Identification of three-qubit entanglement

Ming-Jing Zhao,[1] Ting-Gui Zhang,[1] Xianqing Li-Jost,[1] and Shao-Ming Fei[1,2]
[1] *Max-Planck-Institute for Mathematics in the Sciences, Leipzig 04103, Germany*
[2] *School of Mathematical Sciences, Capital Normal University, Beijing 100048, China*
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We present a way of identifying all kinds of entanglement for three-qubit pure states in terms of the expectation values of Pauli operators. The necessary and sufficient conditions to classify the fully separable, biseparable, and genuine entangled states are explicitly given. The approach can be generalized to multipartite high-dimensional cases. For three-qubit mixed states, we propose two kinds of inequalities in terms of the expectation values of complementary observables. One inequality has advantages in entanglement detection of the quantum state with positive partial transpositions, and the other is able to detect genuine entanglement. The results give an effective method for experimental entanglement identification.

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I. INTRODUCTION

Entanglement is the essential resource for many tasks in quantum information processing [1,2]. As a result, various approaches have been proposed to characterize entanglement. But as of yet, there are no operational necessary and sufficient separability criteria for high-dimensional states and multipartite states in general.

For unknown quantum states, separability can only be determined by measuring some suitable quantum mechanical observables. An important approach in characterizing entanglement is the Bell inequality [3–8]. For instance, Gisin has proven that all two-qubit pure entangled states violate the Clauser-Horne-Shimony-Holt (CHSH) inequality [4]. In the three-qubit system, Ref. [5] presents a Bell-type inequality that would be violated by all three-qubit pure entangled states, and Ref. [6] shows analytically that all pure entangled states violate another Bell-type inequality by exploiting Hardy's nonlocality argument. Quite recently, Ref. [8] showed that all multipartite high-dimensional entangled pure states violate a single Bell inequality. For general mixed two-qubit states, Bell-type inequality has been proposed to give the necessary and sufficient criterion of separability [9,10]. Besides Bell inequality, the entanglement witness could also be used for experimental detection of quantum entanglement for some special states, such as the W state [11], the GHZ state [11], and the cluster state [12]. Some of these witnesses can be implemented with the present technology [13,14]. Another method to detect entanglement is to measure the entanglement measures experimentally [15–18], which have been implemented for the two-qubit pure state [16,17].

In multipartite systems, there are many kinds of entanglement. For the simplest case, in the three-qubit system, all pure states are classified into six types in terms of stochastic local operations and classical communication (SLOCC) equivalence [19]. They can also be classified into nine types by the canonical form of the pure three-qubit state [20]. Then for three-qubit mixed states, they could be classified into four types if one demands that each type consist of a compact and convex set [21]. Different types of entanglement have different features. But it is generally difficult to characterize different types of multipartite entanglement and distinguish them from each other completely. Entanglement witness and the Bell inequality have been proposed to distinguish important classes of qubit states [22–25].

In this paper we mainly deal with the separability of quantum states and distinguish different entanglement in the three-qubit system. We first express the bipartite entanglement of the three-qubit pure state in terms of expectation values of Pauli operators. Based on this, we derive some inequalities which can be viewed as entanglement witness to detect the separability of three-qubit pure states completely. Therefore, one can recognize whether a three-qubit pure state is fully separable, biseparable, or genuinely entangled by measuring some particular expectation values. For the entanglement detection of three-qubit mixed states, we propose two kinds of inequalities in terms of the expectation values of complementary observables. One inequality is able to detect entanglement in the quantum state with positive partial transpositions (PPT's), and the other is able to detect genuine entanglement. These inequalities may help experimental entanglement detection and differentiation in the three-qubit system.

The paper is organized as follows. In Sec. II, we express the bipartite entanglement of the three-qubit pure state in terms of expectation values of Pauli operators. Then necessary and sufficient conditions to classify the fully separable, biseparable, and genuine entangled states are explicitly given for the three-qubit pure state. In Sec. III, we provide two kinds of inequalities in terms of the expectation values of complementary observables to detect entanglement in the three-qubit mixed state. These inequalities are shown to have the ability to detect some PPT entanglement and genuine entanglement. Conclusions are given in Sec. IV.

