Topological invariance and global Berry phase in non-Hermitian systems

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By studying the topological invariance and Berry phase in non-Hermitian systems, we reveal the basic properties of the complex Berry phase and generalize the global Berry phases Q to identify the topological invariance for non-Hermitian systems. We find that Q can identify topological invariance in two kinds of non-Hermitian model, the two-level non-Hermitian Hamiltonian and the bipartite dissipative model. For the bipartite dissipative model, an abrupt change of the Berry phase in the parameter space reveals a quantum phase transition and is related to the exceptional points. These results give the basic relationships between the Berry phase and the quantum and topological phase transitions of non-Hermitian systems.

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I. INTRODUCTION

Quantum state storage and transfer are central issues of quantum technology. The basic idea is to find an efficient way to generate robust quantum states [1]. Several schemes of quantum devices, such as quantum adiabatic pumps [2] and geometric quantum computation, exhibit excellent features [3,4]. The geometric phase plays a crucial role in realizing robust quantum states and detecting quantum phase transitions (QPTs) [5,6]. The geometric phase generalized to non-Hermitian systems provides a geometrical description of the quantum evolution of non-Hermitian systems [7] and give the relationship between the geometric phase and the QPT [8]. For a non-Hermitian quantum walk, it has been found that the topological invariance depends on the ratio of the hopping amplitude between A and B sites [9]. The topological invariance provides some hints toward realizing robust quantum states and opens some fundamental issues [10]. The challenging problem is how to give a unified definition of the topological invariance for different quantum systems. In general, a topological phase transition (TPT) is characterized by a topological index instead of symmetry breaking, such as the winding number or Chern number [11]. For example, the integer conductance plateau in the integer quantum Hall effect is related to the topological quantity the Chern class and the geometric phase [12], which reveals the relationship between the TPT and the geometric phase.

However, there has been no general method to define the topological invariance for different quantum systems, particularly for non-Hermitian systems. Actually, how to give a unified definition of topological invariance and how to characterize TPTs for a general system are still challenging problems even though there have been some schemes for some specific systems [10,11,13]. Hence, any effort to construct a novel paradigm to describe topological invariance and the TPT for both Hermitian and non-Hermitian systems is valuable.

In this paper, we will study the basic properties of the complex Berry phase in the quantum evolution of non-Hermitian models. We propose the global Berry phase Q of all states to identify the topological invariance; it is used in two kinds of non-Hermitian model. One is a general 2×2 non-Hermitian model; the other is a bipartite dissipative model. They have different mathematical structures. We find that Q can be a topological index to identify the topological invariance in these two models. We will also give the phase diagram of the complex geometric phase in the parameter space that indicates the relationships between topological invariance, quantum phase transitions, and the complex Berry phase.

II. BERRY PHASE IN THE QUANTUM EVOLUTION OF NON-HERMITIAN SYSTEMS

We consider a general quantum system described by the parameter-dependent Hamiltonian $H(\alpha)$, where the parameters $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_m)$ are functions of time, which leads to an implicit time dependence of $H(\alpha)$ and describes the state evolution. For a general non-Hermitian system, $H(\alpha)\neq H^\dagger(\alpha)$. The state vectors $|\Psi(\alpha)\rangle$ and $|\Lambda(\alpha)\rangle$ within the Hilbert space $|\Psi(\alpha)\rangle\in\mathcal{H}$ and its dual Hilbert space $|\Lambda(\alpha)\rangle\in\mathcal{H}^\dagger$ satisfy the Schrödinger equation [14]

$$i\hbar\partial_t|\Psi(\alpha)\rangle = H(\alpha)|\Psi(\alpha)\rangle,\quad i\hbar\partial_t|\Lambda(\alpha)\rangle = H^\dagger(\alpha)|\Lambda(\alpha)\rangle. \tag{1}$$

In the adiabatic approximation, the state evolution can be expanded into the instantaneous right eigenstates

$$|\Psi(\alpha)\rangle = \sum_\mu c_\mu(t)|\psi_\mu(\alpha)\rangle, \tag{2}$$

where $|\psi_\mu(\alpha)\rangle$ satisfies the instantaneous eigenequation $H(\alpha)|\psi_\mu(\alpha)\rangle = E_\mu(\alpha)|\psi_\mu(\alpha)\rangle$. Neglecting the off-diagonal terms in the adiabatic approximation, the coefficient can be expressed as [14]

$$c_\mu(t) = c_\mu(0)e^{-(i/\hbar)\int_0^t E_\mu(t')dt'}e^{i\int_0^t\langle\lambda_\mu(\alpha)|\nabla_\alpha|\psi_\mu(\alpha)\rangle d\alpha}, \tag{3}$$

where $\langle\lambda_\mu(\alpha)|$ is the corresponding eigenstate of $|\psi_\mu(\alpha)\rangle$ in the dual space. The parameters α are varied in a cyclic way and the initial state of the system remains at an instantaneous eigenstate $|\Psi(0)\rangle = |\psi_\mu(\alpha(0))\rangle$; the state evolution follows [14]

$$|\Psi(T)\rangle = e^{i[\gamma_\mu^D(T)+\gamma_\mu^G(T)]}|\Psi(0)\rangle, \tag{4}$$

where $\gamma_\mu^D = -\frac{1}{\hbar}\int_0^T E_\mu(t)dt$ is the complex dynamical phase. T is the periodicity of time evolution such that $\alpha(T)=\alpha(0)$.

