Sub–shot-noise sensitivities without entanglement

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We critically analyze the commonly maintained statement that entanglement is necessary to beat the shot-noise limit or standard quantum limit (SQL) in the sensitivity with which certain parameters can be measured in interferometric experiments. We consider in detail three different physical realizations of a beam splitter and a Mach-Zehnder interferometer using photons, massive bosons, and distinguishable spins or qubits. We study several input states and the corresponding definitions of entanglement. We show that mode entanglement is not required for photons or massive bosons for beating the SQL. In particular, with a fluctuating number of two-mode bosons, the shot-noise limit can be beaten by nonentangled bosonic states with all bosons in one mode. As a consequence, the trivially present entanglement due to symmetrization that appears when writing bosonic states in terms of individual particles is a useful resource for precision measurements.

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I. INTRODUCTION

Suppose a parameter-dependent probability distribution $\mu_\theta(\xi)$ arises from the description of a classical system consisting of N independent parties. No matter what estimator one uses for estimating the parameter θ from measured values ξ_i drawn from the probability distribution, a universal lower bound to the best mean-square error in the determination of θ is given by the inverse of the classical Fisher information $F[\mu,\theta]$: this at best behaves as $1/N$, a scaling known as the shot-noise limit or standard quantum limit [1,2] (SQL), as a simple consequence of the central limit theorem.

The field of quantum metrology concerns the use of quantum mechanical features to improve on the above classical limitation [3,4]. In particular, by using systems consisting of N subsystems prepared in entangled states, the shot-noise limit can be beaten [5]. The squared sensitivity of the determination of a parameter θ has been proved to be bounded from below by the inverse of a quantity known as quantum Fisher information [6,7]. The quantum Fisher information can scale as fast as N^2 if the state ρ_θ of the system represents certain specific N-partite entangled states, a scaling known as the Heisenberg limit. Based on this, in the literature one often finds stated that, albeit not sufficient, entanglement is necessary for overcoming the shot-noise limit (see [8] for a recent review).

However, in experimental contexts where identical particles are used for metrological purposes, as for instance ultracold atoms trapped in double-well potentials which can be effectively described as two-mode bosons [9,10], the very notion of entanglement has to be generalized with respect to the case when the constituent parties, say qubits, are distinguishable. Indeed, in this latter case, there is a natural tensor product structure related to the particle aspect of first quantization. For instance, for two qubits, the Hilbert space is $\mathbb{C}^2 \otimes \mathbb{C}^2$ and the algebra of observables is $M_2 \otimes M_2$, where M_2 is the algebra of 2×2 matrices for the first and second qubit, respectively. Instead, in the case of identical particles, such a structure is no more available: the fermionic sector of the Hilbert space is one dimensional, while the bosonic one is $\mathbb{C}^3$. A way out is to address entanglement always in relation to a given algebraic context specified by a suitable mode description typical of the second quantization formalism [11,12].

In the following, after recalling the basics of quantum parameter estimation theory and entanglement of indistinguishable particles, we compare three different physical realizations of a beam splitter (BS) and a Mach-Zehnder (MZ) interferometer based on (1) distinguishable spins or qubits, (2) photons, and (3) massive bosons, with respect to the presence or absence of entanglement of equivalent states. We show that the naturally occurring entanglement of massive bosons due to symmetrization is sufficient for beating the SQL. In terms of mode entanglement, such states can be separable, as demonstrated with an N-boson two-mode Fock state. In the case when the number of identical bosons is not fixed, the SQL can be beaten by nonentangled states with all bosons in one mode, without the need of partially populating the other mode. Due to the particle-number superselection rule, the exact formal equivalence of different physical realizations of the MZ breaks down in this case. We also identify the optimal one-mode and two-mode pure states in the sense of maximum quantum Fisher information for a given maximum number of bosons and show that for two modes in the absence of decoherence, these are maximally entangled (NOON) states.

II. QUANTUM METROLOGY WITH INDISTINGUISHABLE PARTICLES

A. Basic quantum parameter estimation theory

Consider a (possibly mixed) quantum state ρ_θ that depends on the parameter θ whose value we want to find out as precisely as possible. N repeated generalized measurements with positive operator-valued measure (POVM) elements [non-negative Hermitian operators $E(\xi)$, $\int d\xi\, E(\xi) = \mathbf{1}$] in the identically prepared state ρ_θ lead to N measurement outcomes ξ_i $(i = 1,\dots,N)$, distributed according to $\mu_\theta(\xi) = \mathrm{tr}\rho_\theta E(\xi)$. One estimates the value of θ based on these N outcomes ξ_i with an estimator function $\theta_{\mathrm{est}}(\xi_1,\dots,\xi_N)$. The squared sensitivity with which θ can be estimated from the data is defined as $\langle(\delta\theta)^2\rangle$, where

$$\delta\theta = \frac{\theta_{\rm est}}{\left|\frac{d\langle\theta_{\rm est}\rangle}{d\theta}\right|} - \theta, \tag{1}$$

and the average $\langle\cdot\rangle$ is over $\mu_\theta(\xi)$, $\langle\theta_{\rm est}(\xi_1,\ldots,\xi_N)\rangle = \int(\prod_{i=1}^N \mu_\theta(\xi_i)d\xi_i)\theta_{\rm est}(\xi_1,\ldots,\xi_N)$.

For an unbiased estimator ($\langle\theta_{\rm est}\rangle = \theta$ locally at the value of θ we are interested in, which we take without restriction of generality as $\theta = 0$ in the following), a universal lower bound of $\langle(\delta\theta)^2\rangle$ is provided by the inverse of the quantum Fisher information $F[\rho]$,

$$\langle(\delta\theta)^2\rangle \geqslant \frac{1}{F[\rho]}, \tag{2}$$

where $F[\rho] = {\rm Tr}(\rho\, L^2)$ and

$$\partial_\theta\rho_\theta|_{\theta=0} = \tfrac{1}{2}(\rho L + L\rho) \tag{3}$$

defines the symmetric logarithmic derivative L of the quantum state. This so-called quantum Cramér-Rao bound [6,7] limits the best sensitivity achievable for a given parameter-dependent state ρ_θ, regardless of the choice of measurements and the data analysis, as it is optimized over all POVM measurements, in addition to the optimization over all possible estimators used for the derivation of the classical Cramér-Rao bound [1]. According to Fisher's theorem, the bound can be saturated in the limit of $N \to \infty$ [13]. Beating the shot-noise limit that is making

$$\langle(\delta\theta)^2\rangle < 1/N \tag{4}$$

necessarily requires $F[\rho] > N$. Note that instead of measuring the same system N times with an identical initial preparation for each measurement, one can equivalently measure once a composite system consisting of N identical subsystems in an initial product state.

The quantum Fisher information can be written as $F[\rho_\theta] = 4d^2_{\rm Bures}(\rho_\theta,\rho_{\theta+d\theta})$ in terms of the Bures distance $d_{\rm Bures}(\rho,\sigma) = \sqrt{2[1-f(\rho,\sigma)]}$, where the fidelity $f(\rho,\sigma) = {\rm tr}[(\rho^{1/2}\sigma\rho^{1/2})^{1/2}]$ [7,14]. For pure states $\rho = |\psi\rangle\langle\psi|$, $\sigma = |\phi\rangle\langle\phi|$, the fidelity f reduces to the overlap $f(\rho,\sigma) = |\langle\psi|\phi\rangle|$. Therefore, one has the intuitive and information-theoretically plausible interpretation that the distinguishability of two neighboring states whose parameters θ and $\theta + d\theta$ differ by an infinitesimal amount determines the best sensitivity with which θ can be obtained through measurement of whatever observables.

