Geometrical aspects of $\mathcal{PT}$-invariant transfer matrices

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(Received 27 July 2012; published 14 January 2013)

We show that the transfer matrix for a $\mathcal{PT}$-invariant system, when recast in the appropriate variables, can be interpreted as a point in the $(3+1)$-dimensional de Sitter space. We introduce a natural $\mathcal{PT}$-invariant composition law for these matrices and confirm that their action appears as a Lorentz transformation. We elucidate the geometrical meaning of the $\mathcal{PT}$ symmetry breaking and suggest that the cosmological event horizon arising in the de Sitter metric can be can be unraveled with a simple optical scheme.

DOI: 10.1103/PhysRevA.87.012111 PACS number(s): 11.30.Er, 42.25.−p, 03.65.Nk, 03.30.+p

I. INTRODUCTION

After the work of Bender and co-workers [1], considerable effort has been invested in the study of non-Hermitian potentials that have neither parity ($\mathcal{P}$) nor time-reversal symmetry ($\mathcal{T}$), yet they retain combined $\mathcal{PT}$ invariance [2–10]. These systems can exhibit real energy eigenvalues, thus suggesting a possible generalization of quantum mechanics. Moreover, they can also display a spontaneous $\mathcal{PT}$ symmetry breaking, at which the reality of the eigenvalues is lost [11]. This speculative concept has motivated an ongoing debate in several forefronts, including quantum field theories [12], Anderson models [13–15], complex crystals [16–19], Lie algebras [20–22], and open quantum systems [23], to mention a few.

Quite recently, the prospect of realizing $\mathcal{PT}$-symmetric potentials within the framework of optics has been suggested [24] and experimentally tested [25]. The complex refractive index takes on here the role of the potential so they can be accomplished through a judicious inclusion of index guiding and gain and loss regions. Besides, $\mathcal{PT}$-synthetic materials can exhibit several intriguing features; these include, among others, power oscillations [26], nonreciprocity of light propagation [27], Bloch oscillations [28], coherent perfect absorbers [29,30], nonlinear switching structures [31], or unidirectional invisibility [32].

Interesting as they are, these developments have one aspect in common that might be considered as a flaw: the physical interpretation of $\mathcal{PT}$ symmetry remains obscure [33]. Although complex potentials have been used to phenomenologically describe loss mechanisms [34], there are further subtleties in the $\mathcal{PT}$ invariance. It is our purpose to put forth a simple feature of these systems that, possibly, may help to answer this criticism. We argue that under $\mathcal{PT}$ symmetry, the transfer matrix may be understood as a point in the de Sitter space and its action manifest as a Lorentz transformation.

Apart from a relativistic presentation of the topic, which has interest on its own, this gives rise to a nice picture in terms of hyperbolic geometry, which is a fundamental aspect of modern physics. As an illustration of this geometrical scenario, we reanalyze the existence of a spontaneous $\mathcal{PT}$ symmetry breaking and we also suggest that the existence of a cosmological horizon might be unraveled by using a simple optical setup.

II. $\mathcal{PT}$-INVARIANT TRANSFER MATRIX

The main ideas we wish to put forward can be captured by considering the monochromatic wave propagation in a dielectric structure with a spatially dependent complex permittivity $\varepsilon(x)$, in the plane-wave and scalar approximations. This is fully equivalent to the scattering by a one-dimensional complex potential in quantum mechanics [35–37].

As sketched in Fig. 1, the structure is embedded in the region $|x| < L/2$, where $\varepsilon(x)$ is complex; it is the imaginary part which describes the local gain or loss of the medium. Outside this region, $\varepsilon(x)$ is assumed to be real and equal to $\varepsilon(x) = n_0^2$, where n_0 represents a constant background index (in a practical implementation, n_0 is the refractive index of the waveguide or the fiber in which the system is embedded).

