

## Observable vortex properties in finite-temperature Bose gases

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(Received 30 October 2012; published 25 January 2013)

We study the dynamics of vortices in finite temperature atomic Bose-Einstein condensates, focusing on decay rates, precession frequencies, and core brightness, motivated by a recent experiment [Science **329**, 1182 (2010)] in which real-time dynamics of a single vortex was observed. Using the Zaremba, Nikuni, and Griffin (ZNG) formalism based on a dissipative Gross-Pitaevskii equation for the condensate coupled to a semiclassical Boltzmann equation for the thermal cloud, we find a rapid nonlinear increase of both the decay rate and precession frequency with increasing temperatures. The increase, which is dominated by the dynamical condensate-thermal coupling is also dependent on the intrinsic thermal cloud collisional dynamics; the precession frequency also varies with the initial radial coordinate. The integrated thermal cloud density in the vortex core is for the most part independent of the position of the vortex (except when it is near the condensate edge), with its value increasing with temperature. This could potentially be used as a variant to the method of Coddington *et al.* [Phys. Rev. A **70**, 063607 (2004)] for experimentally determining the temperature.

DOI: 10.1103/PhysRevA.87.013630

PACS number(s): 03.75.Kk, 67.85.-d

### I. INTRODUCTION

The existence of quantized vortices in Bose-Einstein condensates (BECs) is a feature of the superfluid nature of these systems [1]. Since the first experimental realization of vortices in BECs [2–4], there have been numerous investigations into the nature of their dynamics (see, e.g., Ref. [5]), ranging from their nucleation as a result of rotation [6] (see also Ref. [7] and references therein), to the effect of the inhomogeneous density on the velocity of a vortex [8–10], to the decay of a single vortex induced by sound emission [11–15], and the formation and structure of vortices in multicomponent [16–19] and spinor BECs [20–22]. Vortices are now routinely created in a variety of ways, including stirring the condensate using a laser beam [23,24], letting a soliton decay via the snake instability [25], phase imprinting [26], and by a rapid quench through the transition temperature for the onset of Bose-Einstein condensation (i.e., the Kibble-Zurek mechanism) [27,28]. These techniques have opened the door to the study of more complicated configurations; for instance, the motion of vortex dipoles [28,29], the formation of multiply charged vortices [26,30], vortex lattices [3,4,31,32] and, more recently, the creation of a small tangle of vortices [33–36]. The lifetime of vortex structures created in the laboratory can range up to several seconds. The reason for their eventual demise is thought to be thermal dissipation [32,37]. At finite temperature, atom-atom interactions cause the vortex to lose energy; this results in the vortex spiralling out of a harmonically trapped condensate (see Fig. 3, top); thus, the lifetime of a vortex is severely reduced with increasing temperature [38]. Earlier work on finite-temperature vortex dynamics has confirmed this effect [38–46]. Along with the decay, the precession frequency of a singly charged, harmonically trapped vortex has also been found to vary with temperature [38,42,47].

The motivation behind our work is twofold. First, we revisit the problem of vortex decay rate and precession frequency using the only implementable model to date which includes the

full thermal cloud dynamics [48]; other approaches typically only include the dynamics up to a cutoff within classical field theory (which is more suitable in the fluctuation-dominated regime, very close to  $T_c$ —see, for example, Ref. [49]). Our study is motivated by a recent experiment [28] which followed the motion of a single vortex in a harmonically trapped condensate using a new imaging technique. Second, we are interested in revisiting the experiment proposal of Coddington *et al.* [50] regarding the use of finite-temperature effects on vortex core “brightness” as a possible tool for thermometry; at low temperatures it is difficult to accurately extract the temperature of a system of bosons, and since the mean field felt by the thermal cloud in the region of the vortex core is much less than anywhere else in the condensate, the vortex core was said to act as a “thermal-atom concentration pit” [50]. Virtanen *et al.* [51] have theoretically investigated the core brightness over a small range of temperatures, for a central vortex using the Hartree-Fock Bogoliubov theory within the “Popov” approximation [52] and found it to increase with temperature, as expected. We revisit this problem, using a fully dynamical theory [48,53], and determine the dependence of core contrast on both the radial component of the vortex and the temperature.

Our approach is based on the formalism of Zaremba, Nikuni, and Griffin (ZNG) [48,53], where the Gross-Pitaevskii equation (GPE) is generalized by the inclusion of the thermal cloud mean field, and a dissipative or source term which is associated with a collision term in a semiclassical Boltzmann equation for the thermal cloud. This approach has previously been used to successfully describe the damping of condensate collective modes [54–58] and macroscopic excitations [38,59,60] in the mean-field-dominated regime at finite temperature; the method reduces to the damped two-fluid equations of  $^4\text{He}$  in the hydrodynamic limit [48,61–68]. This is the only model implemented to date which self-consistently accounts for all collisional dynamics of the system and is suited to elevated temperatures excluding the region of critical fluctuations.

The plan of this article is the following: In Sec. II we briefly review the ZNG equations (with some further details in Appendix) and apply them to the problem of vortex decay at finite temperatures. Following the procedure of Jackson *et al.* [38], we extract a decay rate for various temperatures for a *fixed* number of condensate atoms, and analyze the effect of collisions on the decay rate. In Sec. III, we use the same parameters to assess how increasing temperature affects the precession frequency of a vortex. In particular, in Sec. III A we extract precession frequencies of vortices for the parameters of the Freilich *et al.* [28] experiment, where the total number of atoms is fixed. In Sec. IV we investigate vortex core brightness, and how it changes with radial position and temperature, and conclude by summarizing our findings in Sec. V.

## II. BACKGROUND, THEORY, AND MOTIVATION

In typical vortex experiments, the vortex core size is smaller than the wavelength of the light used to image it. As a result, to make the vortex visible it is necessary to expand the condensate [23]. However, as a consequence of this expansion, the condensate is destroyed, requiring successive reproducible runs to observe time evolution of the vortex.

Freilich *et al.* [28] developed an imaging technique, which involves the repeated extraction and expansion of approximately 5% of the condensate atoms, thus enabling a series of images of the same condensate containing the vortex to be created. This technique allowed the precession frequency of the vortex to be measured in a single experimental run rather than repeatedly over several runs with reproducible initial conditions [69].