If $\rho = \rho_{\theta=0}$ is pure, then $\rho = |\psi\rangle\langle\psi|$ and $\rho_{d\theta}$ is created from ρ through a unitary rotation with self-adjoint generator $J = J^\dagger$ from ρ,

$$\rho \mapsto \rho_{d\theta} = e^{-id\theta J}\rho e^{id\theta J}, \tag{5}$$

and one shows that

$$F[\psi] = 4\Delta^2_\psi J = 4[\langle\psi|J^2|\psi\rangle - (\langle\psi|J|\psi\rangle)^2] \equiv F[\psi,J], \tag{6}$$

where in the last step and from now on we make the dependence on the generator J explicit [7]. For a fully separable pure state $|\psi\rangle$ of N distinguishable subsystems and a generator J that is a sum of operators of the individual subsystems, it turns out that $\Delta^2_\Psi J \leqslant N/4$ [7]. It follows that in such a situation, entanglement is necessary to achieve sensitivities beyond the shot-noise limit, otherwise

$$\langle(\delta\theta)^2\rangle \geqslant \frac{1}{F[\psi,J]} \geqslant \frac{1}{N}. \tag{7}$$

Several ways are known by now of how this limitation can be surpassed. One of them is to have N distinguishable subsystems interact with a single $N+1$st system (a "quantum bus") and read out the latter [15,16]. This method has the advantage that the system needs to accommodate only N interaction terms, as opposed to "nonlinear schemes" that employ k-body interactions [17–24]) and require that N particles all interact with each other. The scaling with N of the "quantum bus scheme" is stable under local decoherence, and even decoherence itself can be used as a signal, if the $N+1$st system is an environment.

In the following, we explore a third option, namely, the use of indistinguishable particles. Before addressing this possibility, it is necessary to stress that for identical particles, the notion of separability (entanglement) cannot be given independently of the modes that are selected for the description of the system.

B. Separability and entanglement for identical particles

Identical bosons are best addressed within the second quantization formalism by means of the Fock representation: we shall denote by $|{\rm vac}\rangle$ the vacuum state and by a_i, $a_i^\dagger$ the annihilation and creation operators relative to an orthonormal basis $\{|i\rangle\}_{i\in I}$ in the single-particle Hilbert space. They satisfy the commutation relations $[a_i, a_j^\dagger] = \delta_{ij}$, $[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0$; furthermore, states $|n_1,n_2,\ldots,n_k\rangle$ with n_i bosons in the single-particle states $|i\rangle$, $i = 1,2,\ldots,k$, are generated by acting on the vacuum as follows:

$$|n_1,n_2,\ldots,n_k\rangle = \frac{\prod_{i=1}^k (a_i^\dagger)^{n_i}}{\sqrt{\prod_{i=1}^k n_i!}}|{\rm vac}\rangle. \tag{8}$$

In the following, we shall be dealing with identical bosons that can be found in two modes identified by pairs of creation and annihilation operators $a,a^\dagger$ and $b,b^\dagger$, respectively, satisfying the canonical commutation relations $[a,a^\dagger] = [b,b^\dagger] = 1$, while the remaining ones all vanish. We shall consider the Fock representation based on a vacuum state $|{\rm vac}\rangle$ so that the states

$$|n_a,n_b\rangle = \frac{(a^\dagger)^{n_a}(b^\dagger)^{n_b}}{\sqrt{n_a!n_b!}}|{\rm vac}\rangle, \quad n_{a,b} \in \mathbb{N}, \tag{9}$$

constitute the orthonormal basis of eigenstates of the Fock number operator $a^\dagger a + b^\dagger b$, with n_a bosons in one mode and n_b bosons in the other one.

In the second quantization formalism, there is no predefined algebraic tensor product structure as for distinguishable particles. In the latter case, one starts out with the tensor product of the algebras of operators acting on the Hilbert spaces of the single particles: for instance, in the case of one qubit, the operator algebra is the 2×2 complex matrix algebra M_2 and, in the case of two distinguishable qubits, it is the 4×4 matrix algebra $M_2 \otimes M_2$.

In the absence of a definite tensor product structure, an alternative approach to locality (of observables) and separability (of states) must be developed [11,12]: observe that the main property of local observables $A \otimes 1$ and $1 \otimes B$ for a bipartite system consisting of distinguishable particles is that they commute.

We shall replace the tensor product structure by pairs of commuting subalgebras $(\mathcal{A},\mathcal{B})$ generated by $\{a,a^\dagger\}$ and $\{b,b^\dagger\}$, respectively, and define

(i) $(\mathcal{A},\mathcal{B})$ local all operators of the form $A\,B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$; and

(ii) $(\mathcal{A},\mathcal{B})$ separable all N-boson density matrices ρ such that the associated mean values of local operators split into convex combinations of products of mean values with respect to other N-boson density matrices [see (11) and (29) for concrete examples], namely, if

$$\mathrm{Tr}(\rho\, A\, B) = \sum_i \lambda_i\, \mathrm{Tr}\big(\rho_i^{(1)} A\big)\, \mathrm{Tr}\big(\rho_i^{(2)}\, B\big), \quad \lambda_i \geqslant 0, \qquad \sum \lambda_i = 1, \tag{10}$$

for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Remark 1. The above algebraic formulation of locality and separability is fully general and neither restricted to finitely many bosons with finite dimensional single-particle Hilbert space nor to states described by density matrices. Indeed, in quantum statistical systems with infinitely many degrees of freedom, as in quantum field theory, states on the algebra of operators are generic linear, positive functionals $X \mapsto \omega(X)$ associating to an observable X its mean value $\omega(X)$. These functionals need not be representable as $\omega(X) = \mathrm{Tr}(\rho X)$ by means of a density matrix ρ [25].

The simplest examples of $(\mathcal{A},\mathcal{B})$ separable states are the Fock states in (9); indeed, expectations of $(\mathcal{A},\mathcal{B})$ local observables $A\,B$, $A \in \mathcal{A}$, and $B \in \mathcal{B}$ factorize,

$$\langle n_a,n_b|A\,B|n_a,n_b\rangle = \langle n_a,n_b|A|n_a,n_b\rangle\,\langle n_a,n_b|B|n_a,n_b\rangle, \tag{11}$$

and show no correlations among these commuting observables.

For N indistinguishable two-mode bosons, the Hilbert space is $\mathbb{C}^{N+1}$, and the Fock number states $|k,N-k\rangle$, $0 \leqslant k \leqslant N$, form an orthonormal basis with respect to which a generic state is represented by a density matrix,

$$\rho = \sum_{k,\ell=0}^{N} \rho_{k\ell}\, |k,N-k\rangle\langle \ell,N-\ell|. \tag{12}$$

It turns out [11] that any such density matrix is $(\mathcal{A},\mathcal{B})$ separable if and only if

$$\rho = \sum_{k=0}^{N} \rho_{kk}\, |k,N-k\rangle\langle k,N-k|; \tag{13}$$

namely, if and only if it is diagonal in the $(\mathcal{A},\mathcal{B})$ Fock number states.