By writing the electric field in the structure as $\mathcal{E}(x,t) = E(x)\exp(-i\omega t) + \text{c.c.}$, where c.c. is the complex conjugate and ω is the (complex) frequency of the field, the spatial mode envelope $E(x)$ satisfies the Helmholtz equation

$$\left[\frac{d^2}{dx^2} + \frac{\omega^2}{c^2}\varepsilon(x)\right] E(x) = 0\,, \tag{1}$$

with c being the speed of light in vacuum. The most general solution of Eq. (1) can be written as

$$E(x) = \begin{cases} A_+ \exp(ikn_0x) + A_- \exp(-ikn_0x)\,, & x < -L/2, \\ B_+ \exp(ikn_0x) + B_- \exp(-ikn_0x)\,, & x > L/2, \end{cases} \tag{2}$$

where $k = \omega/c$ is the wave vector in vacuum, the subscripts $+$ and $-$ indicate that the waves propagate to the right and to the left, respectively, and the amplitudes A and B refer to the end points a and b of the structure, as marked in Fig. 1.

The linearity of the problem allows one to relate the wave amplitudes on both sides of the structure by

$$\begin{pmatrix} A_- \\ A_+ \end{pmatrix} = \mathbf{M} \begin{pmatrix} B_- \\ B_+ \end{pmatrix}, \tag{3}$$

where $\mathbf{M}$ is the transfer matrix, which can be written as [38]

$$\mathbf{M} = \begin{pmatrix} 1/T^* & R_l/T \\ -R_r/T & 1/T \end{pmatrix}, \tag{4}$$

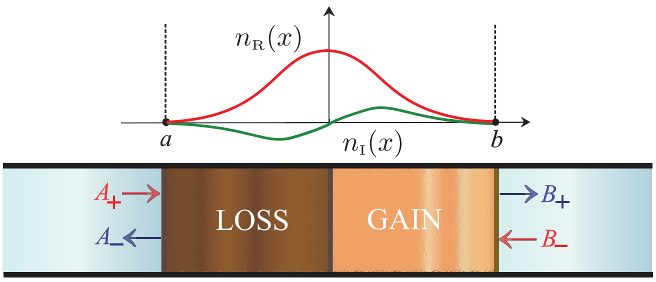

FIG. 1. (Color online) Illustration of the wave scattering in a one-dimensional optical $\mathcal{PT}$-symmetric structure with complex dielectric constant $\epsilon(x)$, displaying the input (A_+ and B_-, in red) and output (A_- and B_+, in blue) complex amplitudes. In the upper panel we schematize a typical behavior for the real and imaginary parts of the complex refractive index $n(x)$.

with the constraint det $\mathbf{M} = 1$. Here, R_l and R_r stand for the reflection coefficients for left ($a \to b$) and right ($b \to a$) incidence, whereas $T \equiv T_l = T_r$ is the direction-independent transmission coefficient. They must be determined from the boundary conditions and, in general, are frequency dependent. In fact, there might exist spectral singularities for those frequencies where T and consequently, R_l and R_r, diverge [37]. We can look at a spectral singularity as a frequency for which the two solutions in Eq. (2) become linearly dependent, i.e., they have a vanishing Wronskian.

The $\mathcal{PT}$ invariance leads to the requirement $\varepsilon(x) = \varepsilon^*(-x)$. In terms of the complex refractive index $\varepsilon^2(x) = n(x)$, the real part is then an even function of position $n_R(x) = n_R(-x)$, while the imaginary is odd $n_I(x) = -n_I(-x)$. In physical words, this indicates that there is a balance of absorption and amplification in parity-related regions.

The $\mathcal{PT}$ operation on $\mathbf{M}$ can be formulated as $\mathcal{PT}\mathbf{M}(\omega) = \sigma_x \, \mathbf{M}(\omega^*) \, \sigma_x$, where σ_x is the corresponding Pauli matrix [35,36]. Hence one works out the condition

$$\mathrm{Re}\left(\frac{R_l}{T}\right) = \mathrm{Re}\left(\frac{R_r}{T}\right) = 0. \tag{5}$$

Alternatively, we can rewrite this as

$$\rho_l - \tau = \pm\pi/2, \quad \rho_r - \tau = \pm\pi/2, \tag{6}$$

where $\tau = \arg(T)$ and $\rho_{l,r} = \arg(R_{l,r})$. If we look at the complex numbers R_l, R_r, and T as phasors, Eq. (6) tells us that R_l and R_r are always collinear, while T is simultaneously perpendicular to them. We draw attention to the fact that the same expressions have been derived for lossless symmetric beam splitters [39]; here we have shown that they hold true for any $\mathcal{PT}$ structure.