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γ_μ^D can be written as $\gamma_\mu^D \equiv \gamma_\mu^d + i\xi_\mu^d$, where the real part γ_μ^d is the dynamical phase, and the imaginary part ξ_μ^d is the dissipative effect induced by the energy shift for non-Hermitian systems. The complex Berry phase in the cyclic evolution is

$$\gamma_\mu^B = \oint_C A_\mu, \tag{5}$$

where $A_\mu = i\langle\lambda_\mu(\alpha)|d|\psi_\mu(\alpha)\rangle$ is the Berry potential (connection), where d is the exterior derivative. Similarly, $\gamma_\mu^B \equiv \gamma_\mu^b + i\xi_\mu^b$, where the real part γ_μ^b is the Berry phase, and the imaginary part ξ_μ^b is the dissipative effect induced by the geometric potential.

For non-Hermitian systems, the dissipative effect originates from ξ_μ^d and ξ_μ^b, where ξ_μ^d depends on the eigenvalues of H and describes the dynamical dissipation. ξ_μ^b depends on the geometry of the evolution path. We refer to this as the geometrical dissipative effect.

We may define a non-Abelian Berry connection (a gauge potential one-form) by [15]

$$A = A_k d\alpha^k = i\begin{pmatrix} \langle\lambda_1|d\psi_1\rangle & \cdots & \langle\lambda_1|d\psi_M\rangle \\ \vdots & \ddots & \vdots \\ \langle\lambda_M|d\psi_1\rangle & \cdots & \langle\lambda_M|d\psi_M\rangle \end{pmatrix}, \tag{6}$$

where

$$A_k = i\begin{pmatrix} \langle\lambda_1|\partial_k\psi_1\rangle & \cdots & \langle\lambda_1|\partial_k\psi_M\rangle \\ \vdots & \ddots & \vdots \\ \langle\lambda_M|\partial_k\psi_1\rangle & \dots & \langle\lambda_M|\partial_k\psi_M\rangle \end{pmatrix}. \tag{7}$$

It should be remarked that the non-Abelian Berry connection involves all eigenstates of the system such that it can describe the global properties of the system. We may define the global Berry phase of the system [2]

$$Q = \frac{1}{2\pi}\oint \mathrm{Tr}A. \tag{8}$$

We will demonstrate that the global Berry phase Q can be used to identify the topological invariance for Hermitian and two kinds of non-Hermitian systems. We call Q the global Berry phase or the ground-state Berry phase. As in Hermitian systems, we can obtain a few basic properties of the complex Berry phase.

Claim 2.1. For the gauge transformation $|\psi_\mu'\rangle = e^{-if(\alpha)}|\psi_\mu\rangle$ where $e^{-if(\alpha)}$ is a single-value function modulo 2π, namely, $f[\alpha(t)+\alpha(t+T)] = f[\alpha(t)] + 2n\pi$, where n is integer,

(a) the Berry potential is $A_\nu' = A_\nu + df$;

(b) the complex Berry phase is invariant mod 2π, $\gamma_\nu'^G = \gamma_\nu^G + 2n_\nu\pi$;

(c) $Q' = Q + \sum_\nu n_\nu$, where n_ν is integer corresponding to the state ν.

Proof. (a) $A_\nu' = i\langle\lambda_\nu(\alpha)|e^{if(\alpha)}de^{-if(\alpha)}|\psi_\nu(\alpha)\rangle = i\langle\lambda_\nu(\alpha)|d|\psi_\nu(\alpha)\rangle + i\langle\lambda_\nu(\alpha)|\psi_\nu(\alpha)\rangle e^{if(\alpha)}de^{-if(\alpha)} = A_\nu + df$; (b) noting that $\oint df = 2n_\nu\pi$, we have $\gamma_\nu'^G = \oint A_\nu' = \gamma_\nu^G + 2n_\nu\pi$; (c) straightforwardly since $\mathrm{Tr}A' = \mathrm{Tr}A + \mathrm{Tr}(df)$ and $\oint \mathrm{Tr}df = \mathrm{Tr}\oint df = \sum_\mu n_\mu$; hence $Q' = \frac{1}{2\pi}\oint \mathrm{Tr}A' = Q + \sum_\mu n_\mu$. ■

It can be seen that γ_ν^G and Q are gauge invariant modulo 2π. They will be used to describe quantum phase transitions and the topological invariance of systems.