II. ENTANGLEMENT DETECTION OF THE THREE-QUBIT PURE STATE

Any pure three-qubit state $|\psi\rangle$ can be either fully separable, biseparable, or genuinely entangled. A fully separable pure three-qubit state $|\psi\rangle$ can be written as a tensor product of three pure states, $|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes |\phi_3\rangle$, while biseparable states have three different kinds depending on the partitions. If $|\psi\rangle$ is separable under partition of the first qubit and the rest of the qubits, it has the form $|\psi\rangle = |\phi_1\rangle \otimes |\phi_{23}\rangle$, with $|\phi_{23}\rangle$ an entangled state of the second and third qubits. We denote this kind of biseparable state as a 1|23 separable state. Analogously, there are 2|13 and 12|3 separable states. These 1|23, 2|13, and 12|3 separable states are biseparable ones. If state $|\psi\rangle$ is neither fully separable nor biseparable, then it is genuinely entangled. There are two kinds of genuine entangled states under SLOCC classification [19]: $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ and $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$.

For the three-qubit mixed state ρ, it is fully separable if it is a convex combination of fully separable pure states; ρ is biseparable if it can be written as the convex combination of biseparable pure states. Furthermore, a mixed state ρ is called separable under partition 1|23 if it is a convex combination of a 1|23 separable pure state. Analogously, there are 2|13 and 12|3 separable mixed states. In this respect, a general biseparable state ρ is also a convex combination of 1|23, 2|13, and 12|3 separable mixed states. ρ is genuinely entangled if it is neither fully separable nor biseparable.

We first deal with the problem of identifying bipartite entanglement of three-qubit pure states by realizing entanglement measure in terms of the expectation values of local observables. Here we adopt concurrence as the bipartite entanglement measure [26–31]. For a bipartite pure state $|\psi\rangle$, its concurrence is defined by $C(|\psi\rangle) = \sqrt{1 - \mathrm{tr}\rho_1^2}$ with $\rho_1 = \mathrm{tr}_2(|\psi\rangle\langle\psi|)$ the reduced density matrix. For a three-qubit state $|\psi\rangle = \sum_{i,j,k=0}^{1} a_{ijk}|ijk\rangle$, $\sum_{i,j,k=0}^{1} |a_{ijk}|^2 = 1$, if we view it as a bipartite state under the partition of the first qubit and the rest of the qubits, its squared concurrence is given by

$$C_{1|23}^2(|\psi\rangle) = \left(\sum_{j,k=0}^{1} |a_{0jk}|^2\right)\left(\sum_{j,k=0}^{1} |a_{1jk}|^2\right) - \left|\sum_{j,k=0}^{1} a_{0jk}a_{1jk}^*\right|^2. \tag{1}$$

After a lengthy calculation, we get that the right-hand side of Eq. (1) can be expressed as the quadratic polynomial of the expectation values of Pauli operators,

$$\begin{aligned}\langle G_1\rangle_{|\psi\rangle\langle\psi|} \equiv \tfrac{1}{16}(3 &- \langle II\sigma_3\rangle^2 - \langle I\sigma_3I\rangle^2 - 3\langle\sigma_3II\rangle^2 + \langle\sigma_3\sigma_3I\rangle^2\\ &+ \langle\sigma_3I\sigma_3\rangle^2 - \langle I\sigma_3\sigma_3\rangle^2 + \langle\sigma_3\sigma_3\sigma_3\rangle^2 - 3\langle\sigma_1II\rangle^2\\ &+ \langle\sigma_1I\sigma_3\rangle^2 + \langle\sigma_1\sigma_3I\rangle^2 + \langle\sigma_1\sigma_3\sigma_3\rangle^2 - 3\langle\sigma_2II\rangle^2\\ &+ \langle\sigma_2I\sigma_3\rangle^2 + \langle\sigma_2\sigma_3I\rangle^2 + \langle\sigma_2\sigma_3\sigma_3\rangle^2),\end{aligned} \tag{2}$$

where $\sigma_1 = |0\rangle\langle 1| + |1\rangle\langle 0|$, $\sigma_2 = i(|0\rangle\langle 1| - |1\rangle\langle 0|)$, and $\sigma_3 = |0\rangle\langle 0| - |1\rangle\langle 1|$ are Pauli operators, I is the identity operator, $\langle II\sigma_3\rangle$ stands for $\langle I \otimes I \otimes \sigma_3\rangle$, and so on.

