Four-dimensional photonic lattices and discrete tesseract solitons

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We theoretically study discrete photonic lattices in more than three dimensions and point out that such systems can exist in continuous three-dimensional space. We study discrete diffraction in the linear regime and demonstrate the existence of four-dimensional (4D) *tesseract* solitons in nonlinear 4D periodic photonic lattices. Finally, we propose a design towards a potential realization of such periodic 4D lattices in experiments.

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Dimensionality is one of the key attributes of physical systems which determine their properties. This particularly holds for photonic lattices—linear or nonlinear photonic structures conventionally considered in one- and two-dimensional geometry, where the flow of light exhibits a plethora of intriguing phenomena which hold great potential for applications [1,2]. Paradigmatic phenomena occurring in nonlinear photonic lattices are discrete solitons [1,2]. Their prediction in one-dimensional (1D) photonic lattices in 1988 [3] has waited 10 years for the first experimental observation [4]. With the suggestion [5] and the experimental realization of optically induced photonic lattices, both in one [6] and two [7] spatial dimensions, novel excitations such as vortices [8,9] (that cannot occur in 1D) were discovered and explored. The one- and two-dimensional (2D) photonic lattices discussed here [1,2] are also referred to as waveguide arrays, because they are continuous along another dimension (second or third) along which the light propagates. Photonic crystals provide the opportunity to study versatile linear (e.g., see Ref. [10]) and nonlinear phenomena [11] in three spatial dimensions. In those systems one usually applies (continuous) Maxwell equations, but discrete lattice models may also be used when the so-called tight-binding approximation (TBA) is applicable [12]. Here we theoretically study discrete photonic lattices in more than three dimensions. We point out that such systems can exist in continuous three-dimensional (3D) space; that is, their experimental realization is not hindered due to the properties of our space. The properties of discrete diffraction and four-dimensional (4D) tesseract solitons are presented in 4D linear and nonlinear periodic discrete lattices. Finally, we propose a design towards a potential realization of such periodic 4D lattices in experiments.

The fact that a discrete lattice can have its dimension larger than the continuous space it is embedded in is known from complex networks [13]. For example, the Internet can be regarded as a network of dimension close to 4.5, and the network of airports close to 3, even though they are embedded in the 2D surface of Earth [13]. Complex networks have been scarcely considered in optical systems. We point out an interesting concept of complex networks of interacting fields called solitonets [14,15], where the interaction dynamics at each individual node in the system has infinite degrees of freedom [14,15]. Furthermore, it should be pointed out that methods for increasing dimensionality in (general) optical systems have been studied previously, for example, by using a delayed feedback of nonlinear optical resonators [16] or global nonlinear coupling of modes in a resonator [17].

We begin with a paradigmatic model—the discrete nonlinear Schrödinger equation (DNLS)—which describes the dynamics of light in lattices of various dimensions:

$$i\frac{d\psi_\alpha(t)}{dt} = -J\sum_{\beta\in N_\alpha}\psi_\beta - \gamma|\psi_\alpha|^2\psi_\alpha, \qquad (1)$$

where ψ_α is the complex amplitude describing the field, N_α denotes the sites that are coupled to the site α, J is the coupling (hopping) parameter that we assume to be equal between all coupled sites, and γ is the strength of the nonlinearity. Throughout the paper we use the following normalization: $\sum_\alpha |\psi_\alpha|^2 = 1$. This model was successfully used to describe the dynamics of light in 1D and 2D photonic lattices (waveguide arrays) [1,2]. It is applicable when the lattice wells are sufficiently deep, such that each well has a well-defined resonance, and the coupling between different lattice sites is weak. Thus, one can think of this model as describing a system of weakly coupled high-Q resonators.