III. TOPOLOGICAL INVARIANCE AND THE QUANTUM PHASE TRANSITION IN NON-HERMITIAN MODELS

A. Two-level non-Hermitian model

Let us consider a 2×2 non-Hermitian Hamiltonian,

$$H = H_{\mathrm{Hermi}} + H_{\text{non-Hermi}}, \tag{9}$$

where the Hermitian part of the Hamiltonian is given by

$$H_{\mathrm{Hermi}} = \mathbf{h}\cdot\boldsymbol{\sigma}, \tag{10}$$

where $\mathbf{h}(\alpha) = (h_x\sin\theta_\alpha\cos\varphi_\alpha, h_y\sin\theta_\alpha\sin\varphi_\alpha, h_z\cos\theta_\alpha)$ describes the Hermitian properties. Thus,

$$H_{\mathrm{Hermi}} = \begin{bmatrix} h_z\cos\theta_\alpha & (h_x\cos\varphi_\alpha - ih_y\sin\varphi_\alpha)\sin\theta_\alpha \\ (h_x\cos\varphi_\alpha + ih_y\sin\varphi_\alpha)\sin\theta_\alpha & -h_z\cos\theta_\alpha \end{bmatrix}. \tag{11}$$

The non-Hermitian part is

$$H_{\text{non-Hermi}} = \Delta\cdot\mathbf{n}(\alpha), \tag{12}$$

where

$$\Delta = \begin{pmatrix} 0 & \Delta_x \\ -\Delta_x & 0 \end{pmatrix}\mathbf{i} + \begin{pmatrix} 0 & -i\Delta_y \\ -i\Delta_y & 0 \end{pmatrix}\mathbf{j} + \begin{pmatrix} i\Delta_z & 0 \\ 0 & -i\Delta_z \end{pmatrix}\mathbf{k} \tag{13}$$

describes the non-Hermitian properties, and $\mathbf{n}(\alpha) = (\sin\theta_\alpha\cos\varphi_\alpha, \sin\theta_\alpha\sin\varphi_\alpha, \cos\theta_\alpha)$ is a unit vector of the Bloch sphere. We have

$$H_{\text{non-Hermi}} = \begin{bmatrix} i\Delta_z\cos\theta_\alpha & (\Delta_x\cos\varphi_\alpha - i\Delta_y\sin\varphi_\alpha)\sin\theta_\alpha \\ (-\Delta_x\cos\varphi_\alpha - i\Delta_y\sin\varphi_\alpha)\sin\theta_\alpha & -i\Delta_z\cos\theta_\alpha \end{bmatrix}, \tag{14}$$

where α is the evolution parameter. The total Hamiltonian is rewritten as

$$H = \begin{bmatrix} Z\cos\theta_\alpha & r_\alpha^{(+)} e^{i\nu_{\alpha,1}} \sin\theta_\alpha \\ r_\alpha^{(-)} e^{i\nu_{\alpha,2}} \sin\theta_\alpha & -Z\cos\theta_\alpha \end{bmatrix}; \quad (15)$$

where $Z = h_z + i\Delta_z$; $r_\alpha^{(\pm)} = \sqrt{(h_x \pm \Delta_x)^2 \cos^2\varphi_\alpha + (h_y \pm \Delta_y)^2 \sin^2\varphi_\alpha}$; $\nu_{\alpha,1} = \arctan[-\frac{h_y+\Delta_y}{h_x+\Delta_x}\tan\varphi_\alpha]$; $\nu_{\alpha,2} = \arctan[\frac{h_y-\Delta_y}{h_x-\Delta_x}\tan\varphi_\alpha]$. To solve the above Hamiltonian, the eigenvalues are obtained:

$$E_\pm = \pm\sqrt{r_\alpha^2 e^{i2\nu_\alpha^{(+)}} \sin^2\theta_\alpha + Z^2\cos^2\theta_\alpha} \quad (16)$$

where $r_\alpha \equiv \sqrt{r_\alpha^{(+)} r_\alpha^{(-)}}$, and $\nu_\alpha^{(+)} \equiv \frac{\nu_{\alpha,2}+\nu_{\alpha,1}}{2}$. We define $\tan\phi_\alpha \equiv \frac{r_\alpha e^{i\nu_\alpha^{(+)}}}{Z}\tan\theta_\alpha$, The corresponding wave functions are

$$|\psi_+\rangle = \begin{pmatrix} \rho_\alpha e^{-i\nu_\alpha^{(-)}} \cos\frac{\phi_\alpha}{2} \\ \sin\frac{\phi_\alpha}{2} \end{pmatrix}; \quad |\psi_-\rangle = \begin{pmatrix} -\rho_\alpha e^{-i\nu_\alpha^{(-)}} \sin\frac{\phi_\alpha}{2} \\ \cos\frac{\phi_\alpha}{2} \end{pmatrix}, \quad (17)$$

where $\nu_\alpha^{(-)} \equiv \frac{\nu_{\alpha,2}-\nu_{\alpha,1}}{2}$ and $\rho_\alpha \equiv \sqrt{r_\alpha^{(+)}/r_\alpha^{(-)}}$. Similarly the wave functions in the dual space are

$$|\lambda_+\rangle = \begin{pmatrix} \frac{e^{-i\nu_\alpha^{(-)}}}{\rho_\alpha} \cos^*\frac{\phi_\alpha}{2} \\ \sin^*\frac{\phi_\alpha}{2} \end{pmatrix}; \quad |\lambda_-\rangle = \begin{pmatrix} -\frac{e^{-i\nu_\alpha^{(-)}}}{\rho_\alpha} \sin^*\frac{\phi_\alpha}{2} \\ \cos^*\frac{\phi_\alpha}{2} \end{pmatrix}. \quad (18)$$