The definition of separable states for N bosons given above is a direct extension of the standard one for distinguishable particles. What should be remarked is that while in the case of distinguishable particles the tensor product structure is somewhat taken for granted and one need not specify that locality and separability always refer to it, it is not so in the case of identical bosons: in such a case, it must always be specified with respect to which pair $(\mathcal{A},\mathcal{B})$ a state is separable. Indeed, it is easy to see that Bogoliubov transformations, such as those implemented by beam splitters in quantum optics or in cold-atom interferometry, transform the mode operators $\{a,a^\dagger\}$, $\{b,b^\dagger\}$ into new mode operators $\{c,c^\dagger\}$, $\{d,d^\dagger\}$ such that the $(\mathcal{A},\mathcal{B})$ separable state in (9) turns out not to be separable with respect to the new pair of commuting subalgebras generated by $\{c,c^\dagger\}$ and $\{d,d^\dagger\}$. Consider, for instance, the transformation

$$a = \frac{c+d}{\sqrt{2}}, \quad b = \frac{c-d}{\sqrt{2}}, \tag{14}$$

associated with the action of a 50/50 beam splitter: the $(\mathcal{A},\mathcal{B})$ separable state $|n_a,n_b\rangle$ becomes

$$|n_a,n_b\rangle = \frac{2^{-(n_a+n_b)/2}}{\sqrt{n_a!n_b!}} \sum_{q=0}^{n_a}\sum_{p=0}^{n_b} (-1)^{n_b-q} \binom{n_a}{p}\binom{n_b}{q} \times (c^\dagger)^{p+q} (d^\dagger)^{n_a+n_b-p-q}\, |\mathrm{vac}\rangle.$$

This vector state is a superposition of eigenstates of $c^\dagger c$ and $d^\dagger d$ and thus it turns out to be $(\mathcal{A},\mathcal{B})$ separable, but $(\mathcal{C},\mathcal{D})$ entangled. Notice also that the Bogoliubov transformation is generated by a unitary operator which is not $(\mathcal{A},\mathcal{B})$ local since its generator is of the form $-i(\alpha a^\dagger b - \alpha^* a b^\dagger)$ [see (17) below].

Other definitions of entanglement for indistinguishable particles have been discussed in the literature, e.g., in the context of "generalized entanglement" (see [26–28] and references therein).

III. PARAMETER ESTIMATION WITH A SINGLE BEAM SPLITTER

A major part of the literature on quantum parameter estimation deals with the estimation of a phase shift in one arm of a Mach-Zehnder interferometer. We will consider that situation in the next section, but for the sake of clarity, we first focus on an even simpler example: the estimation of the transparency of a single beam splitter. Different physical realizations of a beam splitter are possible, with corresponding different interpretations as to the presence or absence of entanglement for a given input state, which we discuss now.

A. Different physical realizations of a beam splitter

The most obvious realization of a beam splitter is found in quantum optics, where the BS creates a coupling between two different modes a and b of the light field (represented typically by two orthogonal directions under which light can fall onto the BS). More precisely, the BS generates a Bogoliubov rotation of the two modes a and b,

$$U_{\mathrm{BS}}(\alpha) a U_{\mathrm{BS}}(-\alpha) = a\cos(|\alpha|) - b e^{i\theta}\sin(|\alpha|), \tag{15}$$

$$U_{\mathrm{BS}}(\alpha) b U_{\mathrm{BS}}(-\alpha) = b\cos(|\alpha|) - a e^{i\theta}\sin(|\alpha|), \tag{16}$$

with $\alpha = |\alpha|\exp(i\theta)$ as a complex parameter characteristic of the BS, via the unitary operator

$$U_{\mathrm{BS}}(\alpha) = e^{\alpha a^\dagger b - \alpha^* a b^\dagger}. \tag{17}$$

Each of the two orthonormal mode functions is a solution of the classical Maxwell equations with appropriate boundary conditions, such as plane waves with a given wave vector and polarization in the case of vacuum and periodic boundary conditions. Each mode corresponds to a harmonic oscillator due to the fact that the energy of the electromagnetic field is quadratic in both the electric and magnetic fields. A state $(a_i^\dagger)^n|\text{vac}\rangle$ means the ith oscillator being excited in the nth one of its excited states, i.e., it corresponds to n photons in mode i (see, e.g., [29] or any other textbook on quantum optics). When talking about the need of entanglement for surpassing the SQL, entanglement means most naturally the entanglement of these two harmonic oscillators.

The interaction of the two modes represented by the BS can lead to entanglement when acting on a product state. This is most easily seen from a simple one-photon example: the state $|1\rangle|0\rangle$ (with one photon in mode a and no photon in mode b) is mapped by a 50:50 BS with $|\alpha| = \pi/4$, $\theta = \pi$ onto the state $(|10\rangle + |01\rangle)/\sqrt{2}$, which is clearly entangled. Another simple physical example that shows that a simple BS can create entanglement is the input state $|1,1\rangle$, with one photon in each of the two input modes, which after the BS becomes $(|2,0\rangle + |0,2\rangle)/\sqrt{2}$, i.e., due to the bosonic nature of the photons, a bunching of the two photons occurs (Hong-Ou-Mandel effect). Interestingly, however, a product state of two coherent states in the two modes remains a product state (see [30], p. 354).

Entanglement that is purely due to the (anti)symmetrization required by the indistinguishability of the particles involved has traditionally been considered as trivial and useless. In particular, it does not allow demonstrating the violation of a Bell inequality, since for doing so one needs local unitary operations. These would map state $|0\rangle$ in a superposition of $|0\rangle$ and $|1\rangle$, which is, however, impossible due to the particle-number conservation superselection rule (In the context of mesoscopic systems [31,32], one finds the term "fluffy bunny entanglement"). Recently, particle-number nonconserving superpositions have attracted renewed interest in the context of cold atoms. It was suggested that when the two modes are coupled to a third one containing a Bose-Einstein condensate that forms a "coherent" reservoir, such superpositions might become possible and might have applications for quantum communication [33,34]. Below we will argue that this kind of "naturally occurring" entanglement is very useful for precision measurements.

As pointed out, e.g., in Ref. [35], the mathematically exact same operation of a BS can also be achieved in multispin-1/2 systems or multiqubit systems. Formally, this goes back to the Schwinger representation that associates to the two-mode bosons the angular-momentum-like operators [36]

$$J_x = \frac{a^\dagger b + ab^\dagger}{2}, \quad J_y = \frac{a^\dagger b - ab^\dagger}{2i}, \quad J_z = \frac{a^\dagger a - b^\dagger b}{2}. \tag{18}$$

One checks that with these definitions, the usual commutation relations between angular-momentum components are reproduced (with $\hbar = 1$), i.e., $[J_x, J_y] = iJ_z$, etc. Focusing momentarily on purely imaginary parameters $\alpha = i|\alpha|$, we see that Eq. (17) can be reexpressed as

$$U_{\text{BS}}(|\alpha|) = e^{i2|\alpha|J_x}. \tag{19}$$

J_x in turn can be obtained in the case of N distinguishable spins or qubits as the sum of the x components of the individual (pseudo)angular momenta $j_x^{(i)} = \sigma_x^{(i)}/2$, $J_x = \sum_{i=1}^N j_x^{(i)}$, where $\sigma_x^{(i)}$ is a Pauli spin operator. The very use of a label i signals the spins as distinguishable. Therefore, the unitary operator $U_{\text{BS}}(|\alpha|)$ on $\mathbb{C}^{2^N}$ is a local transformation as it splits into the tensor product of individual unitary operators, $U_{\text{BS}}(|\alpha|) = \bigotimes_{i=1}^N e^{2i|\alpha|\, j_x^{(i)}}$. For this incorporation of the BS, it is thus clear that in order to outperform the SQL, one needs to operate with the BS on an entangled initial state. Physically, the local unitaries can be obtained, for example, by microwave pulses in electron spin resonance experiments. However, since a spin-1/2 is equivalent to a qubit, it can also be interpreted in purely information-theoretical terms. Thus, we may describe a BS in terms of qubits, such that a single qubit encodes the information whether a particle that passes the BS is in mode a or mode b (states $|0\rangle$ or $|1\rangle$, respectively, in qubit parlance, or $|\uparrow\rangle$ or $|\downarrow\rangle$ for spins-1/2). This realization of the BS allows for the most transparent assessment of the effect of indistinguishability, as one may evaluate the Fisher information separately for N-particle states that are symmetrized (as is relevant for identical bosons) or not (as is possible for distinguishable spins).