We will now briefly discuss the ZNG equations and the advantages of using such a model before presenting our results.

### A. Zaremba, Nikuni, and Griffin formalism

An extensive review of the ZNG formalism and its derivation can be found elsewhere [48,53]. Here it suffices to briefly outline the methodology. The formalism is based on the following closed set of equations (where the explicit dependence on  $\mathbf{r}$  and  $t$  is suppressed for convenience:

$$i\hbar \frac{\partial \phi}{\partial t} = \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{ext}} + g[n_c + 2\tilde{n}] - iR \right) \phi, \quad (1)$$

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{r}} f - (\nabla_{\mathbf{r}} U_{\text{eff}}) \cdot (\nabla_{\mathbf{p}} f) = C_{12}[f, \phi] + C_{22}[f]. \quad (2)$$

Here,  $\phi$  is the condensate wave function and  $f$  is a phase-space distribution function. Equation (1) is a finite temperature generalization of the usual Gross-Pitaevskii equation which is modified by the addition of a thermal-cloud mean-field potential  $2g\tilde{n}$ , and a dissipative or source term  $-iR$ . It is coupled to the quantum Boltzmann equation [QBE, Eq. (2)] for the thermal cloud phase-space distribution. The condensate density is written as  $n_c = |\phi|^2$ ,  $V_{\text{ext}}(\mathbf{r})$  is the external trapping potential, and the interaction strength between the atoms is  $g = 4\pi\hbar^2 a_s/m$ , where  $a_s$  is the  $s$ -wave scattering length

and  $m$  is the atomic mass. The thermal cloud density is recovered from the phase-space distribution function via an integration over all momenta,  $\tilde{n}(\mathbf{r}, t) = \int d\mathbf{p}/(2\pi\hbar)^3 f(\mathbf{p}, \mathbf{r}, t)$  and  $U_{\text{eff}} = V_{\text{ext}}(\mathbf{r}) + 2g[n_c(\mathbf{r}, t) + \tilde{n}(\mathbf{r}, t)]$  is the mean-field potential acting on the thermal atoms. The quantities  $C_{22}$  and  $C_{12}$  are collision integrals (definitions of these terms can be found in Appendix).  $C_{22}$  describes the redistribution of thermal atoms as a result of collisions between two thermal atoms (i.e., the usual Boltzmann equation collision integral), while  $C_{12}$ , which is closely related to the dissipation term  $iR$ , describes the change in the phase-space distribution function  $f(\mathbf{p}, \mathbf{r}, t)$  as a result of particle-exchanging thermal atom-condensate collisions. Although other methods have been put forward for finite-temperature vortex dynamics, importantly, this method self-consistently accounts for the redistribution of thermal particles and the effect this has on the condensate dynamics [70].

### B. Vortex decay

Equations (1) and (2) have previously been used to study finite-temperature vortex dynamics by Jackson *et al.* [38]. In that work, the authors show that the decay rate of a vortex increases rapidly with increasing temperature. In these simulations, the system size was fixed to a constant total number of atoms,  $N_{\text{TOT}} = 10\,000$ ; by increasing the temperature of the system, the number of thermal atoms increased and consequently the condensate size decreased. As both of these variations affect the vortex dynamics, in order to isolate the effect of increasing thermal atom number, in this work we instead initially perform simulations at different temperatures for a *fixed* number of condensate atoms,  $N_c = 10\,000$   $^{87}\text{Rb}$  atoms. By fixing the number of condensate atoms, the total number of atoms,  $N_{\text{TOT}}$  increases with temperature  $T$ . As a result, the critical temperature  $T_c$  for the onset of Bose-Einstein condensation is a function of  $T$  in these simulations. We estimate  $T_c(N_{\text{TOT}})$  by means of the ideal gas expression (note that, for  $N_{\text{TOT}} = 10\,000$  atoms, the  $T = 0$  value of the critical temperature is  $T_c = 177$  nK).

The geometry is a fully three-dimensional (3D) harmonic trap,  $V_{\text{ext}}(\mathbf{r}) = \frac{m}{2}[\omega_{\perp}^2(x^2 + y^2) + \omega_z^2 z^2]$  with trapping frequencies,  $\omega_{\perp} = 2\pi \times 129$  Hz and  $\omega_z = \sqrt{8}\omega_{\perp}$  [38]. The purpose of the significantly tighter trapping frequency in the axial direction is to ensure that the vortex remains relatively straight along its length, as shown in Fig. 1 (left). This figure contains 3D isosurface plots of the condensate (left) and thermal cloud (right) densities. We have chosen a high-density surface of the

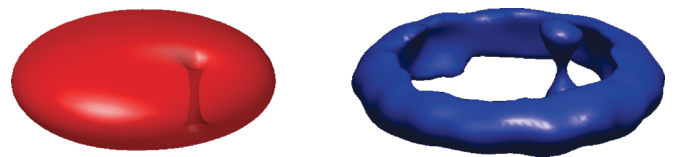


FIG. 1. (Color online) 3D isosurface plots of low condensate density (left, red) and high thermal cloud density (right, blue) for a cloud of  $N_c = 10\,000$   $^{87}\text{Rb}$  atoms at a temperature of  $0.5T_c$  for the trapping parameters as described in the text. Notice the tubular isosurface of the thermal cloud at a position corresponding to the vortex core in the condensate.

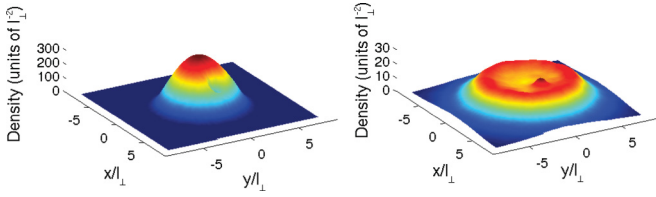


FIG. 2. (Color online) Two-dimensional (2D) densities, integrated along the  $z$  direction for the condensate (left) and thermal cloud (right). Notice the effect of the vortex core on the thermal cloud. Trapping parameters are as in text for  $T/T_c = 0.5$ ,  $N_c = 10\,000$   $^{87}\text{Rb}$  atoms.

thermal cloud in order to show how the noncondensate atoms fill the vortex core and concentrate around the edges of the condensate. Figure 2 shows the densities of both components integrated along the  $z$  direction. The relatively flat profile of integrated thermal density in Fig. 2 (right) illustrates how the thermal cloud surrounds the condensate, with the regions of highest concentration of noncondensate atoms corresponding to the condensate edge as well as the position of the vortex (these are the areas where the thermal cloud feels the lowest mean-field repulsion from the condensate).