In the same way, the non-Abelian Berry connection may be given by

$$A = i\begin{pmatrix} \langle\lambda_+|d\psi_+\rangle & \langle\lambda_+|d\psi_-\rangle \\ \langle\lambda_-|d\psi_+\rangle & \langle\lambda_-|d\psi_-\rangle \end{pmatrix} \quad (19)$$

where

$$A = \frac{1}{2}(\sigma_0 + \sigma_z\cos\phi_\alpha - \sigma_x\sin\phi_\alpha)\left(i\frac{d\rho_\alpha}{\rho_\alpha} + d\nu_\alpha^{(-)}\right) + \sigma_y d\phi_\alpha$$

where $\sigma_0 = \mathbf{1}_{2\times 2}$, $\sigma_{x,y,z}$ are the Pauli matrices. It can be seen that ρ_α, ϕ_α, and $\nu_\alpha^{(-)}$ vary periodically with φ_α and θ_α, which are related to the evolution parameters α. Thus, similarly, the global Berry phase can be obtained:

$$Q = \frac{1}{2\pi}\oint \mathrm{Tr}A = \frac{i}{2\pi}\oint d\ln\rho_\alpha + \frac{1}{2\pi}\oint d\nu_\alpha^{(-)}. \quad (20)$$

Notice that ρ_α is a periodic function of α, with periodicity $\oint d\ln\rho_\alpha = \ln\rho_\alpha|_{\varphi=0}^{\varphi=2\pi} = 0$. Integrating the second term in Eq. (20), we can obtain

(1) $Q = 1$ for $(\Delta_x^2 - h_x^2)(\Delta_y^2 - h_y^2) > 0$;

(2) $Q = 0$ for $(\Delta_x^2 - h_x^2)(\Delta_y^2 - h_y^2) < 0$;

(3) $Q = 1$ for $\Delta_x = \Delta_y = \Delta_z = 0$, when the system reduces to a general two-level Hermitian system.

It can be seen that the global Berry phase Q is equal to 1 or 0, depending only on $\Delta_{x,y}$ at $\pm h_{x,y}$, but is independent of the values of Δ_x and Δ_y beyond $\pm h_{x,y}$. However, $\Delta_{x,y} = \pm h_{x,y}$ is a singularity and Q has no definition at these points. Therefore, Q can be regarded as a topological index identifying the topological invariance. Thus, when the parameters Δ_x and Δ_y vary on crossing $\pm h_{x,y}$, it implies that a topological phase transition occurs.

B. Bipartite dissipative model

To clearly see the topological invariance and the meaning of the complex Berry phase of non-Hermitian systems, we consider a bipartite one-dimensional (1D) lattice model with dissipation on one of its sublattices, which can be realized by double quantum dots [9]. An electron can hop between sites and is initially localized on any nondecaying site. The Hamiltonian of this system can be written as [9]

$$H = \sum_m [\varepsilon_A c_m^\dagger c_m + \varepsilon_B d_m^\dagger d_m + v(c_m^\dagger d_m + d_m^\dagger c_m) + v'(c_m^\dagger d_{m+1} + d_{m+1}^\dagger c_m)], \quad (21)$$

where ε_A is the on-site energy of A sites, while $\varepsilon_B = \varepsilon_A - i2\Gamma$ is the on-site energy of B sites, and the imaginary part describes the dissipation. v and v' are the transition amplitudes of the intracell and intercell hopping processes, respectively. They are independent of the site index m; namely, the translation symmetry is preserved. Thus, using Fourier transformation, we can rewrite the Hamiltonian in Eq. (21) in reciprocal space,

$$H = \sum_k (c_k^\dagger, d_k^\dagger)\begin{bmatrix} \varepsilon_A & v_k \\ v_k^* & \varepsilon_B \end{bmatrix}\begin{pmatrix} c_k \\ d_k \end{pmatrix}, \quad (22)$$

where $v_k = v + v' e^{ik}$. It can be seen that this Hamiltonian is non-Hermitian, but has different mathematical structure from the first model in Eq. (9). By solving the eigenequation in Eq. (22), we can obtain the eigenvalues

$$E_{k,\pm} = \varepsilon_A - i\Gamma \pm \sqrt{|v_k|^2 - \Gamma^2}, \quad (23)$$

where s labels the pseudospin ± 1. The corresponding eigenfunctions are

$$|\psi_+\rangle = \begin{pmatrix} \frac{v_k}{|v_k|}\cos\frac{\phi}{2} \\ \sin\frac{\phi}{2} \end{pmatrix}; \quad |\psi_-\rangle = \begin{pmatrix} -\frac{v_k}{|v_k|}\sin\frac{\phi}{2} \\ \cos\frac{\phi}{2} \end{pmatrix}, \quad (24)$$

and their dual wave functions in the dual space are

$$|\lambda_+\rangle = \begin{pmatrix} \frac{v_k}{|v_k|}\cos^*\frac{\phi}{2} \\ \sin^*\frac{\phi}{2} \end{pmatrix}; \quad |\lambda_-\rangle = \begin{pmatrix} -\frac{v_k}{|v_k|}\sin^*\frac{\phi}{2} \\ \cos^*\frac{\phi}{2} \end{pmatrix}, \quad (25)$$

where $\tan\phi = \frac{|v_k|}{i\Gamma}$, ϕ is a complex variable, and k is a wave vector within the Brillouin zone, $k \in (-\pi,\pi)$. Similarly, the non-Abelian Berry potential of the system can be given by

$$A = \tfrac{1}{2}(\sigma_0 + \sigma_z\cos\phi - \sigma_x\sin\phi)\partial_k\theta_k dk - \tfrac{i}{2}\sigma_y\partial_k\phi dk, \quad (26)$$

where θ_k is defined by $v_k = |v_k|e^{-i\theta_k}$, and $\cos\phi \equiv \frac{i\Gamma}{\sqrt{|v_k|^2-\Gamma^2}}$. Since the wave function is periodic in k, the wave vector k can be regarded as a parameter inducing the Berry phase [16]. The complex geometric phases in two bands can be expressed as

$$\gamma_\pm^G = \oint_C A^\pm = \frac{1}{2}\oint(1 \pm \cos\phi)d\theta_k. \quad (27)$$