In theory, any two pairs of resonators can be coupled, thus yielding versatile structures of complex networks of resonators, which calls for a more rigorous definition of dimension. The dimensionality D of any such network can be calculated by the following procedure: Let us choose one resonator and calculate the number of resonators [call it $N(l)$] that one can reach in l or fewer connections. The number $N(l)$ scales as $N(l) \sim l^D$ when $l \to \infty$. This procedure is somewhat altered from that used in Ref. [13] for usual complex networks due to the fact that it is the possibility of coupling rather than the Euclidean distance between the resonators that matters here. It is straightforward to verify that the dimensionality of a simple cubic lattice corresponds to half the number of nearest neighbors, but we emphasize that this is not a generally valid prescription. To see that, note that the well-known body-centered-cubic or face-centered-cubic lattices have dimension three as expected.

Up to this point the theoretical model (1) was general in the sense that coupling between any two pairs of resonators was possible. From this point on we focus on 4D "simple cubic" lattices defined as follows: every resonator is labeled by four indices, $\alpha = (i,j,k,l)$, and it is coupled to eight resonators labeled by $(i+1,j,k,l)$, $(i-1,j,k,l)$, $(i,j+1,k,l)$, $(i,j-1,k,l)$, $(i,j,k+1,l)$, $(i,j,k-1,l)$,

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$(i,j,k,l+1)$, and $(i,j,k,l-1)$. We proceed with a discussion of light propagation phenomena encountered in these 4D simple-cubic lattices. The first question concerns the diffraction of light, i.e., dynamics when nonlinearity is absent ($\gamma = 0$). The phenomenon of discrete diffraction has been addressed many times in 1D and 2D systems (e.g., see Refs. [4,7]), yielding a characteristic pattern of lobes spreading during propagation. In 4D it is impossible to visualize such a pattern and therefore to characterize diffraction we utilize the concept of the inverse participation ratio, $I(t) = 1/\sum_\alpha I(t)_\alpha^2$, where $I(\alpha) = |\psi_\alpha(t)|^2$ is the intensity of light. A typical question that we wish to address is the following: if we excite a single resonator, how does the excitation spread through the system? For a D-dimensional simple-cubic lattice, propagation of the complex amplitude is given by

$$\psi_{j_1 j_2 \dots j_D} = \prod_{\alpha=j_1 \dots j_D} i^\alpha \mathcal{J}_\alpha(2Jt), \tag{2}$$

where the initially excited (at $t=0$) site is $\alpha_0 = (0,\dots,0)$ with amplitude 1, and $\mathcal{J}_n$ is a Bessel function of order n. For the sake of the clarity it should be pointed out that Eq. (2) results from the fact that the linear problem is factorizable with respect to different dimensions of the system [each α factor in the product of Eq. (2) is a solution for one dimension]. The evolution of the inverse participation ratio (IPR) is asymptotically then

$$I(t) = \left(\sum_{n=-\infty}^{\infty} \mathcal{J}_n(2Jt)^4\right)^{-D} \sim \left(\frac{2Jt}{\ln 2Jt}\right)^D, \tag{3}$$

where we have utilized a formula from Ref. [18] to express the sum (the base of the logarithm is e throughout the manuscript). In what follows, for simplicity we take $J = 1$.

It is evident that the inverse participation ratio asymptotically increases as a power law with a slow logarithmic modulation, where the power-law exponent equals dimension. The logarithmic modulation would not have been present in continuous systems and its existence occurs because of the presence of the lobes. The effective number of sites excited by the light is smaller during diffraction in discrete systems in comparison to volume in continuous systems where the lobes are not present. Figure 1(a) illustrates diffraction IPR dynamics for a finite amount of time in a simple-cubic lattice in 1D, 2D, 3D, and 4D. Superimposed on the results for an infinite lattice (red dashed lines), we also show the dynamics for a finite size lattice (blue solid lines) with $N = 2^{24}$ sites and periodic boundary conditions. This is important because finite size effects will occur in practical realizations of 4D lattices. For example, our 4D lattice has the length of only $2^6 = 64$ sites along one dimension and the effect is clearly visible for large enough times.