We extract the decay rate  $\gamma$  for vortices of different initial positions by fitting the vortex radial variable to the exponential  $r_v(t) = r_0 e^{\gamma t}$ , where  $r_0$  is the initial vortex position. Results corresponding to three temperatures are plotted in Fig. 3, where  $(x_v, y_v)$  trajectories and the time evolution of  $r_v$  (including the corresponding exponential fits) for a particular value of  $r_0$  are also shown. We express the vortex radial coordinate in terms of the Thomas-Fermi (TF) radius which is defined as  $R_{\text{TF}} = \sqrt{2\mu/(m\omega_{\perp})}$ , where  $\mu$  is the chemical potential.

Increasing the temperature from  $0.3T_c$  to  $0.7T_c$  increases the decay rate significantly, in agreement with previous findings [38,39,41,71]. It is also apparent that, at all temperatures, moving the vortex initial radial coordinate  $r_0$  away from the center and closer to the condensate edge ( $R_{\text{TF}} \simeq 5l_{\perp}$ ) results in an increase in the vortex decay rate  $\gamma$ . This feature is notably more pronounced at the higher temperatures and is caused by the higher thermal cloud density towards the edge of the condensate (the maximum thermal cloud density occurs at  $\approx 0.85R_{\text{TF}}$ ) which arises because of the weaker mean-field repulsion. We now discuss the effect of the individual collision terms of the QBE (2) on the vortex decay rate as well as on the precession frequency of a vortex. We measure the precession frequency of a vortex as the inverse time it takes for one oscillation of the trap, calculated by averaging the oscillation time over the first three oscillations.

### 1. Effect of collisions on vortex decay rate

We can carry out numerical simulations with the full QBE (2); however, we can also simulate the effect of the thermal cloud with a combination of these collision terms, or without their inclusion at all.

In this way, we can determine the contributions to the vortex decay rate coming from different collisional processes. Figure 4 gives the results of vortex decay rate  $\gamma$  and precession frequency  $\omega_v$  for a vortex having an initial radial offset from the trap center of  $r_0 \simeq 0.26R_{\text{TF}}$ , for various temperatures. The

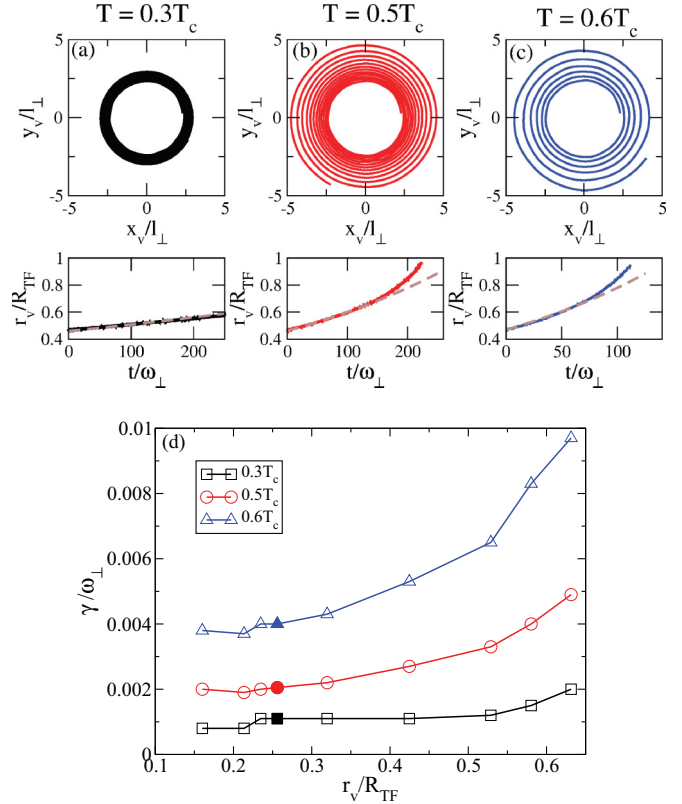


FIG. 3. (Color online) Panels (a)–(c) top show  $(x_v, y_v)$  trajectories of vortex initially at  $r_0 \simeq 0.4R_{\text{TF}}$  at temperatures  $T/T_c = 0.3$  [53 nK (a), black],  $0.5$  [89 nK (b), red], and  $0.6$  [124 nK (c), blue]. In panels (a)–(c) bottom we plot for these temperatures the corresponding radial coordinate  $r_v$  (solid lines) and the exponential fit  $r_v = r_0 e^{\gamma t}$  (dashed lines) with  $\gamma = 0.001$ ,  $0.0026$ , and  $0.005$ , respectively. These trajectories have been smoothed to remove any numerical “jitter” arising as a result of our tracking routine. Panel (d) shows the corresponding values of the decay rate  $\gamma$  for vortices of variable initial position,  $r_0 = (x_0^2 + y_0^2)^{1/2}$ , at these temperatures. The filled symbols indicate the position of the vortex subsequently analyzed in Fig. 4. Trapping parameters are as in text for  $N_c = 10\,000$   $^{87}\text{Rb}$  atoms,  $T_c$  varies between 175 and 205 nK for the range of temperatures shown.  $R_{\text{TF}} \simeq 5l_{\perp}$  with the maximum values of the thermal cloud density occurring at  $\approx 0.85R_{\text{TF}}$  in all cases.

filled circles show the results for the full QBE simulation. Results are also shown for when the thermal-thermal  $C_{22}$  collisions are neglected but the particle-exchanging  $C_{12}$  collisions are included (magenta squares), and vice versa (blue crosses; consequently, when  $C_{12} = 0$  the dissipative or source term  $iR$  will also be zero). In the case when both of these terms are neglected (open, green circles), the QBE is propagated in time according to free streaming terms [left-hand side of Eq. (2)], which ensures that the value of the phase-space distribution function remains the same along a trajectory in phase space. We have also obtained results for a static thermal cloud (red stars), in which the mean field of the equilibrium thermal cloud density (obtained in the absence of a vortex) is included in the solution of the GPE. These results highlight the crucial role of all collision processes in determining the actual decay rate and precession frequency of a vortex.