Notice that the second terms for different states may cancel, and the global Berry phase becomes

$$Q = \frac{1}{2\pi}\oint A = \frac{1}{2\pi}\oint d\theta_k. \quad (28)$$

Let $q = v'/v$ be the ratio of the hopping amplitudes between the A and B sites. For $q > 1$, as k varies from $-\pi$ to π, the closed integral path goes around the zero point, which corresponds to the winding number, namely, $Q = 1$. For $0 < q < 1$, the integral closed path does not go around the zero point, implying that $Q = 0$. Q is the winding number of the relative phase between components of the Bloch wave function, which identifies the topological invariance of this quantum dissipative system. This topological invariance identified by Q is robust against local variation of q because whether $Q = 0$ or 1 depends only on whether $q < 1$ or $q > 1$ instead of the q value. This implies a topological invariance and Q can be regarded as a topological index. The TPT occurs at $q = 1$ in the parameter space. Namely, this non-Hermitian system has two states that have topological invariance distinguished by $0 < q < 1$ and $q > 1$ in the parameter space.

On the other hand, an abrupt change of the Berry phase in the ground state implies a quantum phase transition for Hermitian systems [5]. For non-Hermitian systems an abrupt change of the complex Berry phase in a quantum state also implies a QPT. Suppose that electrons occupy the lower energy band $E_{k,-}$, namely, the half-filling case, where $|\psi_-\rangle$ is the ground state. Thus, the abrupt change of the complex Berry phase γ_-^G implies a QPT.

In the integration of Eq. (27), for $q < 1$, the closed integral path does not go around the zero point such that the first term of the closed path integration equals zero, and for $q > 1$, the closed integral path goes around the zero point such that the first term of the integration equals 1. Thus, we have $\frac{1}{2}\oint d\theta_k = \pi\Theta(q-1)$, where Θ is the step function. For the second term, notice that

$$\frac{d\theta}{dk} = \frac{q(q+\cos k)}{1+q^2+2q\cos k} = \frac{1}{2}\left(1 + \frac{q^2-1}{1+q^2+2q\cos k}\right), \tag{29}$$

so that the second term in Eq. (27) can be written

$$\begin{aligned}\oint \cos\phi d\theta &= \frac{i\eta}{2}\int_{-\pi}^{\pi}\frac{dk}{\sqrt{1+q^2+2q\cos k-\eta^2}} + \frac{i\eta(q^2-1)}{2}\int_{-\pi}^{\pi}\frac{dk}{(1+q^2+2q\cos k)\sqrt{1+q^2+2q\cos k-\eta^2}}\\ &= \frac{i\eta}{\sqrt{(1+q^2)-\eta^2}}\int_0^{\pi/2}\frac{dk}{\sqrt{1-\frac{4q}{(1+q)^2-\eta^2}\sin^2 k}} + \frac{i\eta(q-1)}{(q+1)\sqrt{(1+q^2)-\eta^2}}\\ &\quad\times\int_0^{\pi/2}\frac{dk}{\left[1-\frac{4q}{(1+q)^2}\sin^2 k\right]\sqrt{1-\frac{4q}{(1+q)^2-\eta^2}\sin^2 k}},\end{aligned} \tag{30}$$

where $\eta = \frac{\Gamma}{v}$. Letting $x = \frac{4q}{(q+1)^2}$ and $y = \frac{4q}{(q+1)^2-\eta^2}$, the above integration can be rewritten as the elliptic integral, and the complex Berry phase γ_μ^G can be obtained:

$$\gamma_\pm^G = \pi\Theta(q-1) \pm i\frac{\eta}{2}\sqrt{\frac{y}{q}}\left(K(y) + \frac{q-1}{q+1}\Pi(x,y)\right), \tag{31}$$

where $K(y) = \int_0^{\pi/2}\frac{dk}{\sqrt{1-y\sin^2 k}}$ and $\Pi(x,y) = \int_0^{\pi/2}\frac{dk}{(1-x\sin^2 k)\sqrt{1-y\sin^2 k}}$ are the first and third kinds of the complete elliptic integral, respectively.

The analytic properties of $\gamma_\pm^G$ in Eq. (31) reveal the relationship between the complex Berry phase, the QPT, and the TPT for non-Hermitian systems. We plot $\gamma_\pm^G$ in the parameter space (q,η) in Figs. 1 and 2 for two pseudospin states.