A third physical realization of the BS is obtained in the context of ultracold trapped atoms, due to the formal equivalence of the creation and annihilation operators for bosons in second quantization formalism with the raising and lowering operators of harmonic oscillators in first quantization, such that $(a_i^\dagger)^n|\text{vac}\rangle$ means that n identical bosons are created in the ith single-particle state, where i now labels different single-particle orthonormal basis vectors, i.e., two orthonormal solutions of the single-particle Schrödinger equation. In the Bose-Hubbard approximation, instances of single atom orthonormal bases for ultracold atoms trapped in a double-well potential are states corresponding to an atom being localized in either one or the other of the two wells, or the first two energy eigenstates of the single-particle Hamiltonian. Within the context of indistinguishable particles, $U(|\alpha|)$ is $(\mathcal{A},\mathcal{B})$ nonlocal as it cannot be split into the product of some $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

B. Fock states in the three different settings

In order to illustrate the physical consequences of the indistinguishability of particles on the sensitivity of measurements, we consider as the input state of the BS an $(\mathcal{A},\mathcal{B})$ separable number Fock state as in (9), with k bosons in mode a and $N-k$ bosons in mode b,

$$|k, N-k\rangle = \frac{(a^\dagger)^k (b^\dagger)^{N-k}}{\sqrt{k!(N-k)!}}|\text{vac}\rangle. \tag{20}$$

We call a state with $k = N$ or $k = 0$ a one-mode state, as all bosons are in the first or second mode, respectively. It is straightforward to calculate the Fisher information corresponding to the action $U(|\alpha|)$ of the BS in state (20),

$$\begin{aligned} F[|k,N-k\rangle, 2J_x] &= 16\,\Delta^2_{|k,N-k\rangle} J_x \\ &= 16\langle k,N-k|J_x^2|k,N-k\rangle \\ &= 4[N(2k+1) - 2k^2], \end{aligned} \tag{21}$$

which exceeds $4N$ for all $k \neq 0$ and $k \neq N$.

Therefore, except when all identical bosons are in one mode and none in the other, despite the $(\mathcal{A},\mathcal{B})$ separability of the Fock state, one can beat the shot-noise limit.

Remark 2. The additional factor 4 arises from the fact that α is multiplied with a factor 2 in the exponent of $U_{\mathrm{BS}}(\alpha)$, such that the more natural angle to be considered would in fact be 2α (with a SQL of $F[|k,N-k\rangle,J_x] = N$).

In the just-studied case, it is the $(\mathcal{A},\mathcal{B})$ nonlocality of the rotation $U_{\mathrm{BS}}(\alpha)$ in (19) generated by J_x that allows beating the SQL. Indeed, if the state ρ in $F[\rho,J]$ is $(\mathcal{A},\mathcal{B})$ separable and subjected to an $(\mathcal{A},\mathcal{B})$ local continuous unitary transformation $U(\theta) = \exp(i\theta J)$, then the Fisher information identically vanishes. This can be seen as follows: we know from (13) that $(\mathcal{A},\mathcal{B})$ separable states are diagonal in the $(\mathcal{A},\mathcal{B})$ Fock number state basis. Thus, the convexity of the Fisher information [37,38] gives

$$F[\rho^{\mathrm{sep}},J] \leqslant \sum_{k=0}^{N} \rho_{kk}\, F[|k,N-k\rangle,J],$$

for all $(\mathcal{A},\mathcal{B})$ separable two-mode boson states. Furthermore, the locality assumption on $U(\theta)$ implies that its generator J must be of the form $J = A(a,a^\dagger) + B(b,b^\dagger)$, where A and B are (Hermitian) functions of $(a,a^\dagger)$ and $(b,b^\dagger)$. Indeed, by the locality assumption $U(\theta) = U_a(\theta)U_b(\theta)$, where $U_a(\theta)$, $U_b(\theta)$ involve operators $a,a^\dagger$ and $b,b^\dagger$ only; then by differentiating, $U(\theta)$ at $\theta = 0$ one gets $J = A(a,a^\dagger) + B(b,b^\dagger)$, since $U(0) = U_a(0) = U_b(0) = 1$. Furthermore, the assumed fixed boson number N in the context considered forces $U(\theta)$ to preserve the total boson number $\hat{N} = a^\dagger a + b^\dagger b = N$. Consequently,

$$[U_a(\theta),a^\dagger a] = [U_a(\theta),N - b^\dagger b] = 0,$$

and analogously $[U_b(\theta),b^\dagger b] = 0$. Therefore, the eigenvectors of $U_a(\theta)$ and $A(a,a^\dagger)$ are the eigenvectors $|k\rangle = (a^\dagger)^k|\mathrm{vac}\rangle/\sqrt{k!}$ of $a^\dagger a$; similarly, the eigenvectors of $b^\dagger b$, $|N-k\rangle = (b^\dagger)^{N-k}|\mathrm{vac}\rangle/\sqrt{(N-k)!}$ are eigenvectors of $U_b(\theta)$ and $B(b,b^\dagger)$. Therefore, the mean-square errors $\Delta^2_{|k\rangle}A(a,a^\dagger) = \Delta^2_{|N-k\rangle}B(b,b^\dagger) = 0$. Finally, from (6) and $[A(a,a^\dagger),B(b,b^\dagger)] = 0$, it follows that

$$F[|k,N-k\rangle,J] = 4\big[\Delta^2_{|k\rangle}A(a^\dagger a) + \Delta^2_{|N-k\rangle}B(b^\dagger b)\big] = 0.$$

The fact that no information can be extracted in such a case is due to the absence of phase relations between the two modes either embodied by the state or induced by the rotation.

When considering a BS from the point of view of distinguishable particles, a separable state of N qubits, which matches the Fock state (20) in terms of the expectation values of J_x, J_y, and J_z, and in terms of the interpretation of k particles in mode a and $N-k$ particles in mode b, is

$$|\varphi\rangle = |0\rangle_1 \ldots |0\rangle_k |1\rangle_{k+1} \ldots |1\rangle_N. \tag{22}$$

Clearly, $\langle J_x\rangle = 0$, and $\langle J_x^2\rangle = \frac{1}{4}\langle(\sum_{i=1}^{N}\mathbf{1}^{(i)} + \sum_{i\neq j}\sigma_x^{(i)}\sigma_x^{(j)})\rangle = N/4$. Thus, $F[|\varphi\rangle,2J_x] = 4N$, and the SQL can therefore not be beaten with this state, as expected. The lack of interaction between the particles in a BS process implies that before undergoing an interferometric protocol, the initial state should be first made entangled by some suitable procedure, for instance by spin squeezing as suggested by Wineland [9,10,39].