By permutation we can similarly get the squared concurrence $C_{2|13}^2(|\psi\rangle)$ of $|\psi\rangle$, denoted by

$$\begin{aligned}\langle G_2\rangle_{|\psi\rangle\langle\psi|} = \tfrac{1}{16}(3 &- \langle II\sigma_3\rangle^2 - \langle\sigma_3II\rangle^2 - 3\langle I\sigma_3I\rangle^2 + \langle\sigma_3\sigma_3I\rangle^2\\ &+ \langle I\sigma_3\sigma_3\rangle^2 - \langle\sigma_3I\sigma_3\rangle^2 + \langle\sigma_3\sigma_3\sigma_3\rangle^2 - 3\langle I\sigma_1I\rangle^2\\ &+ \langle I\sigma_1\sigma_3\rangle^2 + \langle\sigma_3\sigma_1I\rangle^2 + \langle\sigma_3\sigma_1\sigma_3\rangle^2 - 3\langle I\sigma_2I\rangle^2\\ &+ \langle I\sigma_2\sigma_3\rangle^2 + \langle\sigma_3\sigma_2I\rangle^2 + \langle\sigma_3\sigma_2\sigma_3\rangle^2),\end{aligned} \tag{3}$$

and the squared concurrence $C_{3|12}^2(|\psi\rangle)$ of $|\psi\rangle$,

$$\begin{aligned}\langle G_3\rangle_{|\psi\rangle\langle\psi|} = \tfrac{1}{16}(3 &- \langle\sigma_3II\rangle^2 - \langle I\sigma_3I\rangle^2 - 3\langle II\sigma_3\rangle^2 + \langle I\sigma_3\sigma_3\rangle^2\\ &+ \langle\sigma_3I\sigma_3\rangle^2 - \langle\sigma_3\sigma_3I\rangle^2 + \langle\sigma_3\sigma_3\sigma_3\rangle^2 - 3\langle II\sigma_1\rangle^2\\ &+ \langle\sigma_3I\sigma_1\rangle^2 + \langle I\sigma_3\sigma_1\rangle^2 + \langle\sigma_3\sigma_3\sigma_1\rangle^2 - 3\langle II\sigma_2\rangle^2\\ &+ \langle\sigma_3I\sigma_2\rangle^2 + \langle I\sigma_3\sigma_2\rangle^2 + \langle\sigma_3\sigma_3\sigma_2\rangle^2).\end{aligned} \tag{4}$$

Equations (2)–(4) give a realization of experimental measurement of bipartite entanglement of three-qubit pure states. One can obtain the value of concurrence by measuring the expectation values of Pauli operators. If $\langle G_i\rangle_{|\psi\rangle\langle\psi|} > 0$, then the three-qubit pure state $|\psi\rangle$ is not separable between the ith qubit and the rest. If $\langle G_i\rangle_{|\psi\rangle\langle\psi|} = 0$, then the three-qubit pure state $|\psi\rangle$ is at least biseparable, $i = 1,2,3$.

Note that any three-qubit pure state $|\psi\rangle$ is fully separable if and only if its concurrence under all biseparable partitions is zero; $|\psi\rangle$ is biseparable if and only if its concurrence between one fixed qubit and the other two qubits is zero, while the other two bipartite concurrences are not zero. At last, $|\psi\rangle$ is genuinely entangled if and only if its concurrence for all bipartite partitions is nonzero. Therefore, employing the nonlinear operators G_j, $j = 1,2,3$, we have the following result for experimentally identifying different kinds of entanglement in arbitrary unknown three-qubit pure states.

Theorem 1. For any pure three-qubit state $|\psi\rangle$, we have

(i) $|\psi\rangle$ is fully separable if and only if $\langle G_j\rangle_{|\psi\rangle\langle\psi|} = 0$, for $j = 1,2$, or $j = 2,3$, or $j = 1,3$.

(ii) $|\psi\rangle$ is separable between the ith qubit and the rest if and only if $\langle G_i\rangle_{|\psi\rangle\langle\psi|} = 0$ and $\langle G_j\rangle_{|\psi\rangle\langle\psi|} > 0$, $j \in \{1,2,3\}$ and $j \neq i$, $i = 1,2,3$.