From the topological point of view, the parameter space is divided into two parts, $0 < q < 1$ and $q > 1$, corresponding to topological phases (TPs) I $Q = 0$ and II $Q = 1$, respectively, and the Berry phase $\gamma_\pm^g$ has a step of π at $q = 1$. The abrupt change of $\gamma_\pm^G$ in the parameter space implies a QPT. In quantum evolution, for Hermitian systems the system properties are sensitive near the exceptional points, where the energy levels have avoided or diabolic crossings, with the parameter varying near these points. For non-Hermitian systems, the eigenenergy is complex. When the real part of the eigenenergy shows an avoided crossing and the imaginary part a true crossing at the exceptional point, it is called a type-I exceptional point, and the reverse is called a type-II exceptional point [14]. We find three basic properties of the exceptional points for this model.

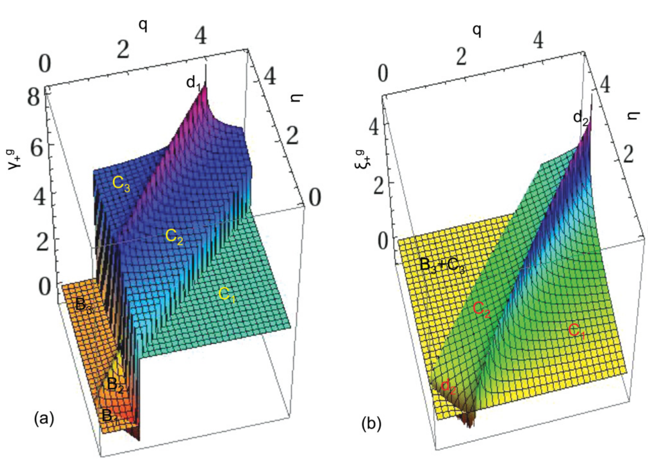

FIG. 1. (Color online) (a) The geometric phases in the up-pseudospin state in the parameter space (q,η). The TPT occurs at $q = 1$, where γ_+^g has an abrupt change of π. The states in $0 < q < 1$ and $q > 1$ are two topological phases. γ_+^g diverges at the line d_1. $\eta = q + 1$ implies a quantum phase transition. (b) The geometric dissipation ξ_+^g in the parameter space. $\xi_+^g = 0$ in B_3 and C_3, meaning there is no dissipative effect from the gauge potential. ξ_+^g diverges at the line d_2.

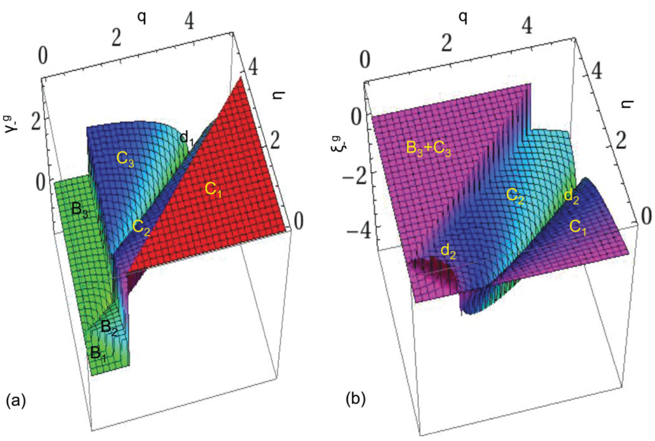

FIG. 2. (Color online) (a) The geometric phases in the down-pseudospin state in the parameter space (q,η). As in Fig. 1, the TPT occurs at $q=1$, where γ_-^g has an abrupt change of π. (b) The geometric dissipation ξ_-^g in the parameter space. $\xi_-^g=0$ in B_3 and C_3 and $\xi_-^g\neq 0$ in other regions. ξ_-^g also diverges at the line d_2.

Claim 3.1. In the region $|q-1|\leqslant\eta\leqslant q+1$ (B_2+C_2 in Figs. 1 and 2) there exists some k such that the complex energy level has a true crossing, $E_+=E_-$, namely, it is gapless.

Proof. $E_+=E_-$ leads to $1+q^2+2q\cos k=\eta^2$. Namely, $1+q^2\pm 2q=\eta^2_{\max(\min)}$. It implies that there exists some k satisfying $1+q^2+2q\cos k=\eta^2$ for η within $|q-1|\leqslant\eta\leqslant q+1$.

Claim 3.2. In the region $\eta\leqslant|1-q|$ (C_1 in Figs. 1 and 2) there exists some k such that the energy-level crossing is type I, $\mathrm{Im}(E_+)=\mathrm{Im}(E_-)$, but $\mathrm{Re}(E_+)\neq\mathrm{Re}(E_-)$, implying the existence of a gap.

Proof. For $\eta\leqslant|1-q|$, $\sqrt{|v_k|^2-\Gamma^2}\in\mathbb{R}$ such that $\mathrm{Re}(E_+)\neq\mathrm{Re}(E_-)$ and $\mathrm{Im}(E_+)=\mathrm{Im}(E_-)$.

Claim 3.3. In the region $\eta\geqslant 1+q$ (B_3+C_3 in Figs. 1 and 2) there exists some k such that the energy-level crossing is type II, $\mathrm{Re}(E_+)=\mathrm{Re}(E_-)$, but $\mathrm{Im}(E_+)\neq\mathrm{Im}(E_-)$.