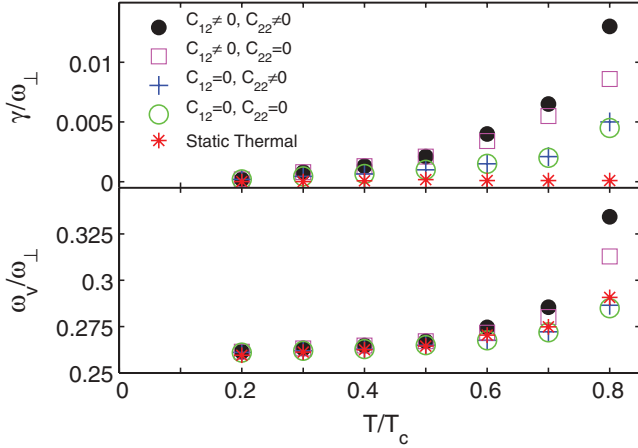


FIG. 4. (Color online) Vortex decay dynamics as a function of temperature: values of vortex decay rate  $\gamma$  (top) and corresponding vortex precession frequencies  $\omega_v$  (bottom). Results for different levels of approximation are indicated by (i) black filled circles for all collision ( $C_{12}, C_{22}$ ) processes, (ii) open, magenta boxes for particle-transferring ( $C_{12}$ ) collisions only, (iii) blue crosses for thermal-thermal ( $C_{22}$ ) collisions only, (iv) open, green circles for no collisions, and (v) red stars for static thermal cloud approximation. An initial radial vortex offset of  $r_0 \simeq 0.26R_{TF}$ , as highlighted in Fig. 3, is used for all simulations. Clearly, all collision mechanisms contribute significantly to the decay. Trapping parameters as in text for  $N_c = 10\,000$   $^{87}\text{Rb}$  atoms with  $T_c$  varying between 175 and 280 nK for the results shown here.

Focussing initially on the decay rate (Fig. 4, top), we see that the largest contribution to vortex decay comes from the particle-exchanging  $C_{12}$  collisions. When the  $C_{12}$  collisions are not included, the decay rate is reduced significantly. The thermal-thermal  $C_{22}$  collisions have a noticeable effect only when they are included together with the particle-exchanging  $C_{12}$  collisions. This implies that  $C_{22}$  collisions affect the decay rate indirectly through a modification of the  $C_{12}$  collision rates.

For the precession frequency of a vortex (Fig. 4, bottom), we see increasing values with increasing temperature, with the effect of the inclusion or absence of the different collision terms being more apparent at the higher temperatures. The largest influence on the precession frequency again arises from the particle-exchanging  $C_{12}$  collisions—an effect which is intuitive since these collisions cause the vortex to lose energy and move out of the condensate radially.

To summarize, our analysis demonstrates the importance of including the full dynamics of the thermal cloud; that is all of the collision terms of the QBE when modeling vortex dynamics. All further results quoted have been simulated with all collision terms included in the propagation of Eq. (2). In the next section, we analyze further the dependence of the vortex precession frequency on temperature, as well as with increasing vortex radial coordinate.

### III. PRECESSION FREQUENCY OF VORTEX

Our motivation for investigating further the effect of finite temperature on vortex precession arises from the development of an imaging technique [28] enabling real-time vortex dynamics to be observed. To first understand the effect of the initial

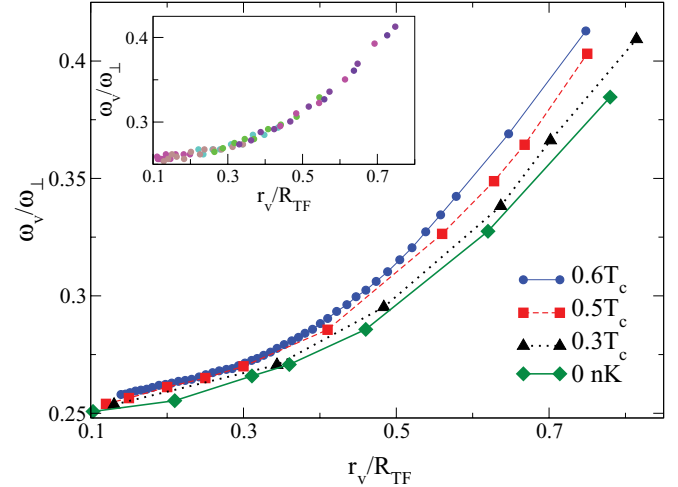


FIG. 5. (Color online) Inset shows the frequencies of vortices with different initial positions  $r_0$  in the range  $0.1R_{TF}$  to  $0.8R_{TF}$  (shown by filled circles of different colors) at a temperature of  $T/T_c = 0.6$ . Main panel shows the corresponding averaged curve for this temperature  $T/T_c = 0.6$  (124 nK; blue filled circles and connecting line). For the remaining temperatures, from bottom to top with increasing temperature,  $T/T_c = 0$  (green diamonds),  $0.3$  (53 nK, black triangles), and  $0.5$  (88.5 nK, red squares), precession frequencies have been extracted as a function of radial coordinates. Trapping parameters are as in text for  $N_c = 10\,000$   $^{87}\text{Rb}$  atoms,  $R_{TF} \approx 5l_{\perp}$  for these temperatures and  $T_c$  varies between 175 and 205 nK for the results shown here.

vortex position on its subsequent precession, we begin our analysis by extracting the precession frequencies for vortices having various different initial positions for a fixed temperature (Fig. 5 inset). The simulations are again carried out using the trapping parameters quoted in Sec. II B and a fixed number of condensate atoms,  $N_c = 10\,000$  for various temperatures.