(iii) $|\psi\rangle$ is genuinely entangled if and only if $\langle G_j\rangle_{|\psi\rangle\langle\psi|} > 0$, $j = 1,2$, or $j = 2,3$, or $j = 1,3$.

In fact, to determine the type of entanglement existing in the three-qubit pure state, one can resort to the Schmidt decomposition across the bipartition 1|23, 2|13, and 12|3, and then conclude whether it is fully separable, biseparable, or genuinely entangled. However, this method works in theory and requires that we have already known precisely all the coefficients of the pure state. In contrast, Theorem 1 works for any unknown pure three-qubit states, and it is operational experimentally.

From the view of the entanglement witness, G_i, $i = 1,2,3$ can be regarded as nonlinear entanglement witness operators. Theorem 1 shows that there exists a complete set of entanglement witnesses to identify all kinds of possible pure three-qubit entanglement: fully separable, three types of biseparable entanglement, and genuinely entangled states. Compared with the usual Bell inequality, which requires infinitely many measurements of observables, our local operators are fixed. In other words, to detect and differentiate pure three-qubit entanglement, one only needs to measure the coincidence probabilities: $\sigma_3 \otimes \sigma_3 \otimes \sigma_3, \sigma_3 \otimes \sigma_3 \otimes \sigma_1, \sigma_3 \otimes \sigma_3 \otimes \sigma_2, \sigma_3 \otimes \sigma_1 \otimes \sigma_3, \sigma_3 \otimes \sigma_2 \otimes \sigma_3, \sigma_1 \otimes \sigma_3 \otimes \sigma_3, \sigma_2 \otimes \sigma_3 \otimes \sigma_3$ in G_1, G_2, and G_3. These finite and deterministic measurements make the experimental entanglement detection simpler.

For the multipartite high-dimensional pure state, there are many different kinds of entanglement. For example, an N-partite system can be genuinely entangled, $\binom{N}{2}$ different biseparable, $\binom{N}{3}$ different tripartite separable, ..., fully separable. Different kinds of entanglement could be detected by expanding the $\binom{N}{2}$ different bipartite concurrence, $\binom{N}{3}$ different tripartite generalized concurrence, ..., and $\binom{N}{N-1}$ different $N-1$ partite generalized concurrence in terms of the expectation values of Hermitian operators. Hence all pure-state entanglement could be detected completely by measuring the expectation values of local observables.

III. ENTANGLEMENT DETECTION OF THREE-QUBIT MIXED STATES

It is much more complicated to detect entanglement of mixed states. Even for a known mixed state, one has no general approach to judge its separability. In order to identify the entanglement of a three-qubit mixed state ρ, here we give two kinds inequalities to detect three-qubit entanglement in terms of the expectation values of complementary observables. First, let A, B, and C denote the observables acting on the first, second, and third qubits, respectively. $\{A_i = \vec{a}_i \cdot \vec{\sigma}\}_{i=1}^3$, $\{B_j = \vec{b}_j \cdot \vec{\sigma}\}_{j=1}^3$, and $\{C_k = \vec{c}_k \cdot \vec{\sigma}\}_{k=1}^3$ are arbitrary complete sets of complementary observables with the same orientations, and $\vec{\sigma}$ is the vector composed by Pauli operators [9]. For a set of three mutual complementary observables $\{A_i\}_{i=1}^3$, we denote $\mu_A = -iA_1A_2A_3$ as its orientation, which can assume only two values ± 1. If $\mu_A = 1$ the orientation of the basis formed by the three real vectors $\vec{a}_i$ is right-handed; the same definition of orientation applies to $\vec{\sigma}$. The orientations of $\{B_j\}_{j=1}^3$ and $\{C_k\}_{k=1}^3$ are defined similarly. In the following, when we refer to the complementary observables $\{A_i\}_{i=1}^3$, $\{B_j\}_{j=1}^3$ and $\{C_k\}_{k=1}^3$, we mean that they have the same orientations as the default.

Theorem 2. For any three-qubit mixed state ρ, if it is separable under the partition 1|23 and 12|3, then it satisfies

$$\begin{aligned}\langle T_1\rangle_\rho = {}& \langle 1 + B_3 + A_3C_3 + A_3B_3C_3\rangle_\rho^2 \\ & - \langle C_3 + B_3C_3 + A_3 + A_3B_3\rangle_\rho^2 \\ & - \langle A_1C_1 + A_1B_3C_1 + A_2C_2 + A_2B_3C_2\rangle_\rho^2 \geqslant 0 \quad (5)\end{aligned}$$

for all complementary local observables.