Next we consider a 4D lattice of nonlinear resonators that we model with a DNLS equation, Eq. (1). A paradigmatic nonlinear phenomenon that occurs in nonlinear lattices are discrete or lattice solitons. Versatile types of 1D–3D discrete solitons were predicted and/or observed in optics including bright on-site solitons [3,4,7], staggered solitons [6,19], vortex solitons [8,9], and octopole solitons [20]. The simplest type of soliton that one can consider in 4D lattices is a 4D on-site bright soliton that is centered on a single site. We have found this soliton by self-consistently numerically solving the stationary DNLS equation with the focusing nonlinearity $\gamma > 0$ (not shown). It occurs only above some threshold value of the nonlinearity (this also holds for 2D and 3D solitons [20,21]). The relevant parameter here is in fact the ratio γ/J, because the DNLS equation can be scaled; this means that by reducing the coupling parameter J the effective nonlinearity γ/J can be made stronger. On-site bright solitons are also found in 1D–3D photonic systems, and one can expect that analogs of other types of 1D–3D soliton excitations will also exist in 4D systems.

Here we demonstrate a soliton that occurs in 4D lattices: the *tesseract* solitons equally excite 16 sites of a 4D cube (i.e., tesseract); these solitons can be considered as 4D analogs of 1D dipole, 2D quadrupole, and 3D octopole solitons (e.g., see Ref. [22]). By using the methods outlined in Ref. [20], we find that these solitons exist when the neighboring sites are π out-of-phase as illustrated in Fig. 1(b). The stability of these solitons was checked numerically; Fig. 1(c) shows the inverse participation ratio dynamics of a stable tesseract soliton and its discrete diffraction when propagated without nonlinearity present. Without the π out-of-phase feature the intensity on the neighboring sites would not repel, and the excitation would collapse.

Let us discuss the distinction between the solitons that we found in 4D lattices and a plethora of discrete solitons that were

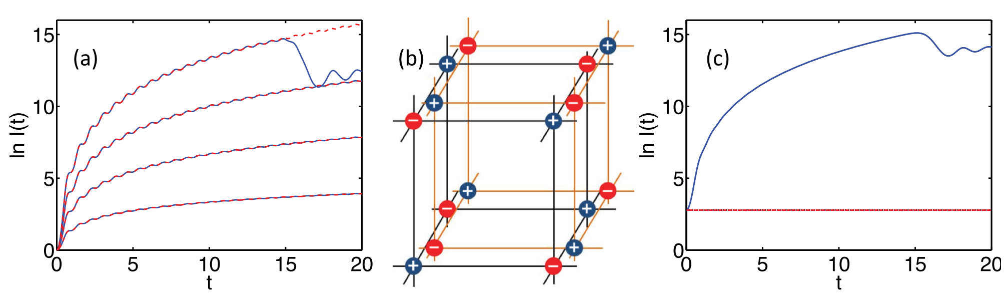

FIG. 1. (Color online) Discrete diffraction in 4D lattices and tesseract solitons. (a) Evolution of the inverse participation ratio $I(t)$ during discrete diffraction for linear lattices with $D = 1$, 2, 3, and 4 (bottom up). The base of the logarithm is e, i.e., we plot $\ln I(t)$. Red dashed lines depict $I(t)$ for an infinite lattice, whereas blue solid lines correspond to a finite size system with $N = 2^{24}$ sites and periodic boundary conditions. For $D = 4$ the boundary is reached at $t \approx 15$ and the IPR starts to oscillate. (b) A schematic illustration of a tesseract soliton. The phases at nearest-neighbor nodes differ by π, $\gamma/J = 630$. (c) Inverse participation ratio dynamics of a tesseract soliton with (black dotted line) and without (red dashed line) noise demonstrating stability (the two lines are on top of each other). When the nonlinearity is turned off, the tesseract initial condition diffracts (solid blue).

predicted and observed previously [1,2]. Solitons occur when the tendency of a wave packet to diffract is in balance with the self-action induced by the nonlinearity. Since diffraction evidently differs depending on the dimensionality, any soliton found in 4D systems in that sense differs from its 1D–3D counterpart (provided that the counterpart exists). However, soliton excitations in higher dimensions can be more or less similar to their counterparts in lower-dimensional lattices, and in that sense more or less interesting. For example, the on-site bright soliton in 4D lattices discussed above is effectively equal to its 1D–3D counterparts; however, its tails spread in a 4D lattice and its diffraction (in the absence of nonlinearity) is different, and therefore it is to some extent a different excitation. By the same line of reasoning, the tesseract soliton exciting a 4D cube in that sense differs more from its 1D–3D analogs mentioned above, and can be regarded as a new type of excitation.