Of course, the same state and the same unitary transformation should yield the same Fisher information, irrespective of how they are realized physically. Clearly, however, $|\varphi\rangle \neq |k,N-k\rangle$, as $|\varphi\rangle$ is not symmetrized, contrary to $|k,N-k\rangle$. It is instructive to check that a symmetrization of $|\varphi\rangle$ does lead back to the exact same result as (21). To show this, consider the symmetrized state

$$|\varphi_s\rangle \equiv \sum_{P\in S_N} |0\rangle_{P(1)} \ldots |0\rangle_{P(k)} |1\rangle_{P(k+1)} \ldots |1\rangle_{P(N)}/\mathcal{N}, \tag{23}$$

where the sum is over all permutations of the N qubits, and $\mathcal{N} = \sqrt{N!k!(N-k)!}$ is the normalization constant. To obtain the latter, observe that out of the $N!$ permutations only $\binom{N}{k}$ lead to distinct states, which are all orthogonal and appear each $k!(N-k)!$ times. We still have $\langle\varphi_s|J_x|\varphi_s\rangle = 0$. As for $J_x^2 = (1/4)(\sum_{i=1}^{N}\mathbf{1}^{(i)} + \sum_{i\neq j}^{N}\sigma_x^{(i)}\sigma_x^{(j)})$, the last term only contributes if the number of qubits in the state $|0\rangle$ (or $|1\rangle$) does not change, i.e., if $\sigma_x^{(i)}$ and $\sigma_x^{(j)}$ act in different subgroups (e.g., $\sigma_x^{(i)}$ on a qubit in state $|0\rangle$ and $\sigma_x^{(i)}$ on a qubit in state $|1\rangle$, or vice versa). If this is enforced, the constraint $i \neq j$ is automatically fulfilled. There are thus $k(N-k) + (N-k)k = 2k(N-k)$ contributions for each component in (23). The first sum over S_N gives a factor $N!$, and from the second sum over S_N there are $(N-k)!k!$ terms that give the same component. Altogether, we therefore have

$$F[|\varphi_s\rangle,J_x] = 4\Delta^2_{|\varphi_s\rangle}J_x = 4\bigg[N + 2k(N-k)\frac{1}{N!k!(N-k)!} \times N!(N-k)!k!\bigg] \tag{24a}$$

$$= 4[N(2k+1) - 2k^2], \tag{24b}$$

i.e., exactly the same result as for the identical bosons, given by Eq. (21). This demonstrates most clearly that the "naturally occurring" entanglement due to the symmetrization for identical bosons is a free resource that does allow the improvement of sensitivity over the SQL. Contrary to distinguishable spins or qubits, no additional entanglement creation (e.g., by squeezing) is required at the input ports of the BS. Thus, while the puffy bunny entanglement might be useless for demonstrating a violation of Bell inequalities as explained above, it *is* useful for precision measurements.

IV. PHASE ESTIMATION WITH A MACH-ZEHNDER INTERFEROMETER

Let us now consider a Mach-Zehnder (MZ) interferometer consisting of two equal beam splitters and a phase shifter.

After the first BS, a unitary rotation by an angle ϕ is generated by the number operator $a^\dagger a$, which implements a phase shift in the a mode, whereas the b mode experiences no phase shift. A second BS recombines the beams by means of $U_{\mathrm{BS}}(-\alpha)$.

The total effect on an incoming state ρ is described by the unitary operator

$$U_{\rm MZ}(\alpha,\phi) = U_{\rm BS}(-\alpha)e^{i\phi a^\dagger a}\,U_{\rm BS}(\alpha) = e^{i\phi J_\alpha},$$
$$J_\alpha = \bigg[\cos^2(|\alpha|)a^\dagger a + \sin^2(|\alpha|)b^\dagger b + \frac{\sin(2|\alpha|)}{2}(e^{i\theta}a^\dagger b + e^{-i\theta}ab^\dagger)\bigg], \tag{25a}$$
$$= \frac{\hat N}{2} + \cos(2|\alpha|)J_z + \sin(2|\alpha|)[\cos(\theta)J_x - \sin(\theta)J_y], \tag{25b}$$

where we have introduced the operator $\hat N = a^\dagger a + b^\dagger b$ for the total number of particles in both modes. One easily checks that it commutes with J_x, J_y, and J_z introduced in (18). Equation (25b) shows that a MZ acts in a very similar fashion as a single BS, but with a rotation generated in a different basis that depends on the individual BSs, and with a rotation angle given by the phase shift (without the factor 2, however, compared to a single BS).

The discussion of the necessity of entanglement is here slightly more subtle, as there is the additional freedom of considering states at different stages of the propagation through the MZ. In Ref. [5], a classification of different scenarios concerning the preparation of an entanglement at the input and detection using possible joint measurements projecting onto entangled states was proposed. It was found that the existence of entanglement is decisive at the stage where the parameter to be measured is imprinted onto the state, i.e., for the MZ after the first BS. In the following, we discuss different input states for the MZ and calculate the resulting best sensitivity, examining at the same time the presence of entanglement of the input state and of the state after the first BS for the three different physical situations described in Sec. III.

A. Two-mode Fock state

If the input state to the MZ is a two-mode Fock state $|k,N-k\rangle$ with k bosons in the first mode and $N-k$ in the second one, then the Fisher information (6) can be computed to be

$$F[|k,N-k\rangle,J_\alpha] = \sin^2(2|\alpha|)[2k(N-k)+N]. \tag{26}$$

For fixed $|\alpha|$, the SQL can be beaten (i.e., $F[|k,N-k\rangle, J_\alpha] > N$) if and only if $N > 2\cot^2(2|\alpha|)$ and

$$\frac{N}{2}\bigg[1-\sqrt{1-\frac{2}{N}\cot^2(2|\alpha|)}\bigg] < k < \frac{N}{2}\bigg[1+\sqrt{1-\frac{2}{N}\cot^2(2|\alpha|)}\bigg].$$

Remark 3. The upper and lower bounds on the number of bosons in the left well, k, make apparent the symmetry between exchanging the boson content of the two wells already present in (26). Also, if the phase shift by ϕ in (25a) is operated by $b^\dagger b$ instead of by $a^\dagger a$, then the generator associated with MZ reads

$$J_\alpha = \frac{\hat N}{2} - \cos(2|\alpha|)J_z - \sin(2|\alpha|)[\cos(\theta)J_x - \sin(\theta)J_y],$$

and leads to the same conclusions.

The entanglement properties of the two-mode Fock state and the state after the first BS are, of course, as discussed in Sec. III: the input state is $(\mathcal{A},\mathcal{B})$ separable, but becomes $(\mathcal{A},\mathcal{B})$ entangled after the first BS for generic BS parameters. If realized in terms of individual qubits (without symmetrization), the corresponding input state (22) as well as the state after the BS are separable, but symmetrization makes the state entangled. In terms of two optical modes in the case of photons, the input state is separable, but the first BS generically entangles it (unless its transparency equals 0 or 1). But the two-mode entanglement cannot explain the improvement of the sensitivity over the SQL: The proof of the necessity of entanglement for beating the SQL is based on N distinguishable particles, using notably a sum of local Hamiltonians, $H = \sum_i h_i$, where h_i acts on particle number i. For two modes, one has $N = 2$ and cannot make any statement about scaling with N. Therefore, this reasoning can only be applied to the realization of the MZ with N distinguishable spins or qubits. For the case of separable qubits at the input of the MZ, it is clear, however, that the SQL cannot be beaten, as the state remains separable after the first BS. The theorem in Ref. [5] applies, thus requiring an entangled multispin input state. It is known that a permutationally symmetric multispin state is pairwise entangled if it is spin squeezed [40]. This is no longer true if permutational symmetry is relaxed, but spin squeezing always implies entanglement (see Sec. 8.1 of Ref. [41] and references therein, and [42]).

