Canonical Naimark extension for generalized measurements involving sets of Pauli quantum observables chosen at random

Carlo Sparaciari[1] and Matteo G. A. Paris[1,2,*]

[1]*Dipartimento di Fisica, Università degli Studi di Milano, I-20133 Milano, Italy*
[2]*Consorzio Nazionale Italiano di Struttura della Materia, UdR Milano, I-20133 Milano, Italy*

We address measurement schemes where certain observables X_k are chosen at random within a set of nondegenerate isospectral observables and then measured on repeated preparations of a physical system. Each observable has a probability z_k to be measured, with $\sum_k z_k = 1$, and the statistics of this generalized measurement is described by a positive operator-valued measure. This kind of scheme is referred to as *quantum roulettes*, since each observable X_k is chosen at random, e.g., according to the fluctuating value of an external parameter. Here we focus on quantum roulettes for qubits involving the measurements of Pauli matrices, and we explicitly evaluate their canonical Naimark extensions, i.e., their implementation as indirect measurements involving an interaction scheme with a probe system. We thus provide a concrete model to realize the roulette without destroying the signal state, which can be measured again after the measurement or can be transmitted. Finally, we apply our results to the description of Stern-Gerlach-like experiments on a two-level system.

I. INTRODUCTION

In this paper we deal with a specific class of generalized quantum measurements usually referred to as *quantum roulettes*. These quantum measurements are achieved through the following procedure. Consider K projective measurements, described by the set $\{X_k\}_{k=1,...,K}$ of nondegenerate isospectral observables in a Hilbert space H. The system is sent to a detector which, at random, performs the measurement of the observable X_k. Each observable has a probability z_k to be measured, with $\sum_k z_k = 1$. This scheme is referred to as *quantum roulette* since the measured observable X_k is chosen at random, e.g., according to the fluctuating value of a physical parameter, as it happens for the outcome of a roulette wheel. The generalized observable actually measured by the detector is described by a positive operator-valued measure (POVM), which provides the probability distribution of the outcomes and the postmeasurement states [1–3].

As a matter of fact, any POVM on a given Hilbert space may be implemented as a projective measurement in a larger one (e.g., see [4] for single-photon qudits). This measurement scheme is usually referred to as a *Naimark extension* of the POVM. Indeed, it is quite straightforward to find a Naimark extension for the POVM of any quantum roulette in terms of a joint measurement performed on the system under investigation and an ancillary one.

On the other hand, for any POVM the Naimark theorem [5] ensures the existence of a *canonical* Naimark extension, i.e., the implementation of the POVM as an indirect measurement involving an *independent* preparation of an ancillary (probe) system [6], an interaction of the probe with the system under investigation, and a final step in which only the probe is subjected to a (projective) measurement [7,8]. A question thus arises on the canonical implementation of the quantum roulette's POVM and on the resources needed to

realize the corresponding interaction scheme. This is the main point of this paper. In particular, we focus on quantum roulettes involving the measurements of Pauli matrices on a qubit system and explicitly evaluate their canonical Naimark extensions.

We recall that having the Naimark extension of a generalized measurement is, in general, highly desirable, since it provides a concrete model to realize an apparatus which performs the measurement without destroying the state of the system under investigation. Thus, the state after the measurement can be measured again, or can be transmitted, and the tradeoff between information gain and measurement disturbance may be evaluated [9–12]. Alternatively, the scheme may serve to perform indirect quantum control [13].

It should be emphasized that the concept of quantum roulette provides a natural framework to describe the measurement scheme in which the measured observable depends on an external parameter, which cannot be fully controlled and fluctuates according to a given probability distribution. A prominent example is given by the Stern-Gerlach apparatus, which allows one to measure a spin component of a particle in the direction individuated by an inhomogeneous magnetic field [14,15]. Indeed, whenever the field is fluctuating, or the uncertainty in the splitting force is taken into account [16], the measurement scheme is described by a quantum roulette. Also, in this case, if we have the canonical Naimark extension for the roulette, then we have a concrete way to realize a measurement without destroying the state [17]. We recall that in the continuous variable regime quantum roulettes involving homodyne detection with randomized phase of the local oscillator have been already studied theoretically [18] and realized experimentally [19].

The paper is structured as follows. In the next section we introduce notation, briefly review the Naimark theorem, and gather all the necessary tools, e.g., the Cartan decomposition of SU(4) transformations, which allows us to greatly reduce the number of parameters involved in the problem of finding

_____
*matteo.paris@fisica.unimi.it

the canonical Naimark extension. In Sec. III we introduce the concept of quantum roulette, derive the corresponding POVM, and illustrate an example of noncanonical Naimark extension. In Sec. IV we derive the canonical extension for one-parameter Pauli quantum roulettes and discuss details of their implementation, whereas Sec. V is devoted to analyze Stern-Gerlach-like experiments as quantum roulettes, i.e., taking into account the possibility that the magnetic field is randomly fluctuating. Finally, Sec. VI closes the paper with some concluding remarks.

II. NOTATION AND TOOLS

A. POVMs and the Naimark theorem

When we measure an observable on a quantum system, we cannot predict which outcome we will obtain in each run. What we know is the spectrum of possible outcomes and their probability distribution. Given a system described by a state ρ in the Hilbert space H, to obtain the probability distribution of the outcomes x we use the Born Rule:

$$p_x = \text{Tr}[\rho \Pi_x].$$

In order to satisfy the properties of the distribution p_x, the operators Π_x do not need to be projectors. The operators Π_x have to be positive, $\Pi_x \geqslant 0$, since the probability distribution p_x has to be positive for every $|\varphi\rangle \in H$, and normalized, $\sum_x \Pi_x = \mathbb{I}$, since p_x is normalized. A decomposition of identity by positive operators Π_x will be referred to as a *positive operator-valued measure* (POVM), and the operators Π_x are the elements of the POVM.