In the inset of Fig. 5 we plot the precession frequency as a function of vortex radial coordinate,  $r_v(t) = [x_v(t)^2 + y_v(t)^2]^{1/2}$ , for initial positions  $r_0$  in the range  $0.1R_{TF}$  to  $0.8R_{TF}$  at the temperature  $T/T_c = 0.6$ .

The general increase of the precession frequency with increasing vortex radial coordinate  $r_v$  is apparent. The fact that all the points appear to lie on a common curve confirms numerically an important anticipated feature of vortex dynamics: the frequency of vortex precession depends on the *instantaneous* radial position and not on the *history* of how the vortex arrived at that position. We can therefore average over all the points to generate a representative curve for the frequency as a function of radial coordinate. This curve is shown in the main part of Fig. 5 (blue filled circles). Similar simulations were performed for the other temperatures displayed in Fig. 5. Fewer points are shown because these fully capture the investigated behavior of the frequencies. It is clear that the precession frequency also increases with temperature as found for a particular value of  $r_v$  (Fig. 4, bottom).

Up until this point in this paper we have focused on systems with a fixed  $N_c$ , however, since experiments routinely conserve  $N_{TOT}$ , all remaining simulations of this paper will be performed with a fixed *total* atom number and variable *condensate* atom number.



In the experiment of Freilich *et al.* [28] and in subsequent experimental runs by the group, the observed precession frequencies in a data set typically averaged a few ( $\lesssim 5$ ) percent higher [72] than those predicted for a zero temperature, Thomas-Fermi condensate in an axisymmetric trap [5,7,73,74]. In the next section we continue our analysis on vortex precession frequencies for parameters of this experiment at both zero and finite temperature in order to investigate the origin of this deviation. We will begin by briefly remarking on the procedure of the experiment before presenting our results for those parameters.

#### A. Experimental parameters of Freilich *et al.*

Vortices arise in this experiment during evaporation via the Kibble-Zurek mechanism [75–78]; the procedure for imaging is as follows: approximately 5% of the condensate atoms are outcoupled along the  $z$  axis so that they are no longer confined by the trap, and they therefore fall with gravity (along  $z$ ). This proportion expands and the position of the vortex can be resolved and measured. This leaves the remaining 95% of the atoms trapped in the condensate and the vortex continues to precess in this slightly depleted condensate. At a later time, this process is repeated and the position of the vortex at that time also measured. The result is a series of images of vortex position which means that the real-time dynamics of the vortex can be assessed. For the data presented by Freilich *et al.* [28] there was no discernible thermal cloud and the temperature was estimated to be  $T/T_c < 0.4$ .

As a result of this technique, a single vortex was observed for approximately 655 ms [79] in a series of snapshots. Using these images, the precession frequency of the vortex line could be measured and the observed frequencies were found to average, in a typical data set, a few percent ( $\lesssim 5\%$ ) higher [72] than those expected for the geometry and condensate parameters, given by the following equation [5]:

$$\omega_v = \frac{2\hbar\omega_\perp^2}{8\mu(1-r_v^2/R_{\text{TF}}^2)} \left( 3 + \frac{\omega_\perp^2}{5\omega_z^2} \right) \ln \left( \frac{2\mu}{\hbar\omega_\perp} \right), \quad (3)$$

which uses the Thomas-Fermi approximation for the shape of the condensate. This experiment was conducted in a disk-shaped condensate with a total atom number of  $N_{\text{TOT}} \approx (4-6) \times 10^5$   $^{87}\text{Rb}$  atoms (we use  $N_{\text{TOT}} = 6 \times 10^5$  for our simulations). The trapping frequencies are as follows:  $\omega_\perp = 2\pi \times 36$  Hz and  $\omega_z = \lambda\omega_\perp$ , where  $\lambda \approx \sqrt{8}$ ; therefore, the aspect ratio is similar to that used in the previous section, ensuring that the vortex stays relatively straight throughout its motion.

Figure 6 (top) shows the ZNG results (dashed lines) for precession frequency as a function of vortex position for various temperatures (increasing from bottom to top). Also shown are the predictions of Eq. (3) (solid lines), where the values of  $\mu$  and  $R_{\text{TF}}$  for each temperature correspond to the number of condensate atoms in the ZNG simulations. For values of  $r_v > 0.2$ , preliminary experimental measurements in the presence of a thermal cloud indicate values of precession frequency around  $0.1\omega_\perp$ , with an error of approximately 15% [72].

The bottom part of this plot gives the relative difference between the ZNG results and those obtained from the solution of the Gross-Pitaevskii equation. For each temperature, the GPE is solved numerically via a time-stepping fourth-order