Next we examine the behavior of the scattering matrix, defined by

$$\mathbf{S} = \begin{pmatrix} R_l & T \\ T & R_r \end{pmatrix}, \tag{7}$$

so it relates outgoing to incoming amplitudes. Indeed, the eigenvalues of $\mathbf{S}$, denoted as $s_\pm$, can be displayed in terms of the matrix elements of $\mathbf{M}$. When $\mathcal{PT}$ symmetry holds, either each eigenvalue of $\mathbf{S}$ is itself unimodular or forms pairs with reciprocal moduli. These two possibilities correspond to symmetric and symmetry-broken scattering behavior [11].

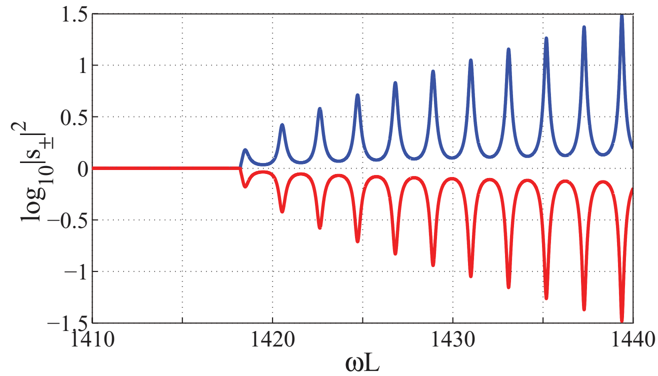

FIG. 2. (Color online) Semilog plot of S-matrix eigenvalue intensities $\log |s_\pm|^2$ as a function of ωL for a $\mathcal{PT}$-symmetric slab of length L with balanced refractive index $n = 3 \pm 0.005i$ in each half. The $\mathcal{PT}$ symmetry is spontaneously broken at $\omega_c \simeq 1418.21/L$.

The criterion for the eigenvalues of $\mathbf{S}$ to be unimodular is $|(R_l - R_r)/T| \leqslant 2$. Upon varying the setup parameters (e.g., the frequency), violating this inequality brings us into the broken-symmetry phase.

To be specific, we shall benefit from the simple model of a single slab of total length L with fixed (and constant) refractive index $n = n_R \pm i n_I$ in each half [30]. In this case, the imaginary part of the index plays the role of the breaking parameter and the critical frequency can be shown to be $\omega_c \simeq c/(n_I L) \ln(2 n_R / n_I)$. Figure 2 shows the appearance of that transition as a function of ωL and how in the broken-symmetry phase a net amplification occurs.

III. GEOMETRICAL INTERPRETATION

In view of the general form of the transfer matrix and the conditions (5) imposed by the $\mathcal{PT}$ invariance, we can generically write $\mathbf{M}$ as

$$\mathbf{M} = \begin{pmatrix} x + iy & i(z+t) \\ i(z-t) & x - iy \end{pmatrix}, \tag{8}$$

where (x,y,z,t) are arbitrary real numbers we shall immediately interpret as spatio-temporal coordinates. In fact, using the transmission and reflection coefficients, they read as

$$x = \mathrm{Re}\left(\frac{1}{T}\right), \quad y = -\mathrm{Im}\left(\frac{1}{T}\right), \quad z = \frac{R_l - R_r}{2iT}, \quad t = \frac{R_l + R_r}{2iT}. \tag{9}$$

The condition of det $\mathbf{M} = 1$ gives now

$$x^2 + y^2 + z^2 - t^2 = 1. \tag{10}$$

In other words, we can regard the matrix $\mathbf{M}$ as defining a point in a single-sheeted unit hyperboloid, which is known as the de Sitter space dS_3. From now on, $\mathbf{M}$ will denote both the transfer matrix and the associated point $(x,y,z,t)^T$ it determines on dS_3 (the superscript T indicates the transpose).