Proof. For $\eta\geqslant 1+q$, $\sqrt{|v_k|^2-\Gamma^2}$ is purely imaginary such that $\mathrm{Re}(E_+)=\mathrm{Re}(E_-)$ and $\mathrm{Im}(E_+)\neq\mathrm{Im}(E_-)$.

The regions in the above three claims are sufficient but not necessary for the conclusion. The exceptional point properties exhibit different energy band structures and dissipative properties.

The Berry phases $\gamma_\pm^g$ diverge logarithmically at the line $\eta=q+1$ labeled by d_1 in Figs. 1(a) and 2(a), but have a finite jump at the other line $\eta=|q-1|$ labeled by d_2 in Figs. 1(b) and 2(b). Similarly, $\xi_\pm^g$ diverge logarithmically at $\eta=|q-1|$ and have a finite jump at $\eta=q+1$. The divergence of $\gamma_\pm^g$ and $\xi_\pm^g$ at these lines implies a quantum phase transition [8]. The divergence of $\gamma_\pm^g$ at the critical line d_1, $\eta=q+1$, corresponds to resonance by interference of the Berry phase, and the divergence of $\xi_\pm^g$ at the line d_2 implies an internal transition between two states by diffusion. In other words, the divergence of the Berry phase $\gamma_\pm^g$ and its dissipative effect $\xi_\pm^g$ at the state transition lines corresponds to a change of the energy band structure, such as when the parameter changes from $\eta\leqslant|q-1|$ to $\eta\geqslant|q-1|$, corresponding to the energy band structure changing from gapless to gapped, as can be seen from the three claims above.

The physical difference between TPs I and II is that charge transfer between the system and environment does not exist in TP I but does in TP II [9]. The bipartite dissipative model is equivalent to the 1D non-Hermitian quantum walk model [9]. Rudner and Levitov use the average displacement of particles $\langle\Delta m\rangle=\sum_m mP_m$ to describe the particle decay, where P_m measures the decay probability distribution [9].

IV. DISCUSSION

It should be remarked that the solutions in Eq. (4) is obtained under the adiabatic approximation. Actually, there are two regimes of the adiabatic theorem in a non-Hermitian Hamiltonian [17]. For the weak non-Hermiticity regime, in which the absolute values of the imaginary parts of the eigenvalues are of the same order of magnitude as the slowness parameter, the adiabatic theorem is valid for the non-Hermitian Hamiltonian. For the strong non-Hermiticity regime, in which at least some of the eigenvalues have imaginary parts much larger (in absolute value) than the slowness parameter, the adiabatic theorem for a non-Hermitian Hamiltonian is not the same as that of the Hermitian Hamiltonian [17]. Nenciu and Rasche generalized the adiabatic theorem for the strong non-Hermiticity regime, proving that an adiabatic expansion exists for evolution restricted to the subspace corresponding to the least dissipative eigenvalues [17]. The high-order terms can modify the adiabatic evolution in the strong non-Hermiticity regime. However, the high-order terms do not disturb the topological invariance we address here because we may define the Berry connection without the high-order terms even though the high-order terms modify the evolution. In fact, even if we define the Berry connection with a first-order correction [17], we can prove that the first-order correction has no contribution to the topological invariance for a 2×2 non-Hermitian Hamiltonian. Namely,

$$\mathrm{Tr}A_1=\frac{i}{E_\ell-E_j}[\langle\partial_k\lambda_\ell|\psi_j\rangle\langle\lambda_j|\partial_k\psi_\ell\rangle-\langle\partial_k\lambda_j|\psi_\ell\rangle\langle\lambda_\ell|\partial_k\psi_j\rangle]=0,\quad(32)$$

where $|\lambda_{\ell(j)}\rangle$ and $|\psi_{\ell(j)}\rangle$ are the eigenvectors in Eqs. (17), (18), (24), and (25). It can be verified that the first-order correction in the adiabatic approximation makes no contribution to the global Berry phase.

In general, the quantum phase transition for an N-level system occurs at an energy-level crossing or avoided crossing (a so-called diabolic or exceptional point) as some parameter is varied, which is related only to two energy levels and is associated with a nonanalytic Berry phase; the other energy levels are invariant with change in the parameters [5,6,8]. Thus, the divergence or jump of the Berry phase γ_-^g implies a quantum phase transition [8]. However, a topological phase transition is characterized by a topological index. A state with topological invariance is robust against local perturbations. A TPT is independent of energy-level crossings and avoided crossings [7], as can be seen from the discussion above. We generalized the global Berry phases Q to identify the TPT for two kinds of non-Hermitian model. Q as a topological index describes the topological invariance of a system. Q involves all of the energy band structure [2,16]. It implies that the topological properties of a system depend on the whole energy band structure which can actually be seen in topological insulators [18]. Interestingly, topological invariance and TPTs can occur in both Hermitian and non-Hermitian systems. The topological invariance and TPT in the first model originate from non-Hermitian off-diagonal elements, while the topological invariance and TPT in the bipartite dissipative model are induced by Hermitian off-diagonal elements of the Hamiltonian. The non-Hermitian diagonal element Γ does not induce topological invariance and a TPT, but induces a QPT.

It should be emphasized that the complex Berry phase was introduced for adiabatic evolution in some specific models [8,14]. The global Berry phase has been used as a topological index for the Hermitian Hamiltonian [2,19]. Here we found the gauge invariance of the complex Berry phase and global Berry phase for a generic non-Hermitian Hamiltonian and generalized the global Berry phase to identify the topological invariance for a non-Hermitian Hamiltonian. Moreover, we used two two-level non-Hermitian models to reveal the relationship between the TPT and QPT.

