.904 063 005 (1)	– 9.932 227 386 (1)	– 3.555 210 080 (1)
130	2.722 686 822 (1)	– 4.904 266 644 (1)	– 9.933 423 414 (1)	– 3.556 045 548 (1)
140	2.722 682 363 (1)	– 4.904 428 641 (1)	– 9.934 376 457 (1)	– 3.556 712 376 (1)
150	2.722 678 763 (1)	– 4.904 559 621 (1)	– 9.935 148 141 (1)	– 3.557 253 075 (1)
160	2.722 675 815 (1)	– 4.904 667 027 (1)	– 9.935 781 725 (1)	– 3.557 697 561 (1)
170	2.722 673 370 (1)	– 4.904 756 193 (1)	– 9.936 308 298 (1)	– 3.558 067 379 (1)
180	2.722 671 320 (1)	– 4.904 831 027 (1)	– 9.936 750 669 (1)	– 3.558 378 362 (1)
190	2.722 669 583 (1)	– 4.904 894 445 (1)	– 9.937 125 879 (1)	– 3.558 642 359 (1)
200	2.722 668 101 (1)	– 4.904 948 655 (1)	– 9.937 446 866 (1)	– 3.558 868 381 (1)
210	2.722 666 824 (1)	– 4.904 999 668 (1)	– 9.937 774 347 (1)	– 3.559 063 377 (1)

function of s symmetry at $r = 0$ has a finite value, whereas it is zero for other states of non-s symmetry. In general, one removes the first B spline to satisfy the bound condition of non-s states. In order to keep the completeness of a basis set, all B splines are included in our calculations. In the present work, all matrix elements are evaluated to near machine precision using Gaussian-Legendre quadratures. Since the Hamiltonian and overlap matrices, obtained using the B-spline basis, are banded, a banded diagonalization subroutine DSBGV from LAPACK [20] is employed. Then the infinite summation and integration over intermediate states in Eqs. (3) and (4) are transformed into a summation over the finite B-spline states.

III. NUMERICAL RESULTS AND DISCUSSION

Table I is a convergence study of the Bethe logarithm for the $1s$ state of hydrogen. We used the B-spline basis set of order $k = 15$ confined in a box with the size of 100 a.u. In Table I, N represents the number of kth-order B splines, i.e., the dimension of the basis set, and C is the denominator in the Bethe-logarithm expression given by Eq. (4), which has the exact value of 2. We also list the result of Huff [5] who used the group theoretical method, and the extrapolated value of Drake and Goldman [10] who used their finite basis-set method with the size of basis set up to $N = 210$. It is shown in Table I that the convergence of $\beta(1s)$ has reached to 23 significant digits using relatively small basis sets. As N increases progressively, $\beta(1s)$ converges very rapidly until $N > 300$, where the rate of convergence starts to drop. When we change k and R to be 14 and 300, respectively, one more significant digit can be obtained for $\beta(1s)$.

The ability of our method to obtain highly accurate results is due to the use of the exponential distribution of the knot sequence, which allows higher-energy states to exist in the energy spectra. To determine a relation among the knot distribution, the energy spectrum, and the precision, we list in Table II the values of the Bethe logarithm for the ground state of hydrogen calculated for 19 different sets of knot distributions. We keep the number of B-spline basis functions to be 300 and fix the size of the box to be 200. In Table II, T_1 represents the first internal knot and $E_{\max}$ is the highest energy in the spectrum, i.e., the last energy eigenvalue obtained by the diagonalization of the Hamiltonian.

One may see from Table II that as the knot distribution becomes closer to the nucleus, higher-energy states appear, resulting in more precise values of $\beta(1s)$. The first internal knot point T_1 can directly reflect the final precision in the Bethe logarithm, as demonstrated by the linear line in Fig. 2. The fitting of the curve gives rise to the following equation:

$$\log[\delta\beta(1s)] = -1.06001 + 1.00146 \times \log(T_1)\,, \qquad (12)$$

which provides a useful estimation for higher-lying s states.

TABLE IV. Bethe logarithms for circular states $(l+1,l)$ with $4 \leqslant l \leqslant 43$.