The proof of Theorem 2 can be derived analogously in the light of the second part of the proof of the main result in Ref. [10]. This theorem tells us that if $\langle T_1\rangle_\rho < 0$ for some complementary local observables, then the quantum state ρ is not separable under the partition 1|23 and 12|3 and it is surely entangled. Similarly, if ρ is separable under the partition 2|13 and 12|3, then it satisfies $\langle T_2\rangle_\rho = \langle 1 + A_3 + B_3C_3 + A_3B_3C_3\rangle_\rho^2 - \langle C_3 + A_3C_3 + B_3 + A_3B_3\rangle_\rho^2 - \langle B_1C_1 + A_3B_1C_1 + B_2C_2 + A_3B_2C_2\rangle_\rho^2 \geqslant 0$ for all complementary local observables, and if ρ is separable under the partition 1|23 and 2|13, then it satisfies $\langle T_3\rangle_\rho = \langle 1 + C_3 + A_3B_3 + A_3B_3C_3\rangle_\rho^2 - \langle B_3 + B_3C_3 + A_3 + A_3C_3\rangle_\rho^2 - \langle A_1B_1 + A_1B_1C_3 + A_2B_2 + A_2B_2C_3\rangle_\rho^2 \geqslant 0$ for all complementary local observables. Next, we illustrate the capability of Theorem 2 in detecting entanglement by some examples.

Example 1. For quantum state

$$\begin{aligned}\rho_1 = {}& \tfrac{1}{3}(|\psi^+\rangle\langle\psi^+|_{AB} \otimes |0\rangle\langle 0|_C + |\psi^+\rangle\langle\psi^+|_{AC} \otimes |0\rangle\langle 0|_B \\ & + |\psi^+\rangle\langle\psi^+|_{BC} \otimes |0\rangle\langle 0|_A),\end{aligned}$$

with $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, it has $\langle T_1\rangle_{\rho_1} = -\frac{16}{9}$ if we take $B_i = C_i = \sigma_i$, and $A_i = U_1\sigma_iU_1^\dagger$ with $U_1 = |0\rangle\langle 1| - |1\rangle\langle 0|$, $i = 1,2,3$. So ρ_1 is identified as an entangled state by Theorem 2.

Example 2. For quantum state

$$\sigma_b = \frac{7b}{7b+1}\sigma_{\text{insep}} + \frac{1}{7b+1}|\phi_b\rangle\langle\phi_b|, \qquad (6)$$

where

$$\begin{aligned}\sigma_{\text{insep}} &= \frac{2}{7}(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3|) + \frac{1}{7}|011\rangle\langle 011|, \\ |\phi_b\rangle &= |1\rangle \otimes \left(\sqrt{\frac{1+b}{2}}|00\rangle + \sqrt{\frac{1-b}{2}}|10\rangle\right), \\ |\psi_1\rangle &= \frac{1}{\sqrt{2}}(|000\rangle + |101\rangle), \\ |\psi_2\rangle &= \frac{1}{\sqrt{2}}(|001\rangle + |110\rangle), \\ |\psi_3\rangle &= \frac{1}{\sqrt{2}}(|010\rangle + |111\rangle),\end{aligned}$$

σ_b is entangled and positive under arbitrary partial transposition for $0 \leqslant b \leqslant 1$. So it is a PPT entangled state in the three-qubit system. Now if we choose $A_i = U_2\sigma_iU_2^\dagger$, $B_i = V_2\sigma_iV_2^\dagger$, $C_i = \sigma_i$, with $U_2 = |0\rangle\langle 1| - |1\rangle\langle 0|$, $V_2 = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|)$, $i = 1,2,3$, then $\langle T_1\rangle_{\sigma_b} = -\frac{32b(-1+b+\sqrt{1-b^2})}{(1+7b)^2} < 0$ for $0 < b < 1$. Therefore, Theorem 2 has advantages in PPT entanglement detection in the three-qubit system.