The symmetrized state of N qubits that corresponds to $|k,N-k\rangle$ is completely equivalent to the latter, is entangled both before and after the first BS, and allows one to beat the SQL. However, its physical creation in terms of distinguishable spins (or qubits) requires substantial resources for creating it (see, e.g., [9,42–44]). As for photons, even a single-mode Fock state is hard to generate (see, e.g., [45] for the creation of microwave Fock states by quantum feedback in a single-mode ultra-high-finesse cavity), as decoherence rapidly tends to mix it with states with different photon numbers. For massive bosons, one might think that a Fock state is the most natural state, as the particle number is fixed, even if it is not precisely known. However, since experiments have to be repeated to collect sufficient statistics, fluctuating numbers of bosons between different runs effectively create mixtures of Fock states. In practice, the best experimental approximation to a Fock state are number squeezed states, with so far a few dB of squeezing demonstrated experimentally [9,10,44]. It remains to be seen to what extent novel measurement techniques such as the quantum gas microscope [46] will enable precise knowledge of boson numbers and thus the preparation of Fock states with a large number of atoms.

B. One-mode states

Suppose a system of identical bosons is prepared as input to the interferometer in a state of the form

$$|\Psi\rangle = \sum_k c_k|k,0\rangle, \quad |k,0\rangle = \frac{(a^\dagger)^k}{\sqrt{k!}}|{\rm vac}\rangle, \tag{27}$$

with all bosons in mode a, that is, $a^\dagger\, a|k,0\rangle = k\,|k,0\rangle$, $b^\dagger\, b|k,0\rangle = 0$. We shall only demand that the mean boson number

$$\overline{N} = \langle\Psi|\,(a^\dagger a + b^\dagger b)|\Psi\rangle = \sum_k p_k k \tag{28}$$

be finite, where $p_k = |c_k|^2$, $\sum_k p_k = 1$.

Observe that such states are $(\mathcal{A},\mathcal{B})$ separable according to the definition of mode or algebraic separability embodied by equality (11). Indeed, on products of the form $(a^\dagger)^p a^q (b^\dagger)^r b^s$ and thus on all polynomials in the mode operators, such states factorize as follows:

$$\begin{aligned}\langle\Psi|(a^\dagger)^p a^q (b^\dagger)^r b^s|\Psi\rangle\\ &= \sum_{k,\ell} c_k^* c_\ell \langle \text{vac}|a^k (a^\dagger)^p a^q (a^\dagger)^\ell (b^\dagger)^r b^s|\text{vac}\rangle\\ &= \langle\Psi|(a^\dagger)^p a^q|\Psi\rangle\langle\Psi|(b^\dagger)^r b^s|\Psi\rangle. \end{aligned}\tag{29}$$

Using (6), the quantum Fisher information corresponding to such a state and the generator J_α in (25a) is readily computed:

$$\begin{aligned}F[\Psi,J_\alpha] = 4\Bigg\{&\cos^4|\alpha|\left[\sum_k p_k k^2 - \left(\sum_k p_k k\right)^2\right]\\ &+ \cos^2|\alpha|\sin^2|\alpha|\sum_k p_k k\Bigg\}.\end{aligned}\tag{30}$$

The term in parentheses in the first line is the mean-square error of a stochastic variable X taking values on the natural numbers $X \in \mathbb{N}$ distributed according to the probabilities $p(X = k) = p_k$. The variance is always non-negative; thereby, choosing $\alpha = \pi/4$ yields $F[\Psi,\hat{N}/2 + J_x] \geqslant \overline{N}$. This already indicates the possibility of beating the shot-noise limit, that is, the bound (4) with the mean photon number $\overline{N}$ in the place of N.

Indeed, consider a balanced BS ($\alpha = \pi/4$) so that Eq. (25a) gives

$$J \equiv J_{\pi/4} = (a^\dagger a + b^\dagger b + a^\dagger b + ab^\dagger)/2 = \frac{\hat{N}}{2} + J_x, \tag{31}$$

and choose an input state with a finite fixed maximum number K of bosons,

$$|\Psi\rangle = \sum_{k=0}^{K} c_k|k,0\rangle. \tag{32}$$

By isolating in (30) the term $k = K$ and using $Kp_K = \overline{N} - \sum_{k=1}^{K-1} p_k k$, one gets

$$F[\Psi,\hat{N}/2 + J_x] = \overline{N}(1 + K - \overline{N}) + \sum_{k=1}^{K-1} p_k k(k - K).$$

Each term in the last sum is negative. The quantum Fisher information is thus optimized by choosing $p_k = 0$ for all $k \neq 0$, K, which in turn implies $p_K = \overline{N}/K$ and

$$|\Psi\rangle = \sqrt{1 - \frac{\overline{N}}{K}}\,|00\rangle + e^{i\chi}\sqrt{\frac{\overline{N}}{K}}\,|K,0\rangle. \tag{33}$$

Then,

$$F[\Psi,\hat{N}/2 + J_x] = \overline{N}(K - \overline{N} + 1). \tag{34}$$

Since the Fisher information is larger than the mean particle number, it thus follows that the shot-noise limit can be beaten by choosing a suitable $\overline{N}$. A similar conclusion has been reached in Ref. [47], where, differently from here, the authors considered a superposition of the vacuum state with a squeezed state acted upon by a rotation generated by the number operator. Earlier work on optimizing states for minimal phase uncertainty, notably in the context of squeezed states, can be found in Refs. [48–52]. Most of this earlier work used the notion of an approximate phase operator. In Ref. [53], it was shown how the optimal sensitivity of a Mach-Zehnder interferometer fed with multimode Gaussian states can be reached without entanglement by appropriate mode engineering.

One might wonder about the importance of the scaling with the average boson number $\overline{N}$ instead of N. Of course, N is only well defined for a Fock state, whereas for all other states, one has to live with fluctuating N. For laser light in a coherent state with $\overline{N} \gg 1$ or even the most squeezed states currently available [54,55], the fluctuations of N still satisfy $\sigma(N)/N \ll 1$, and the average photon number is therefore representative of the photon number in any realization. A state of the form (33) with $\overline{N} \simeq K/2$ maximizes the photon-number fluctuations, however, and obviously the average value of N is never realized (only $N = 0$ or $N = K$ are). Nevertheless, the scaling with $\overline{N}$ is highly relevant practically, as it corresponds to the mean energy in the state, which is indeed what makes producing the state costly. In Ref. [48], a state was proposed that leads to a very sharp maximum in the distribution of measured rotation angles, suggesting even exponential scaling of the phase uncertainty with $\overline{N}$.

From a physical perspective, it makes sense that the optimal state leads to maximum uncertainty in N, as, for a Heisenberg-uncertainty limited state, this corresponds to minimal uncertainty in θ. Since there is no entirely satisfactory definition of a phase operator, "Heisenberg-uncertainty limited" means here a state that saturates the Cramér-Rao bound. Indeed, inequality (2) has been understood from the beginning as a generalization of Heisenberg's uncertainty relation [7]. Equation (6) shows that the relevant "complementary observable" is the generator J of Eq. (5). Furthermore, for the one-mode states considered here, the fluctuations of J are given by the fluctuations of N. Therefore, the optimal one-mode state must be indeed the state that maximizes the photon-number fluctuations in that mode. The single mode "ON" state $(|0\rangle + |N\rangle)/\sqrt{N}$ was also identified as the optimal state for obtaining the best possible sensitivity of mass measurements with nanomechanical harmonic oscillators [56]. It has a Wigner function with N lobes in the azimuthal direction. A practical issue that might arise when using an ON state in an interferometer might be the lack of an intrinsic phase reference, which in the case of a NOON state is conveniently supplied by the second part, the "NO" state. This is particularly challenging at optical frequencies, whereas at lower frequencies stable references are available.