l	$\beta(l+1,l) \times 10^5$	l	$\beta(l+1,l) \times 10^6$	l	$\beta(l+1,l) \times 10^6$	l	$\beta(l+1,l) \times 10^6$
4	− 77.209 890 154 (1)	14	− 20.014 384 872 (1)	24	− 4.033 528 104(1)	34	− 1.427 140 814 (1)
5	− 40.792 616 829 (1)	15	− 16.313 053 763 (1)	25	− 3.571 536 037 (1)	35	− 1.308 801 170 (1)
6	− 24.090 825 872 (1)	16	− 13.470 635 449 (1)	26	− 3.177 478 119 (1)	36	− 1.203 190 769 (1)
7	− 15.386 450 096 (1)	17	− 11.251 829 398 (1)	27	− 2.839 314 416 (1)	37	− 1.108 643 220 (1)
8	− 10.414 809 250 (1)	18	− 9.494 620 181 (1)	28	− 2.547 477 155 (1)	38	− 1.023 747 182 (1)
9	− 7.372 497 858 (1)	19	− 8.084 977 837 (1)	29	− 2.294 298 277 (1)	39	− 0.947 301 967 (1)
10	− 5.407 926 523 (1)	20	− 6.941 065 661 (1)	30	− 2.073 583 590 (1)	40	− 0.878 281 749 (1)
11	− 4.083 367 938 (1)	21	− 6.003 118 199 (1)	31	− 1.880 292 605 (1)	41	− 0.815 806 568 (1)
12	− 3.158 151 892 (1)	22	− 5.226 798 437 (1)	32	− 1.710 295 447 (1)	42	− 0.759 118 688 (1)
13	− 2.492 497 382 (1)	23	− 4.578 744 771 (1)	33	− 1.560 186 576 (1)	43	− 0.707 563 249 (1)

It would be interesting to compare the present method with the method of Drake and Goldman [9,10]. The *B*-spline method is based on a relatively high-order *B*-spline basis set with exponential distribution knots, and the parameter α is determined by a simple linear relation, as shown in Table II. In this respect, our method is similar to the Drake and Goldman's finite basis-set method using a huge range of nonlinear parameters. In their work and in other finite basis-set methods [6–8], one can find that only the ground state and low-lying states have been calculated, but rarely for Rydberg states of $n > 10$. This is because the use of a large number of basis functions with nonlinear parameters may suffer from the problem of near linear dependence. However, a significant advantage of using *B* splines as basis functions is that they are not only complete, but also are numerically stable, highly localized, flexible to construct, and capable of properly describing highly-excited Rydberg states [21]. The Bethe logarithms of states with the principal quantum number $n \leqslant 220$ and the orbital angular momentum quantum number $l \leqslant 200$ were calculated using the present *B*-spline method. The selected results of some Rydberg states are shown in Fig. 3 and Tables III and IV.

The differences between our present values and the exact energy levels of the Rydberg states with the principal quantum number $n \leqslant 220$ and the orbital angular momentum quantum number $l \leqslant 2$ are plotted in Fig. 3. In our calculations, we use 3000 *B*-spline basis functions, together with the radial parameter $R = 110\,000$. From Fig. 3, one can see that our calculated results agree with the exact energy values within 10^{-17}. Note that the exact energy level of hydrogen is $-1/(2n^2)$. The same parameters are used to calculate the Bethe logarithm of these Rydberg states. In Table III, we list the results of the Bethe logarithm for s, p, d, and f states with principal quantum number n up to 210. All digits displayed are significant ones. We have reproduced Jentschura and Mohr's results [18], which are among the most comprehensive calculations of the Bethe logarithm recorded so far in literature. Jentschura and Mohr gave the following asymptotic expression for the s states that was obtained from the data listed in [22]:

$$\beta(n,l=0) \cong 2.72265434(5) + \frac{0.000000(5)}{n} + \frac{0.55365(5)}{n^2} - \frac{0.5993(5)}{n^3} + \frac{0.613(5)}{n^4} - \frac{0.60(5)}{n^5}. \qquad (13)$$

According to this expression, $\beta(210,0) = 2.722666824$. Our result is in good agreement with this value. Compared with their method, our method seems to be computationally easier. For a given angular momentum, we can obtain the Bethe logarithm for all needed states with almost the same accuracy in a single run. More importantly, we do not have any numerical difficulties for Rydberg states, even when the difference $n - l$ is large. Our calculations were carried out in quadruple precision arithmetics.

It is also interesting to calculate the Bethe logarithm of a circular Rydberg state of hydrogen where the principal quantum is $n = l + 1$. Circular Rydberg states play a crucial role in understanding the correspondence limit between classical and quantum physics. Experiments are underway on one-electron systems in circular Rydberg states [17]. The Bethe logarithm of a circular state is quite easy to calculate using our method. In Table IV, we list some circular Rydberg states in the range $4 \leqslant l \leqslant 43$. All decimal digits shown are significant.

IV. SUMMARY

In this paper, a simple and efficient method for calculating the Bethe logarithm of hydrogen has been presented. This method is based on the *B*-spline basis set with an exponential distribution of knot sequence. We have demonstrated that if we choose the knot distribution close enough to the nucleus, we can calculate the Bethe logarithm to high precision. We have also found a relation between the knot distribution and the energy spectrum, which can be used to predict the accuracy we can achieve for the Bethe logarithm. For the ground state, the Bethe logarithm has been evaluated to 24 significant digits. For high-lying Rydberg states, we have calculated the Bethe logarithm for these states with the principal quantum number $n \leqslant 220$ and the orbital angular momentum quantum number $l \leqslant 200$, which has reproduced all the results of Jentschura and Mohr.

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