Example 3. For the quantum state

$$\rho_3 = p\sigma_b + \frac{1-p}{8}I,$$

with $0 \leqslant p$, $b \leqslant 1$, which is a mixture of PPT states σ_b in Eq. (6) with white noise, ρ_3 is still a PPT state with two parameters b and p. Below we plot the expectation value $\langle T_1\rangle_{\rho_3}$ with the help of local observables given in Example 2 (see Fig. 1). The dark (blue) region in the contour plot represents the PPT entangled state ρ_3 that Theorem 2 could detect.

Now we propose another kind of inequality to identify different entanglement in the three-qubit system. Let

$$\begin{aligned}\langle F_1\rangle_\rho &= \langle 1 + B_3C_3\rangle_\rho^2 - \langle B_3 + C_3\rangle_\rho^2 - \langle B_1C_1 + B_2C_2\rangle_\rho^2, \\ \langle F_2\rangle_\rho &= \langle 1 + A_3C_3\rangle_\rho^2 - \langle A_3 + C_3\rangle_\rho^2 - \langle A_1C_1 + A_2C_2\rangle_\rho^2, \\ \langle F_3\rangle_\rho &= \langle 1 + A_3B_3\rangle_\rho^2 - \langle A_3 + B_3\rangle_\rho^2 - \langle A_1B_1 + A_2B_2\rangle_\rho^2,\end{aligned} \qquad (7)$$

then we have the following result.

Theorem 3. For any three-qubit mixed state ρ, we have

(i) if it is fully separable, then it satisfies $\langle F_l\rangle_\rho \geqslant 0$ for all complementary local observables $\{A_i\}_{i=1}^3$, $\{B_j\}_{j=1}^3$, and $\{C_k\}_{k=1}^3$, $l = 1,2,3$.

(ii) if it is biseparable, then

$$\sum_{l=1}^3 \langle F_l\rangle_\rho \geqslant -2 \qquad (8)$$

for all complementary local observables $\{A_i\}_{i=1}^3$, $\{B_j\}_{j=1}^3$, and $\{C_k\}_{k=1}^3$.

(iii) if it violates inequality (8), then it is genuine entangled.

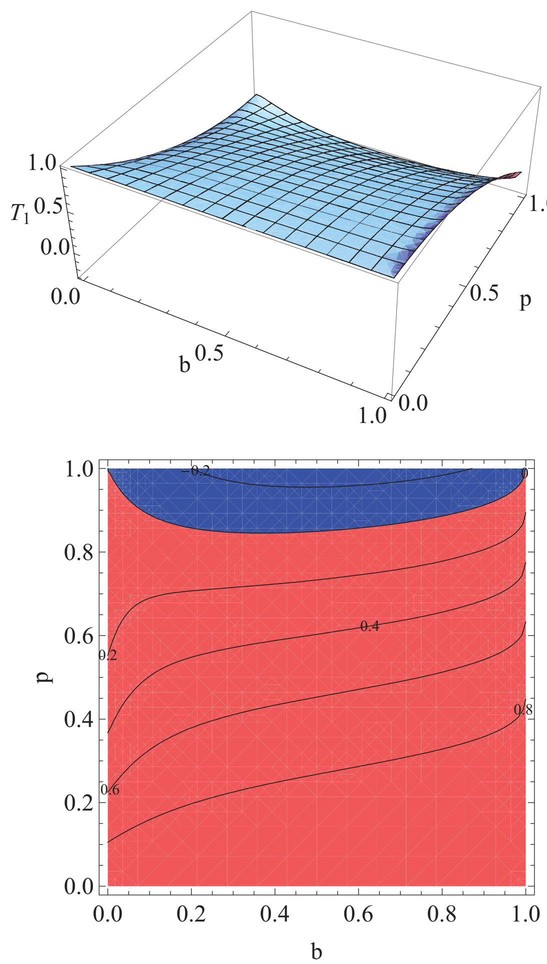

FIG. 1. (Color online) Here we select $A_i = U_2\sigma_i U_2^\dagger$, $B_i = V_2\sigma_i V_2^\dagger$, $C_i = \sigma_i$, with $U_2 = |0\rangle\langle 1| - |1\rangle\langle 0|$, $V_2 = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|)$, $i = 1,2,3$. The first plot describes the expectation value $\langle T_1\rangle_{\rho_3}$ with respect to p and b. The second plot is the contour plot of the first.