We recall that the de Sitter space is perhaps the simplest example of pseudo-Riemanian structure [40], equivalent to a pseudosphere. The causal structure of dS_3 is induced by the restriction of the Lorentzian geometry of the ambient Minkowski space-time [41].

When two conventional Hermitian systems, represented by transfer matrices **M** and **N**, are coupled, the resulting one is given by the matrix product **MN**, taken in the appropriate order. However, when those systems are $\mathcal{PT}$ invariant, to preserve such a symmetry we have to piece them together either as **MNM** or **NMN**. From a mathematical viewpoint it seems thus natural to define the $\mathcal{PT}$ composition law as $\mathbf{M} \odot \mathbf{N} = \mathbf{M}\,\mathbf{N}\,\mathbf{M}$. This resembles the conjugation by matrix **M**, but please note carefully that the inverse of **M** does not appear here. This law is not associative (therefore these matrices do not form a group) and has only left unit element $\mathbb{1} \odot \mathbf{M} = \mathbb{1}\,\mathbf{M}\,\mathbb{1} = \mathbf{M}$, $\mathbf{M} \odot \mathbb{1} = \mathbf{M}\,\mathbb{1}\,\mathbf{M} = \mathbf{M}^2$. The right inverse of **M** is $\mathbf{M}^{-2}$ and the left inverse $\mathbf{M}^{-1/2}$.

Let $(a,b,c,d)^T$ be the coordinates of the matrix **N** in dS_3 and $(a',b',c',d')^T$ the coordinates of $\mathbf{M} \odot \mathbf{N}$. A direct calculation gives

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \Lambda(\mathbf{M}) \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \tag{11}$$

where

$$\Lambda(\mathbf{M}) = \begin{pmatrix} -1+2x^2 & -2xy & -2xz & 2xt \\ 2xy & 1-2y^2 & -2yz & 2yt \\ 2xz & -2yz & 1-2z^2 & 2zt \\ 2xt & -2yt & -2zt & 1+2t^2 \end{pmatrix}. \tag{12}$$

Furthermore, $\Lambda(\mathbf{M})^T\, g\, \Lambda(\mathbf{M}) = g$, with $g = \mathrm{diag}(1,1,1,-1)$ being the metric tensor. This proves that the transformation $\Lambda(\mathbf{M})$ induced by **M** is a Lorentz transformation and maps dS_3 into itself [so that $\Lambda(\mathbf{M})$ realizes an isometry of the de Sitter space]. This must to be taken into account when dealing with periodic $\mathcal{PT}$ systems.

To illustrate our approach, let us analyze from this geometrical perspective the $\mathcal{PT}$ symmetry-breaking point discussed before. Using the space-time coordinates (9), the eigenvalues of the scattering matrix are

$$s_{\pm} = \frac{it \pm \sqrt{1-z^2}}{x - iy}. \tag{13}$$

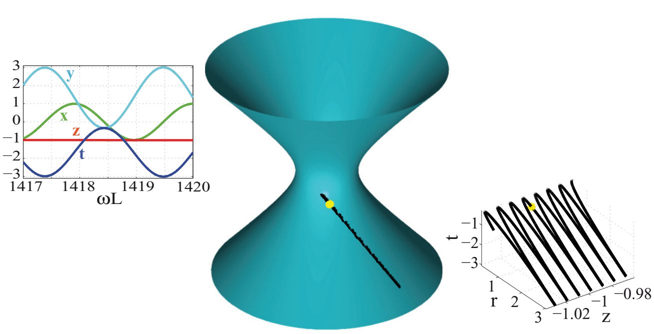

FIG. 3. (Color online) Left panel: Space-time coordinates associated with the same $\mathcal{PT}$-symmetric slab of length L as in Fig. 2, with balanced refractive index $n = 3 \pm 0.005i$ in each half, as a function of ωL. Central panel: The associated trajectory in the de Sitter space dS_3 showing only two of the three space coordinates. The marked yellow point corresponds to the critical frequency ω_c. Right panel: A zoomed version of the previous trajectory, where small oscillations can be appreciated. In this plot, $r^2 = x^2 + y^2$.