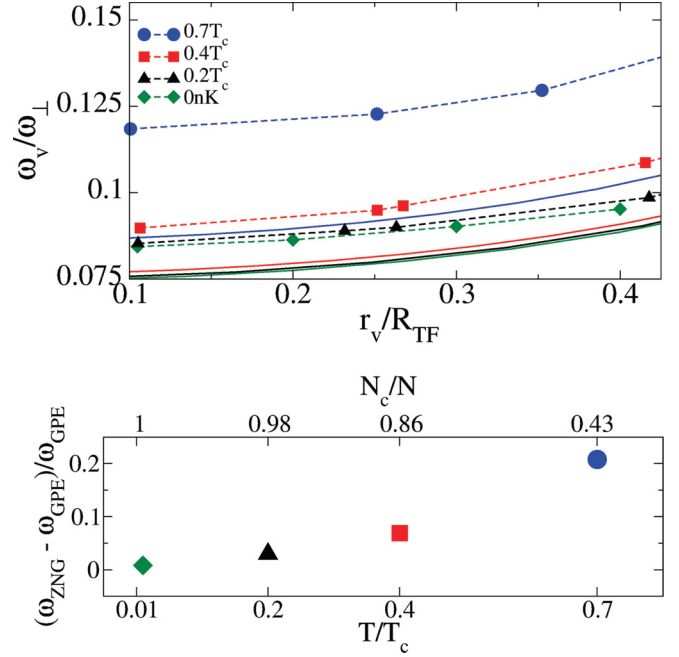


FIG. 6. (Color online) Top panel shows precession frequencies of a vortex for increasing temperature from bottom to top,  $T/T_c = 0$  (green diamonds, dashed line), 0.2 (39 nK, black triangles, dashed line), 0.4 (78 nK, red squares, dashed line), 0.7 (133 nK, blue circles, dashed line), and the respective analytically predicted frequencies from Eq. (3) (correspondingly colored solid lines). Note that the lines for  $T/T_c = 0, 0.2,$  and  $0.4$  lie close together and as a result are difficult to identify individually in this figure. Bottom panel shows relative difference between the ZNG results and the GPE prediction at the vortex radial coordinate  $0.25R_{\text{TF}}$ . For completeness we indicate the values of  $N_c/N$  (top axis) in a nonlinear scale. The GPE results are obtained using the same number of condensate atoms as in the ZNG results. Trapping parameters as in text for  $N_{\text{TOT}} \approx 6 \times 10^5$   $^{87}\text{Rb}$  atoms at  $T_c = 190$  nK with  $R_{\text{TF}}$  in the range  $\approx 9l_\perp$  to  $9.5l_\perp$  for the range of temperatures indicated.

Runge-Kutta scheme for the number of atoms corresponding to the number of condensate atoms in the ZNG calculation. When the number of thermal atoms is small ( $T = 0.01$  nK), the GPE and ZNG predictions agree. We note that if we were to solve the GPE for the total number of atoms of the ZNG simulations, the discrepancy between these results would be a further 20% for  $T/T_c \sim 0.7$ .

For a large condensate, it is natural to approximate using Thomas-Fermi theory via the expression of Eq. (3); however, comparison of the zero-temperature GPE results with the prediction of Eq. (3) reveals that this prediction consistently underestimates the value of precession frequency. We find the relative difference between these results to be in the range 5% to 10% for the parameters shown.

With increasing temperature, the ZNG results increasingly deviate from those obtained using the GPE for the same number of condensate atoms; therefore, the difference between the ZNG results and the predictions of Eq. (3) also increases with temperature (this behavior is in rough agreement with the experimental observations [72]) and is approximately 35% at the highest temperature of  $T/T_c = 0.7$ . Clearly, the TF formula (3) is inherently approximate and should be used with care.

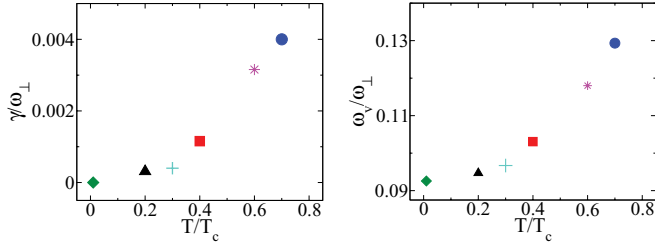


FIG. 7. (Color online) Decay rate (left) and precession frequency (right) of a vortex at a position  $r_v = 0.35R_{\text{TF}}$  at temperatures  $T/T_c = 0.01$  (green diamonds), 0.2 (39 nK, black triangles), 0.3 (59 nK, cyan crosses), 0.4 (78 nK, red squares), 0.6 (114 nK, magenta stars) and 0.7 (133 nK, blue circles). Parameters are as in Fig. 6.

A summary of the results of our simulations for the experimental parameters is shown in Fig. 7. The data for the decay rate  $\gamma$  and precession frequency  $\omega_v$  are all obtained with an initial vortex radial position  $r_0 = 0.35R_{\text{TF}}$ .

#### IV. VORTEX CORE BRIGHTNESS

The temperature of a system of bosons is commonly extracted by fitting a Gaussian profile to the high-energy tails of the velocity distribution of the atoms [80]. These atoms are associated with the high-energy, noncondensed atoms in the system and can be described by a Maxwell-Boltzmann velocity distribution. A limitation of this procedure is that it becomes difficult to accurately extract the temperature at low temperatures when the thermal cloud density is low. Since the thermal cloud density is relatively high in the region of the vortex core, a vortex acts as a “thermal-atom concentration pit” [50]. This led Coddington *et al.* [50] to suggest that measuring the density in the core region could be a possible tool for determining the temperature of the system.

To perform a quantitative analysis of the thermal cloud density in the core (termed vortex core brightness [50]) we revert for computational speed and efficiency to a smaller system of atoms with a fixed total atom number of  $N_{\text{TOT}} = 10000$   $^{87}\text{Rb}$  atoms. Figure 8 shows the condensate and thermal cloud local densities plotted along the radial line in the  $z = 0$  plane (Fig. 8, left) and the densities integrated along the  $z$  direction (Fig. 8, right) in a harmonic trap with trapping frequencies  $\omega_{\perp} = 2\pi \times 129$  Hz and  $\omega_z = \sqrt{8}\omega_{\perp}$ , at a temperature of  $T/T_c = 0.7$ . For both of these quantities, the vortex core

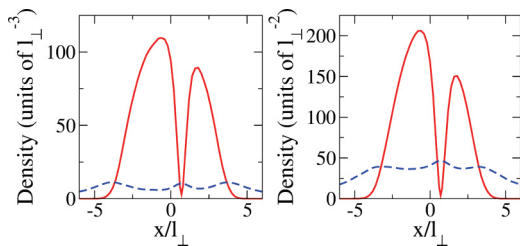


FIG. 8. (Color online) Local density plotted along the radial line in the  $z = 0$  plane (left) and density integrated along the  $z$  direction (right) of the condensate (solid red line) and thermal cloud (dashed blue line). The dip in the condensate density occurs at the position of the vortex. Trapping parameters are as in text for  $N_{\text{TOT}} = 10000$   $^{87}\text{Rb}$  atoms at a temperature of  $T/T_c = 0.7$  where  $T_c = 177$  nK.