V. CONCLUSIONS

In summary, we give the basic properties of the complex Berry phase and the global Berry phases Q of non-Hermitian systems and find that the complex Berry phase and the global Berry phases Q are gauge invariant. We generalize the global Berry phase to non-Hermitian systems as a topological index to identify the topological invariance for two quite general non-Hermitian models. For the bipartite dissipative model, we give the phase diagram of the complex Berry phase in the parameter space, in which an abrupt change of the Berry phase reveals the QPT of the system, which corresponds to an exceptional point. Our findings reveal some topological invariance in non-Hermitian systems, and the relationships between the complex Berry phase, topological invariance, the QPT, and the TPT in non-Hermitian systems.

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[1] G. A. Prinz, Science **282**, 1660 (1998); S. A. Wolf, D. D. Awschalom, A. Buhrman, J. M. Daughton, S. von Molnar, M. L. Roukes, A. Y. Chtchelkanova, and D. M. Treger, *ibid.* **294**, 1488 (2001); S. Murakami, N. Nagaosa, and S. C. Zhang, *ibid.* **301**, 1348 (2003).

[2] Huan-Qiang Zhou, Sam Young Cho, and Ross H. McKenzie, Phys. Rev. Lett. **91**, 186803 (2003); P. W. Brouwer, Phys. Rev. B **58**, R10135 (1998).

[3] Paolo Zanardi, Phys. Lett. A **264**, 94 (1999).

[4] Shi-Liang Zhu and Z. D. Wang, Phys. Rev. Lett. **91**, 187902 (2003); Wang Xiang-bin and Matsumoto Keiji, Phys. Rev. B **65**, 172508 (2002); Shi-Liang Zhu and Z. D. Wang, Phys. Rev. Lett. **89**, 097902 (2002).

[5] Angelo C. M. Carollo and Jiannis K. Pachos, Phys. Rev. Lett. **95**, 157203 (2005); A. Carollo, I. Fuentes-Guridi, M. F. Santos, and V. Vedral, *ibid.* **90**, 160402 (2003); Shi-Liang Zhu, *ibid.* **96**, 077206 (2006).

[6] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, England, 1999), p. 33; I. Zutic, J. Fabian, and S. Das Sarma, Rev. Mod. Phys. **76**, 323 (2004); Y. D. Wang, Z. D. Wang, and C. P. Sun, Phys. Rev. B **72**, 172507 (2005).

[7] Hossein Mehri-Dehnavi and Ali Mostafazadeh, J. Math. Phys. **49**, 082105 (2008); G. Dattoli, R. Mignani, and A. Torre, J. Phys. A **23**, 5795 (1990); Ingrid Rotter, *ibid.* **42**, 153001 (2009).

[8] A. I. Nesterov and S. G. Ovchinnikov, Phys. Rev. E **78**, 015202(R) (2008); S. Massar, Phys. Rev. A **54**, 4770 (1996); Xiao-Liang Qi, Taylor L. Hughes, and Shou-Cheng Zhang, Phys. Rev. B **78**, 195424 (2008).

[9] M. S. Rudner and L. S. Levitov, Phys. Rev. Lett. **102**, 065703 (2009).

[10] Zohar Nussinov and Gerardo Ortiz, Ann. Phys. (NY) **324**, 977 (2009); Proc. Natl. Acad. Sci. USA **106**, 16944 (2009).

[11] X. G. Wen, *Quantum Field Theory of Many-Body Systems* (Oxford University Press, Oxford, 2003).

[12] *Geometric Phases in Physics*, edited by A. Shapere and F. Wilczek (World Scientific, Singapore, 1989).

[13] A. Kitaev, Ann. Phys. (NY) **303**, 2 (2003).

[14] F. Keck, H. J. Korsch, and S. Mossmann, J. Phys. A **36**, 2125 (2003); J. C. Garrison and E. M. Wright, Phys. Lett. A **128**, 177 (1988).

[15] T. Hirano, H. Katsura, and Y. Hatsugai, Phys. Rev. B **78**, 054431 (2008); Yasuhiro Hatsugai, J. Phys. Soc. Jpn. **75**, 123601 (2006); **74**, 1374 (2005); **73**, 2604 (2004).

[16] A. Bohm *et al.*, *The Geometric Phase in Quantum Systems* (Springer, New York, 2003); Jun Goryo and Mahito Kohmoto, Phys. Rev. B **66**, 085118 (2002); **66**, 174503 (2002); Ganesh Sundaram and Qian Niu, *ibid.* **59**, 14915 (1999); F. D. M. Haldane, Phys. Rev. Lett. **93**, 206602 (2004); Kai Sun and Eduardo Fradkin, Phys. Rev. B **78**, 245122 (2008).

[17] G. Nenciu and G. Rasche, J. Phys. A **25**, 5741 (1992).

[18] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 146802 (2005); Liang Fu and C. L. Kane, Phys. Rev. B **76**, 045302 (2007).

[19] I. Maruyama, T. Hirano, and Y. Hatsugai, Phys. Rev. B **79**, 115107 (2009).