Up to this point we have theoretically considered a model which can represent lattices in more than three dimensions and analyzed some linear and nonlinear phenomena in such systems. However, one may say that these theoretical considerations are not more than academic curiosity because experiments are performed in 3D continuous space. We point out that discrete photonic lattices with dimensionality greater than three can exist in continuous 3D space. To illustrate that fact compare a system of coupled resonators schematically illustrated in Figs. 2(a) and 2(b). (the coupled resonators are connected by lines). For the sake of the argument let us assume that the coupling parameters between all coupled resonators are equal, and zero otherwise. The configuration in Fig. 2(a) is evidently a discrete 2D lattice. However, the system sketched in Fig. 2(b) is fully equivalent to that of Fig. 2(a). Thus, a 2D network of resonators can be constructed by embedding these resonators in 1D geometry (on a straight line), provided that connections between distant (in Euclidian sense) resonators can be made. By using this line of reasoning it is evident that the existence of discrete photonic lattices of dimension larger than three, describable with model (1), is not hindered by the dimensionality of our space.

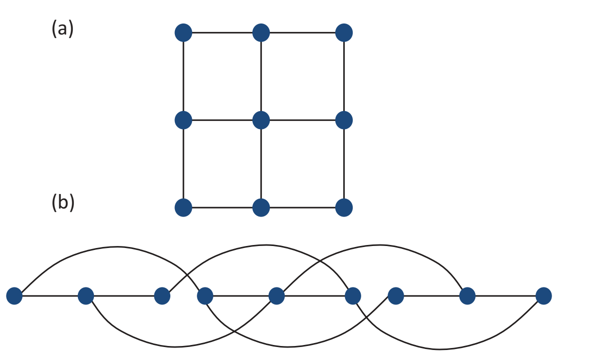

FIG. 2. (Color online) Illustration of the idea for creating photonic lattices in more than three dimensions via complex networks of optical resonators. (a) A schematic illustration of a (finite size) 2D lattice of coupled resonators. (b) The coupling scheme which is topologically fully equivalent to the one in panel (a), despite the fact that resonators are located on a 1D line. By the same token a 4D (discrete) lattice can be embedded in a continuous 3D space, see text for details.

The limitations on the dimensionality and structure of experimentally realizable discrete lattices depend on our ability to construct coupled resonators that may be on distant locations. This is not an easy task because usually high-Q resonators are coupled via evanescent coupling, which implies that they have to be close to each other because the light is tightly bound to the resonators and evanescent fields extend only a few wavelengths. However, in a recent work Sato *et al.* [23] have demonstrated that two distant photonic high-Q cavities can be coupled by using an appropriate waveguide with mirrors at its ends. The realization of the cavities and coupler was made in a photonic crystal structure [23]. Importantly, many Rabi oscillations were observed in that system [23] which provides great promise for the future construction of 4D lattices.

Two ingredients towards the realization of more than 3D discrete photonic lattices are thus present: (i) the fact that more than 3D photonic lattices can in principle exist in our continuous 3D space, and (ii) distant high-Q cavities can be coupled as was demonstrated in Ref. [23]. A proposal of a completely specified photonic structure which would lead to an experimental realization of more than 3D photonic lattices is beyond the scope of the present work. However, here we propose a design which seems to be a viable path towards the realization of 4D simple-cubic photonic lattices. It is based on a combination of waveguide coupling and evanescent coupling between resonators. The design is illustrated in Fig. 3. First consider a 3D simple-cubic lattice of cavities coupled with waveguides. Let the unit vectors of this 3D simple-cubic lattice be $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. The distance between two adjacent sites a_l should be much larger than the size of an individual cavity R. Consider now that we place two such simple-cubic structures next to each other such that each site of the second lattice is displaced by the vector $\mathbf{d} = d(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3)/\sqrt{3}$ from the first one, where d is slightly larger than $2R$, so that evanescent coupling (i.e., hopping) from one sublattice to the other is possible [see Figs. 3(a) and 3(b)]. By adjusting the distances d and $a_l \gg R$, the coupling constant between two sites of