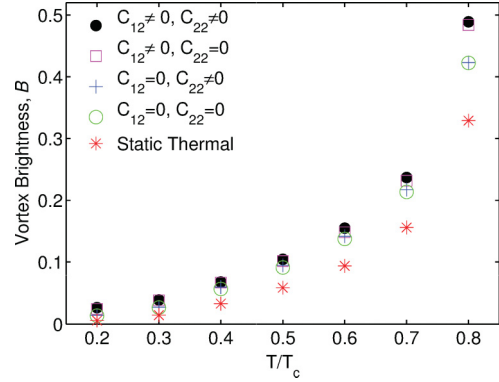


FIG. 9. (Color online) Vortex core brightness  $\mathcal{B}$  for a vortex at position  $r_v = 0.2R_{\text{TF}}$  as a function of temperature. Results for different levels of approximation are indicated by (i) black filled circles for all collision ( $C_{12}, C_{22}$ ) processes, (ii) open, magenta boxes for particle-transferring ( $C_{12}$ ) collisions only, (iii) blue crosses for thermal-thermal ( $C_{22}$ ) collisions only, (iv) open, green circles for no collisions, and (v) red stars for static thermal cloud approximation. Trapping parameters as in text for  $N_{\text{TOT}} = 10000$   $^{87}\text{Rb}$  atoms,  $T_c = 177$  nK.  $R_{\text{TF}}$  varies between  $4.2l_{\perp}$  to  $4.9l_{\perp}$  for these results.

appears as a dip in the condensate density with a corresponding peak in the thermal cloud density. An interesting property of the integrated thermal cloud profile is that, throughout the extent of the condensate, the integrated thermal cloud density is relatively uniform, except for the peak that still emerges in the region of the vortex core.

In Ref. [50], the vortex core brightness was defined as

$$\mathcal{B} = \frac{n_{\text{core}}}{n_{\text{cloud}}}, \quad (4)$$

where  $n_{\text{core}}$  is the *observed* atom density, integrated along the line of sight at the core center (see Fig. 8, right) and  $n_{\text{cloud}}$  is the *projected integrated* density at this point based on a smooth fit of the overall atom cloud. We extract the integrated thermal density  $n_{\text{core}}$  at the central point of the vortex following this during its motion, which we refer to as  $n_{2\text{D}}^T(\mathbf{r}_v, t)$ . We calculate the projected integrated density  $n_{\text{cloud}}$  at the same point as follows: we first run a simulation for the same parameters in the *absence* of a vortex and extract the *total* integrated density.

We use the trapping parameters defined in Sec. II B and above, noting here again that we fix the *total* number of atoms in the system to be consistent with routine experimental procedures.

In Fig. 9 we plot vortex brightness  $\mathcal{B}$  at the vortex position  $r_v = 0.2R_{\text{TF}}$  for various temperatures. We see a clear trend in increasing brightness with increasing temperature for all approximations of the QBE; however, the value of the brightness is largest when the full QBE is solved. We note again the importance of solving the full thermal cloud collisional dynamics in order to avoid underestimating the effect of the thermal cloud on vortex properties.

More detailed information is provided in Fig. 10, where we have also plotted the integrated condensate density  $n_{2\text{D}}^C(\mathbf{r}_v, t)$  in the absence of a vortex. This figure shows the dependence of the integrated condensate  $n_{2\text{D}}^C$  (top row, right) and thermal cloud  $n_{2\text{D}}^T$  (bottom) densities along the vortex trajectory (top row, left). The initial vortex position is  $0.15R_{\text{TF}}$  and the

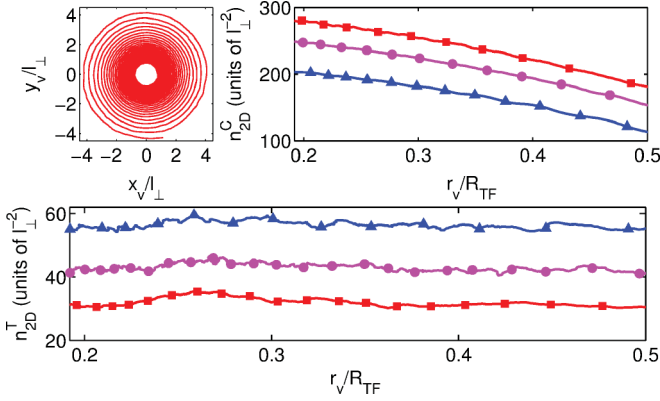


FIG. 10. (Color online) Top-left panel shows vortex trajectory  $r_v = (x_v, y_v)$  for condensate containing a vortex with initial position  $r_0 = 0.15R_{TF}$  at  $T/T_c = 0.5$ . Integrated projected condensate density  $n_{2D}^c(\mathbf{r}_v, t)$  (top-right panel). Bottom panel shows integrated thermal cloud density  $n_{2D}^T(\mathbf{r}_v, t)$  at the center of the vortex core, as a function of radial coordinate, for the temperatures  $T/T_c = 0.5$  (red squares),  $0.6$  (magenta circles), and  $0.7T_c$  (blue triangles). Parameters are as in Fig. 9.

temperature is  $T/T_c = 0.5$ . We also display densities for two other, higher temperatures. There is a clear trend of increasing integrated thermal cloud density with temperature. On the other hand, as one would expect, the integrated condensate density decreases with increasing temperature. Furthermore, as the vortex moves radially outwards, the integrated condensate density decreases, whereas the integrated thermal density stays relatively constant (except near the edge of the condensate, not shown here).

The results presented here indicate that core brightness  $\mathcal{B}$  may indeed be a good candidate for extracting temperature of a condensate containing a vortex. However, in order to make the method quantitative, one would have to take into account the dependence of  $\mathcal{B}$  on the radial position of the vortex. Since

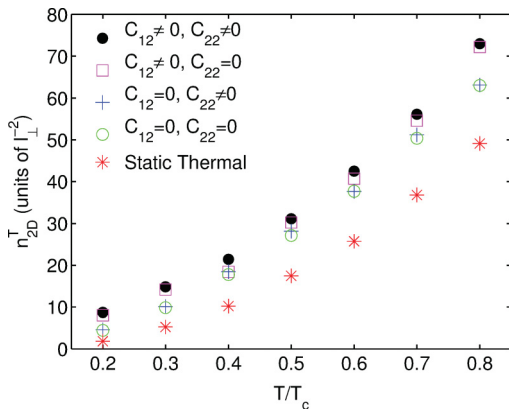


FIG. 11. (Color online) Integrated thermal cloud density  $n_{2D}^T$  at the center of the vortex core as a function of temperature. Results for different levels of approximation are indicated by (i) black filled circles for all collision ( $C_{12}, C_{22}$ ) processes, (ii) open, magenta boxes for particle-transferring ( $C_{12}$ ) collisions only, (iii) blue crosses for thermal-thermal ( $C_{22}$ ) collisions only, (iv) open, green circles for no collisions, and (v) red stars for static thermal cloud approximation. Parameters are as in Fig. 9.