Proof. Let $\rho = \sum_k p_k|\psi_k\rangle\langle\psi_k|$, $\sum_k p_k = 1$, $p_k \geqslant 0$ be an arbitrary three-qubit mixed state.

(i) If ρ is fully separable, then $\{|\psi_k\rangle\}$ are all fully separable, at least in one such pure-state decomposition. Hence the reduced bipartite state $\rho_{\bar{l}}^{(k)} = \mathrm{Tr}_l(|\psi_k\rangle\langle\psi_k|)$ is also separable for all k, $l = 1,2,3$, and $\bar{l}$ denotes the absence of l in the set $\{1,2,3\}$. In the light of the main results in Ref. [9], one gets $\langle F_l\rangle_{|\psi_k\rangle\langle\psi_k|} \geqslant 0$ if and only if $\rho_{\bar{l}}^{(k)}$ is separable. Therefore, if $|\psi_k\rangle$ is fully separable, then $\langle F_l\rangle_{|\psi_k\rangle\langle\psi_k|} \geqslant 0$, $l = 1,2,3$, $\forall k$. Note that if $a_i^2 \geqslant b_i^2 + c_i^2 + x$ holds for arbitrary real numbers b_i and c_i, non-negative a_i and x, $i = 1,\ldots,n$, then $(\sum_{i=1}^n p_i a_i)^2 \geqslant (\sum_{i=1}^n p_i b_i)^2 + (\sum_{i=1}^n p_i c_i)^2 + x$ for $0 \leqslant p_i \leqslant 1$ and $\sum_{i=1}^n p_i = 1$. This observation makes $\langle F_l\rangle_\rho \geqslant 0$, $l = 1,2,3$, for all complementary local observables and for the fully separable quantum state ρ.

(ii) Suppose $\{|\psi_k\rangle\}$ are all biseparable. Without loss of generality, we assume $|\psi_1\rangle$ is $1|23$ separable, then it satisfies $\langle F_2\rangle_{|\psi_1\rangle\langle\psi_1|} \geqslant 0$ and $\langle F_3\rangle_{|\psi_1\rangle\langle\psi_1|} \geqslant 0$ for all complementary local observables. Taking into account that the minimum of $\langle F_1\rangle_{|\psi_1\rangle\langle\psi_1|}$ for arbitrary complementary local observables is two times the minimal eigenvalue of the partial transposed matrix of $\rho_{23}^{(1)}$ [9], one has $\langle F_1\rangle_{|\psi_1\rangle\langle\psi_1|} \geqslant -2$ for all complementary local observables. Therefore we get $\sum_{l=1}^3\langle F_l\rangle_{|\psi_k\rangle\langle\psi_k|} \geqslant -2$ for all complementary local observables and for the arbitrary biseparable state $|\psi_k\rangle$. Consequently, for the biseparable mixed state ρ, we have $\sum_{l=1}^3\langle F_l\rangle_\rho \geqslant -2$ for all complementary local observables.

(iii) The result is obvious. ■

As an example, we consider a state of a mixture of $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ with white noise, $\rho_w = p|W\rangle\langle W| + \frac{1-p}{8}I$, $0 \leqslant p \leqslant 1$. It is detected as genuinely entangled by the entanglement witness in Ref. [14] when $p > 0.62$. If we take $A_k = B_k = C_k = \sigma_k$ in the local operators in Eq. (7), $k = 1,2,3$, we have $\sum_{l=1}^3\langle F_l\rangle_{\rho_w} > -2$ when $p > 0.92$ and $\langle F_l\rangle_{\rho_w} < 0$ when $p > 0.56$, $l = 1,2,3$. Hence we know that ρ_w is genuinely entangled when $p > 0.92$ and entangled when $p > 0.56$ by Theorem 3. Our entanglement witness can better detect the entanglement of ρ_w. Although we cannot detect all genuine entanglement in the mixed state ρ_w by Theorem 3, the advantage of our method here is that it may be experimentally implemented.

IV. CONCLUSIONS

In summary, we have expressed the bipartite entanglement of three-qubit pure states in the form of expectation values of Pauli operators. With the aid of this expression, we have completely solved the problem of entanglement identification for three-qubit pure states by giving the necessary and sufficient conditions for fully separable states, biseparable states, and genuinely entangled states. This approach can be generalized to multipartite high-dimensional cases. Therefore, one can recognize the separability of pure states both theoretically and experimentally. Additionally, we have also derived two inequalities in the form of the expectation values of complementary observables to detect PPT entanglement and genuine entanglement for the three-qubit mixed state. These results may help experimental entanglement detection and identification.