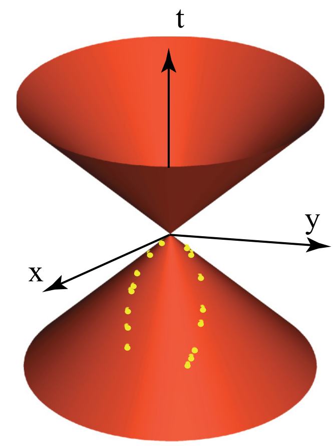

FIG. 4. (Color online) Critical points for a $\mathcal{PT}$-symmetric slab of length L with balanced refractive index $n = 3 \pm n_{\mathrm{I}} i$, when n_{I} varies from 0.005 to 0.105 in steps of 0.005. All the points are located in the light cone (14), which is the intersection of dS_3 with the plane $z^2 = 1$.

Both eigenvalues are unimodular when $z^2 < 1$, while when $z^2 > 1$ they form pairs with reciprocal moduli. The breaking occurs at the points characterized by $z^2 = 1$. This corresponds to the $(2+1)$-dimensional light cone

$$x^2 + y^2 - t^2 = 0, \tag{14}$$

whose vertex is at $(0,0,\pm 1,0)^T$.

In Fig. 3 we have represented the space-time coordinates associated to the slab used before in Fig. 2. The yellow mark corresponds to the breaking point. In the right inset we see that the trajectory on dS_3 is oscillatory when seen with the proper resolution.

In Fig. 4 we have plotted the critical points obtained for the same simple slab model when the imaginary part n_{I} varies, confirming that all of them lie in the light cone (14).

The de Sitter geometry finds its most important physical applications in cosmology, for the induced metric $ds^2 = dx^2 + dy^2 + dz^2 - dt^2$ is a vacuum solution of Einstein's equations with a cosmological constant term. It is customary to introduce in dS_3 static coordinates $x = r\cos\varphi$, $y = r\sin\varphi$, $z = \sqrt{1-r^2}\cosh\lambda$, $t = \sqrt{1-r^2}\sinh\lambda$. In terms of them the metric reads

$$ds^2 = -(1-r^2)d\lambda^2 + \frac{dr^2}{1-r^2} + r^2 d\varphi^2. \tag{15}$$

At $r = 1$ a cosmological horizon appears, which has been under heated debate [42]. The formal analogy drawn in this paper allows one to explore that horizon by means, e.g., of the simple optical $\mathcal{PT}$ slab considered so far. This constitutes yet another instance of an analog for gravitational phenomena [43]. Work in that direction is in progress and will be presented elsewhere.

IV. CONCLUDING REMARKS

Modern geometry provides a useful and, at the same time, simple language in which numerous physical ideas and concepts may be clearly formulated and effectively treated.

In this paper we have devised a geometrical tool to analyze $\mathcal{PT}$ invariance in a concise way that, in addition, can be related to other branches of physics. This picture allows space-time phenomena to be transplanted to the more familiar arena of the optical world. However, note that this gateway works in both directions. Here it has allowed us to establish a relativistic presentation of $\mathcal{PT}$ invariance, but optics can be also used as a powerful instrument for visualizing special relativity [44]. Our paper is one further step in this fruitful interplay between optics and relativity.

ACKNOWLEDGMENTS

We wish to acknowledge discussions with the participants of the 518th Heraeus Seminar, "Quantum-Optical Analogies: A Bridge between Classical and Quantum Physics," at the Physik-Zentrum in Bad Honnef, especially with Gerd Leuchs, Demetrios Christodoulides, and Ali Mostafazadeh. Financial support from the Spanish DGI (Grant FIS2011-26786) and the UCM-BSCH program (Grant GR-920992) is gratefully acknowledged.

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