$n_{2D}^T$  remains practically constant throughout the motion of the vortex through the condensate (Fig. 10, bottom), our analysis suggests that this quantity itself may provide a better measure of the temperature of the system.

We therefore extract the value of  $n_{2D}^T$  for a range of temperatures in Fig. 11. For completeness, we also consider the effect of the various collisional approximations. We see that the sensitivity to the collision processes included is much less than that of the vortex decay rate or precession frequency (see Fig. 4).

## V. CONCLUSION

In summary, we have investigated observable properties of vortices at finite temperature, including decay rate, precession frequency, and vortex core brightness, using a model which accounts for all collision processes between the atoms. Particle-exchanging collisions between the condensate and thermal cloud provide the dominant contribution, while thermal-thermal collisions further affect these, particularly at temperatures in the region  $T > 0.5T_c$ . While the decay rate of a vortex is dependent on *initial* vortex position, the precession frequency of a vortex is instead a function of the *instantaneous* vortex position; both quantities increase with increasing temperature. Furthermore, we have investigated the vortex precession frequency for the experimental parameters of Freilich *et al.* [28] and found that, even at low temperatures, there is a deviation from the predictions of a Thomas-Fermi analysis. This deviation increases with temperature and can be as large as 35% for  $T/T_c = 0.7$  and should be detectable in future experiments.

We also found that the integrated thermal cloud density in the region of the vortex core remains relatively constant as the vortex spirals towards the edge of the condensate. This observation suggests that the integrated thermal density is perhaps a better probe of the temperature than total vortex brightness previously proposed by Coddington *et al.* [50].

## ACKNOWLEDGMENTS

We gratefully acknowledge D. S. Hall for providing us with experimental details and for useful suggestions during the course of this work; we also thank T. M. Wright for useful discussions. A.J.A., C.F.B., and N.N.P. acknowledge funding from the EPSRC (Grant No. EP/I019413/1). E.Z. gratefully acknowledges funding from NSERC of Canada.

## APPENDIX: MEAN-FIELD POTENTIAL AND COLLISION INTEGRALS

Within the ZNG formalism, the thermal excitations are assumed to be semiclassical moving in a Hartree Fock (HF) potential; that is, an excitation with momentum  $\mathbf{p}$  possesses energy  $\tilde{\epsilon}_i = p^2/(2m) + U_{\text{eff}}(\mathbf{r}, t)$  where the effective potential  $U_{\text{eff}}$  is defined as

$$U_{\text{eff}}(\mathbf{r}, t) = V_{\text{ext}}(\mathbf{r}) + 2g[n_c(\mathbf{r}, t) + \tilde{n}(\mathbf{r}, t)]. \quad (\text{A1})$$

The quantities  $C_{22}$  and  $C_{12}$  appearing in Eq. (2) are collision integrals.  $C_{22}$  describes the redistribution of thermal atoms as a result of collisions between thermal atoms while  $C_{12}$  describes the change in the phase-space distribution function  $f(\mathbf{p}, \mathbf{r}, t)$  as a result of particle-exchanging thermal-condensate collisions.

These are respectively defined as

$$C_{22}[f] = \frac{4\pi}{\hbar} g^2 \int \frac{d\mathbf{p}_2}{(2\pi\hbar)^3} \int \frac{d\mathbf{p}_3}{(2\pi\hbar)^3} \int \frac{d\mathbf{p}_4}{(2\pi\hbar)^3} \times (2\pi\hbar)^3 \delta(\mathbf{p} + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \times \delta(\tilde{\epsilon} + \tilde{\epsilon}_2 - \tilde{\epsilon}_3 - \tilde{\epsilon}_4)[(f+1)(f_2+1) \times f_3 f_4 - f f_2 (f_3+1)(f_4+1)], \quad (\text{A2})$$

and

$$C_{12}[f, \phi] = \frac{4\pi}{\hbar} g^2 |\phi|^2 \int \frac{d\mathbf{p}_2}{(2\pi\hbar)^3} \int \frac{d\mathbf{p}_3}{(2\pi\hbar)^3} \int \frac{d\mathbf{p}_4}{(2\pi\hbar)^3} \times (2\pi\hbar)^3 \delta(m\mathbf{v}_c + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \times \delta(\epsilon_c + \tilde{\epsilon}_2 - \tilde{\epsilon}_3 - \tilde{\epsilon}_4) (2\pi\hbar)^3 [\delta(\mathbf{p} - \mathbf{p}_2) - \delta(\mathbf{p} - \mathbf{p}_3) - \delta(\mathbf{p} - \mathbf{p}_4)] \times [(f_2+1)f_3 f_4 - f_2(f_3+1)(f_4+1)]. \quad (\text{A3})$$

The delta functions in these expressions enforce conservation of energy and momentum. In particular, those in the  $C_{12}$  term take into account that the condensate atoms have energy  $\epsilon_c = mv_c^2/2 + \mu_c$ , and momentum  $m\mathbf{v}_c$ , where  $\mu_c$  is the condensate chemical potential.

If the condensed and noncondensed components are in diffusive equilibrium,  $C_{12} = 0$ . When they are out of equilibrium, this term acts to transfer atoms between the condensate and thermal cloud. This exchange results in the source term of Eq. (1),

$$R(\mathbf{r}, t) = \frac{\hbar}{2|\phi(\mathbf{r}, t)|^2} \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} C_{12}[f(\mathbf{p}, \mathbf{r}, t), \phi(\mathbf{r}, t)]. \quad (\text{A4})$$

The effects of  $C_{12}$  in the kinetic equation together with  $R$  in the generalized GPE ensure that the *total* particle number is conserved.

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