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[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).

[2] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).

[3] J. S. Bell, Physics (Long Island, NY) **1**, 195 (1964); J. Clauser, M. Horne, A. Shimony, and R. Holt, Phys. Rev. Lett. **23**, 880 (1969).

[4] N. Gisin, Phys. Lett. A **154**, 201 (1991).

[5] J. L. Chen, C. F. Wu, L. C. Kwek, and C. H. Oh, Phys. Rev. Lett. **93**, 140407 (2004).

[6] S. K. Choudhary, S. Ghosh, G. Kar, and R. Rahaman, Phys. Rev. A **81**, 042107 (2010).

[7] D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, Phys. Rev. Lett. **88**, 040404 (2002).

[8] S. Yu, Q. Chen, C. Zhang, C. H. Lai, and C. H. Oh, Phys. Rev. Lett. **109**, 120402 (2012).

[9] S. Yu, J. W. Pan, Z. B. Chen, and Y. D. Zhang, Phys. Rev. Lett. **91**, 217903 (2003).

[10] M. J. Zhao, T. Ma, S. M. Fei, and Z. X. Wang, Phys. Rev. A **83**, 052120 (2011).

[11] O. Gühne and P. Hyllus, Int. J. Theor. Phys. **42**, 1001 (2003).

[12] Y. Tokunaga, T. Yamamoto, M. Koashi, and N. Imoto, Phys. Rev. A **74**, 020301 (2006).

[13] O. Gühne and G. Tòth, Phys. Rep. **474**, 1 (2009).

[14] M. Bourennane, M. Eibl, C. Kurtsiefer, S. Gaertner, H. Weinfurter, O. Gühne, P. Hyllus, D. Bruß, M. Lewenstein, and A. Spanpera, Phys. Rev. Lett. **92**, 087902 (2004).

[15] F. Mintert, M. Kuś, and A. Buchleitner, Phys. Rev. Lett. **95**, 260502 (2005).

[16] S. P. Walborn, P. H. Souto Ribeiro, L. Davidovich, F. Mintert, and A. Buchleitner, Nature (London) **440**, 1022 (2006).

[17] S. P. Walborn, P. H. Souto Ribeiro, L. Davidovich, F. Mintert, and A. Buchleitner, Phys. Rev. A **75**, 032338 (2007).

[18] S. M. Fei, M. J. Zhao, K. Chen, and Z. X. Wang, Phys. Rev. A **80**, 032320 (2009).

[19] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).

[20] A. Acín, A. Andrianov, E. Jané, and R. Tarrach, J. Phys. A: Math. Gen. **34**, 6725 (2001).

[21] A. Acín, D. Bruß, M. Lewenstein, and A. Sanpera, Phys. Rev. Lett. **87**, 040401 (2001).

[22] C. Schmid, N. Kiesel, W. Laskowski, W. Wieczorek, M. Żukowski, and H. Weinfuter, Phys. Rev. Lett. **100**, 200407 (2008).

[23] M. Huber, H. Schimpf, A. Gabriel, C. Spengler, D. Bruß, and B. C. Hiesmayr, Phys. Rev. A **83**, 022328 (2011).

[24] B. Jungnitsch, T. Moroder, and O. Gühne, Phys. Rev. Lett. **106**, 190502 (2011).

[25] N. Brunner, J. Sharam, and T. Vértesi, Phys. Rev. Lett. **108**, 110501 (2012).

[26] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).

[27] A. Uhlmann, Phys. Rev. A **62**, 032307 (2000).

[28] P. Rungta, V. Bužek, C. M. Caves, M. Hillery, and G. J. Milburn, Phys. Rev. A **64**, 042315 (2001).

[29] S. Albeverio and S. M. Fei, J. Opt. B: Quantum Semiclass. Opt. **3**, 223 (2001).

[30] D. A. Meyer and N. R. Wallach, J. Math. Phys. (NY) **43**, 4273 (2002).

[31] A. R. R. Carvalho, F. Mintert, and A. Buchleitner, Phys. Rev. Lett. **93**, 230501 (2004).