Let us now discuss again the possible realization of a state of the form (33). In the case of ultracold trapped atoms, the additional constraint of number superselection rules seems to exclude the possibility of superposing the vacuum state $|00\rangle$ with $|K,0\rangle$; however, it was argued that the use of BEC reference states coupled to the system of interest offers a way of making the entanglement of modes a concrete available resource [33,34,57–59].

With photons, coherent superpositions of Fock states have been produced for small numbers of photons; for recent circuit QED experiments, see, e.g., [60]. Nevertheless, decoherence becomes more and more rapid with increasing photon numbers, and since photons are cheap, it is unlikely that a state of the form (33) will ever be competitive with a simple coherent state where the photon number can be increased to almost arbitrary levels by just increasing laser power—unless particular constraints for very low-intensity light come into play, e.g., for living biological samples that one may want to study almost in the dark [61]. In terms of spins or qubits, since (33) is a superposition of states with different particle numbers, the particle-number conservation superselection rule again precludes the creation of the state. A fully symmetrized state of K spins does not help either, since then $j = K/2$ is fixed by the maximum possible J_z value.

Moreover, whereas we have seen that for a single BS one can easily create a spin state that exactly reproduces the Fisher information for a symmetric state of two-mode bosons, there is no obvious way of doing so for the MZ, as the term $\hat{N}$ in J_α does not arise in a physical realization of the MZ consisting of single-particle gates acting on spins or qubits. For a state with a fixed number of bosons, this does not matter, as $\hat{N}$ simply gives an irrelevant total phase that we do not have to reproduce with the spins. But for the superposition (27), $\hat{N}$ varies from one component to the other and makes an important contribution to the Fisher information. In fact, without this term, the relevant Fisher information is simply $F[|\psi\rangle, J_x] = \overline{N}$, and the SQL cannot be surpassed.

Can one nevertheless find a spin state that resembles (27) and reproduces the Fisher information in (30)? For large enough $\overline{N}$, when we can neglect linear terms in $\overline{N}$ compared to $\overline{N}^2$, this is indeed possible, as we shall show now—at least for bosonic states in which only even numbers of bosons are superposed. First, observe that for K spins, the maximum value of any spin component is given by $K/2$, reached in the irreducible representation (irrep) that is fully symmetric under permutation of particles. It appears therefore natural to seek a superposition of different irreps (corresponding to different maximum values of J_z), if we want to mimic states with fluctuating boson numbers. This implies immediately, however, that such a state will not be fully symmetric under particle exchange, contrary to the two-mode boson state. Furthermore, observe that adding one boson to a single mode of a two-mode boson state changes the maximum value of J_z as defined by (18) by 1/2, whereas the J_z for a given number of spins can only change in steps of 1. There are therefore twice as many components in (27) compared to a superposition $|\tilde{\psi}\rangle = \sum_{j=0}^{K/2} b_j |j,j\rangle$ (in $|j,m\rangle$ notation). In order to compare the two situations on equal footing, one should therefore restrict (27) to states with components that contain only even numbers of bosons, $|\tilde{\psi}\rangle = \sum_{k=0,2,\ldots}^{K} c_k |k\rangle|0\rangle \equiv \sum_{m=0,1,\ldots}^{K/2} b_m |2m\rangle|0\rangle$. Once this restriction is made, we see that in (30), the first two terms are exactly the fluctuations of angular-momentum component J_z in a state of form $|\tilde{\psi}\rangle$. In order to get the same Fisher information for J_x, one should therefore switch bases and consider $|j,m\rangle$ as an eigenbasis of J_x rather than J_z. In order to keep notations simple, we prefer to keep $|j,m\rangle$ as an eigenbasis of J_z and calculate $F[|\tilde{\psi}\rangle, J_z/2]$ instead, with the obvious result $F[|\tilde{\psi}\rangle, J_z/2] = \sum_{m=0}^{K/2} |b_m|^2 m^2 - (\sum_{m=0}^{K/2} |b_m|^2 m)^2$. Comparing with (30), for $\alpha = \pi/4$ we find that the results agree for large $\overline{N}$, when the last term in (30) can be neglected compared to the quadratic terms. Note that we have calculated the Fisher information for $J_z/2$ instead of J_z for being able to compare to $\hat{N}/2$. Of course, one could have reproduced the same Fisher information with a superposition in the same irrep with $J_z = K/2$, but the point here was to mimic an observable that corresponds to $\hat{N}$, different from but commuting with J_z.

The optimal state (33) with $\overline{N} = K/2$ corresponds to the state of K spins,

$$|\tilde{\psi}\rangle = (|j = 0, m = 0\rangle + |j = K/2, m = K/2\rangle)/\sqrt{2}. \quad (35)$$

The first component is a singlet state, thus not symmetric under particle exchange, preventing the whole state from being symmetric.

In summary, the additional term $\hat{N}/2$ in the Schwinger representation of the MZ prevents an exact reproduction of the bosonic MZ with symmetric spin states once one considers generic superpositions that do not conserve the total boson number. When giving up full permutational symmetry, a corresponding state with asymptotically the same Fisher information can be constructed with spins and a corresponding spin MZ. It is entangled already before the first BS, but has the advantage of being realizable with a fixed number of spins.

C. Optimal state

It has meanwhile become generally accepted that the so-called NOON state [62] is the optimal state for phase estimation in a MZ with a given maximum photon number if decoherence is neglected. In Ref. [63], numerical and some analytical results were shown as evidence for this statement, based on the classical Fisher information and photon counting measurements at the two output ports of the MZ. Here we give a simple demonstration using the quantum Cramér-Rao bound that the NOON state in the absence of losses and dephasing is optimal for a balanced MZ interferometer no matter what measurement is performed in the end. We do so mostly for completeness and for discussing once more the physical realizability and the role of entanglement in the three different settings, as meanwhile it has become clear that NOON states are not very useful in practice. The slightest chance of photon loss leads, in the limit of large N, back to the standard quantum limit [64–66].

We shall consider Eq. (31) and introduce a new class of orthogonal states adapted to the Schwinger representation: these are the pseudoangular momentum states $|jm\rangle_\ell$, $\ell = x,y,z$, that form the eigenbasis of J_l, such that

$$J_\ell |j,m\rangle_\ell = m|j,m\rangle_\ell, \quad \mathbf{J}^2 |j,m\rangle_\ell = j(j+1)|j,m\rangle_\ell. \quad (36)$$

One easily shows that in the Schwinger representation, $\mathbf{J}^2 = (\hat{N}/2)(\hat{N}/2 + 1)$, which implies that the usual pseudo-angular-momentum states $|jm\rangle_\ell$ are also eigenstates of $\hat{N}$, $\hat{N}|jm\rangle_\ell = 2j|jm\rangle_\ell$. Furthermore, if $\ell = z$, using the expression of J_z in Eq. (18), the Fock states $|k, N - k\rangle$ can be recast as common eigenstates $|jm\rangle_z$ of $\mathbf{J}^2$ and J_z with

$$j = \frac{N}{2}, \quad -j \leqslant m = k - \frac{N}{2} \leqslant j. \tag{37}$$

Consider now first a state $|\psi\rangle$ with fixed j. It is useful to write $|\psi\rangle$ in the J_x eigenbasis, $|\psi\rangle = \sum_{m=-j}^{j} c_m |jm\rangle_x$, with $J_x|jm\rangle_x = m|jm\rangle_x$. There are no fluctuations from $\hat{N}/2$ in J, and from Eq. (31) we have thus $\Delta^2_\psi J = \langle J_x^2\rangle - \langle J_x\rangle^2$. Inserting the expansion of $|\psi\rangle$ in the J_x eigenbasis, we are led to

$$\Delta^2_\psi J = \sum_{m=-j}^{j} p_m m^2 - \left(\sum_{m=-j}^{j} p_m m\right)^2, \tag{38}$$

with $p_m = |c_m|^2$.