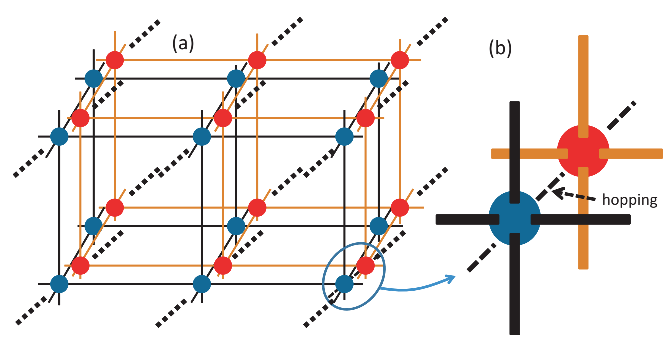

FIG. 3. (Color online) Proposal for the design of a 4D photonic lattice. (a) Illustration of a simple-cubic 4D lattice. Waveguides enable coupling between distant resonators along the edges of 3D cubes. Evanescent coupling (hopping) along the diagonal enables construction of the fourth dimension for the discrete lattices, see encircled region enlarged in panel (b). For clarity only two sublattices are displayed in panel (a); the number of sublattices that can be added along the fourth dimension depends on the size of each individual cavity and the lattice size a_l of every 3D sublattice. See text for details.

the same sublattice (coupled via waveguides) and the hopping parameter between two adjacent sublattices can in principle be made approximately equal. By this procedure we have constructed a bi-cubic lattice. However, since $a_l \gg d \sim 2R$, we can add a number (roughly up to $a_l\sqrt{3}/d$) of sublattices along the direction **d** thereby creating a finite size 4D lattice. If a_l is a bit more than 10 times larger than d, then a finite size 4D simple cubic lattice with 10^4 sites can be constructed. For resonators made of nonlinear materials this construction yields 4D nonlinear photonic lattices.

Let us comment on the finite size effects and scaling associated with realizations of these simple-cubic 4D lattices. From the design scheme presented in Fig. 3 we see that along the fourth discrete dimension the photonic lattice would have to be finite; its size along this direction, i.e., the number of sublattices that one can add, depends on the ratio of the cube edges a_l and the distance d between the resonators along the fourth discrete dimension. Thus, to build a larger 4D lattice one needs larger a_l. In practice all lattices are finite (including 1D–3D) and if they are sufficiently large the finite size effects would be unimportant. Our calculations show that 4D lattices with 10^4 sites (ten sites along one dimension) can already exhibit 4D behavior for some phenomena like solitons.

In conclusion we have studied discrete photonic lattices in more than three dimensions and pointed out that such systems can exist in continuous 3D space. We have studied discrete diffraction in the linear regime and demonstrated the existence of 4D *tesseract* solitons in nonlinear 4D periodic photonic lattices. These structures would open the way for investigating new optical phenomena that one does not encounter in usual 1D–3D systems, but could also provide us with better understanding of dimensionalities beyond 3D which is of fundamental importance. We envision the study of versatile novel types of solitons and instabilities in these systems including vortexlike structures, gap solitons, surface states and surface solitons, incoherent light dynamics, and quantum optical phenomena. The stability of solitons is known to depend on the dimensionality; therefore, one may expect new physical phenomena in this context. These structures also yield opportunities for creating novel optical devices. Greater dimensionality, i.e., greater connectivity of nodes, is expected to yield fundamentally novel schemes for light manipulation not encountered in 1D–3D systems. All of these reasons provide us with the motivation to further address and study these complex systems.

Note added in proof. Recently we became aware of a paper [24] which proposes the realization of a 4D quantum model by using ultracold atoms in optical traps, where the fourth dimension is encoded in the internal states of the atoms providing an extra degree of freedom.

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