Let Δ_π denote the right-hand side of Eq. (38) depending on the distribution $\pi = \{p_m\}_{m=-j}^{j}$; because of the apparent symmetry of the expression, the distribution $\pi' = \{p_{-m}\}_{m=-j}^{j}$ obtained by exchanging p_m with p_{-m} leads to $\Delta_{\pi'} = \Delta_\pi$. Let us then consider the symmetrized distribution $\pi_{\rm sym} = \{\frac{p_m + p_{-m}}{2}\}_{m=-j}^{j}$; convexity yields

$$\left(\sum_{m=-j}^{j} \frac{p_m + p_{-m}}{2} m\right)^2 \leqslant \frac{1}{2}\left(\sum_{m=-j}^{j} p_m m\right)^2 + \frac{1}{2}\left(\sum_{m=-j}^{j} p_{-m} m\right)^2,$$

therefore $\Delta_{\pi_{\rm sym}} \geqslant \Delta_\pi$, i.e., a generic distribution π cannot provide a Δ_π larger than the $\Delta_{\pi_{\rm sym}}$ obtained by symmetrizing it. The maximum Δ_π must then be attained at distributions with the property that $p_m = p_{-m}$; this means maximizing $\Delta^2_\psi J = 2\sum_{m=1}^{j} p_m m^2$ under the constraints $0 \leqslant p_m \leqslant 1/2$ for all m and $\sum_{m=-j}^{j} p_m = 1$. Since $\langle\psi|J_x^2|\psi\rangle \leqslant j^2$, for fixed j, the maximum of the variance is attained at $p_j = 1/2 = p_{-j}$.

Next, consider a general state with at most $j_{\max}$ excitations,

$$|\psi\rangle = \sum_{j=0}^{j_{\max}} \sum_{j=-m}^{m} c_{jm} |jm\rangle_x. \tag{39}$$

Since both $\hat{N}$ and J_x conserve j, we get

$$\langle J\rangle = \sum_j \sum_{m,n} c^*_{jn} c_{jm}\ {}_x\langle jn|J|jm\rangle_x, \tag{40}$$

and similarly for J^2. It is useful to introduce the notation $\langle X\rangle_j = {}_j\langle\psi|X|\psi\rangle_j$, where $|\psi\rangle_j$ is the wave function in the j sector, that is, $|\psi\rangle_j = \sum_m c_{jm}|jm\rangle_x/\sqrt{p_j}$, with $p_j = \sum_m |c_{jm}|^2$ assuring the correct normalization. This gives

$$\Delta^2_\psi J = \sum_j p_j \langle J^2\rangle_j - \left(\sum_j p_j \langle J\rangle_j\right)^2 = \sum_j p_j \langle (J - \langle J\rangle)^2\rangle_j. \tag{41}$$

Since $\max_{\{c_{jm}\}} \langle (J - \langle J\rangle)^2\rangle_j = j^2$ grows monotonically with j, we obtain the maximum of $\Delta^2_\psi J$ over all c_{jm} by choosing $p_{j_{\max}} = 1$ (and, correspondingly, all other $p_j = 0$). Thus, the state that maximizes the quantum Fisher information is

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|j_{\max} j_{\max}\rangle_x + e^{i\chi}|j_{\max} - j_{\max}\rangle_x), \tag{42}$$

where χ is an arbitrary phase.

Since we have derived the state in the $|j,m\rangle_l$ representation and since the number of particles is fixed, it is clear that it can be created at least in principle from N qubits or spins, where it corresponds to a superposition of all spins up and all spins down. This state, commonly called the Greenberger-Horne-Zeilinger (GHZ) state in quantum information theory [67], is clearly entangled. In order to give a physical interpretation to such states in terms of bosons, let us consider the Bogoliubov transformation (14) to new modes described by creation (annihilation) operators c, d ($c^\dagger$, $d^\dagger$). With reference to the new modes, the pseudo-angular-momentum operator J_x becomes

$$J_x = \frac{c^\dagger c - d^\dagger d}{2}. \tag{43}$$

In terms of occupation number states of these modes, it follows that, as expected, the state (42) has the form of a NOON state; see (37), namely, a superposition of all bosons in mode c and all bosons in mode d:

$$|\psi\rangle = \frac{(c^\dagger)^{2j_{\max}} + e^{i\chi}(d^\dagger)^{2j_{\max}}}{\sqrt{2}}|0\rangle. \tag{44}$$

In addition to photons, the above analysis also applies to interferometric setups based on ultracold-atom gases trapped by double-well potentials: the modes a,b describe atoms confined in the left and right well, whereas the modes c,d are related to the first two tunneling split single-particle energy eigenstates (in the limit of high barrier). Clearly, the NOON state of the photons presents two-mode entanglement. The NOON state of cold bosonic atoms is $(\mathcal{A},\mathcal{B})$ entangled.

V. CONCLUSIONS

We have shown that the "trivial" entanglement that comes for free with identical bosons is useful for precision measurements. This is most clearly seen for the task of measuring the transparency of a single beam splitter, where a two-mode Fock state can enable one to beat the standard quantum limit (SQL). We have discussed different states with respect to the presence or absence of entanglement and their physical realizability in three different physical setups, known to implement a beam splitter or Mach-Zehnder interferometer: (1) N spins or qubits acted upon by local unitary operations and state-dependent phase shift, (2) two photonic modes containing N photons, and (3) N massive bosons. Depending on these realizations, entanglement has a different meaning, and a straightforward conclusion about the possibility for beating the SQL based on the entanglement of N distinguishable particles [5] is only possible in the first setup. For identical bosons, we have made use of an extended concept of entanglement based on operator algebras, and its converse, $(\mathcal{A},\mathcal{B})$ separability. In this framework, entanglement vanishes for the trivial state entanglement that is solely due to symmetrization. In the case of fixed boson number N, the two-mode Fock states $|k,N-k\rangle$

with k bosons in mode a and $N-k$ bosons in mode b are $(\mathcal{A},\mathcal{B})$ separable, as there are no correlations between observables of the first and the second mode. Nevertheless, except when $k=0$ or $k=N$ (the single-mode case), these states can achieve squared sensitivities scaling faster than $1/N$ (i.e., beat the SQL) if subjected to beam-splitting transformations generated by pseudo-angular-momentum operators like $J_x = (a^\dagger b + a b^\dagger)/2$ that are nonlocal with respect to the given modes. If one wants to obtain the same result in the first scenario of N spins or qubits, one has to explicitly entangle all of them. Superpositions of Fock states can beat the shot-noise limit even for single-mode states. An ON state in one mode, i.e., a superposition of 0 and N bosons in one mode, leads to a scaling of the squared sensitivity of a Mach-Zehnder interferometer as $\propto 1/\overline{N}^2$, which corresponds to the Heisenberg limit. The latter scaling is also obtained, at least in principle, under ideal unitary evolution for the NOON state. Using the quantum Fisher information we showed that the NOON state is—in such a highly idealized situation—the optimal two-mode state, in the sense that it saturates the quantum Cramér-Rao bound for pure states with the same maximum number of excitations fed into a Mach-Zehnder interferometer.

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