We use Π_x to get information about the probability distribution p_x, but if we are interested in postmeasurement states we have to introduce the set of operators M_x, the *detection operators*. These operators should give the same probability distribution given by Π_x; thus, they are obtained from $p_x = \text{Tr}[M_x \rho M_x^\dagger] = \text{Tr}[\rho \Pi_x]$. Therefore, detection operators that satisfy $\Pi_x = M_x M_x^\dagger$ are $M_x = U_x \sqrt{\Pi_x}$, with U_x being a unitary operator such that $U_x U_x^\dagger = \mathbb{I}$, and this leaves a residual freedom on the postmeasurement states. The postmeasurement states are then given by

$$\rho_x = \frac{1}{p_x} M_x \rho M_x^\dagger.$$

A measurement described by the operators Π_x is referred to as *generalized measurement*.

In order to link general measurements with physical schemes of measurement, we have the Naimark theorem, which states that a generalized measurement in a Hilbert space H_A may be always seen as an indirect measurement in a larger Hilbert space given by the tensor product $H_A \otimes H_B$. This indirect measure is known as *canonical Naimark extension* for the generalized measurement (see Fig. 1). Conversely, when we perform a projective measure on the subsystem H_B of a composite system $H_A \otimes H_B$, the degrees of freedom of H_B may be traced out and we obtain the same probability distribution of the outcomes of the projective measurement and the same postmeasurement states performing a generalized measurement on the subsystem H_A.
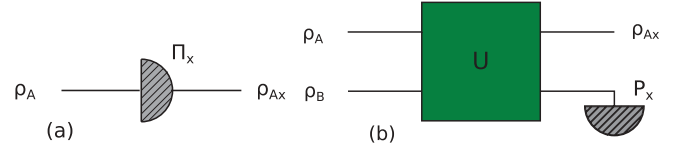

FIG. 1. (Color online) The two measurement schemes linked by the Naimark theorem. (a) A generalized measurement described by the POVM $\Pi_x = M_x^\dagger M_x$ and (b) its canonical Naimark extension, defined by the triple $\{\rho_B, U, \{P_x\}\}$, describing the probe state ρ_B, the evolution operator U, and the projective measurement $\{P_x\}$ on the probe system, respectively.

The Naimark theorem gives a practical recipe to evaluate the canonical extension for a generalized measurement in a Hilbert space H_A:

$$\Pi_x = \text{Tr}_B[\mathbb{I} \otimes \rho_B \, U^\dagger \, \mathbb{I} \otimes P_x \, U], \tag{1}$$

where $\rho_B \in L(H_B)$ describes the state of the probe system (or *ancilla*), the operators $\{P_x\} \in L(H_B)$ are a set of projectors which describe the measurement on the ancilla, and the unitary operator $U \in L(H_A \otimes H_B)$ works on both the system and the ancilla. A canonical Naimark extension for the generalized measurement given by the operators $\{\Pi_x\} \in L(H_A)$ is thus individuated by the triple $\{\rho_B, U, \{P_x\}\}$.

Evaluating the canonical Naimark extension for a generalized measurement is desirable since it gives a concrete model to realize an apparatus which performs the measurement without destroying the state. Then, the postmeasurement state can be transmitted, or measured again.

B. The Cartan decomposition of SU(4) transformations

In the following we are going to deal with two-qubit interactions, i.e., unitary operators (with unit determinant) of the group SU(4), which are individuated by 15 parameters. In order to reduce the number of these parameters we will make use of the Cartan decomposition, which allows us to factor a general operator in SU(4) into local operators working on single qubits and a single two-qubit operator V individuated by three parameters [20–22] (see Fig. 2).

According to the Cartan decomposition any $X \in \text{SU}(4)$ can be rewritten as $X = (R_1 \otimes R_0) V (S_1 \otimes S_0)$, where $R_1, R_0, S_1, S_0 \in \text{SU}(2)$, and $V = \exp\{i \sum_{j=1}^3 k_j \sigma_j \otimes \sigma_j\}$, with $\mathbf{k} \equiv (k_1, k_2, k_3) \in \mathbb{R}^3$. The operators σ_i are the Pauli matrices.
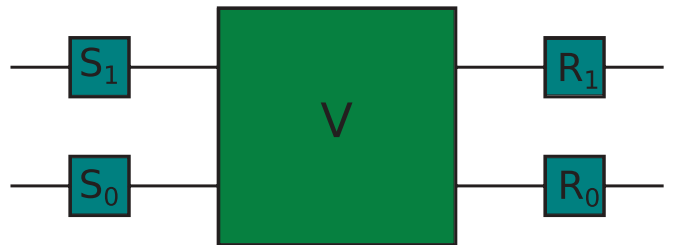

FIG. 2. (Color online) The Cartan decomposition of the operator $X \in \text{SU}(4)$, given by $(R_1 \otimes R_0) V (S_1 \otimes S_0)$.

If we introduce the following equivalence relation in SU(4),

$$A \sim B \quad \text{if} \quad A = (R_1 \otimes R_0)B(S_1 \otimes S_0), \qquad (2)$$

we can split the whole space of unitary operators with unit determinant into equivalence classes. Then, since an operator $X \in$ SU(4) is represented (by the equivalent relation given here) by the matrix V, it is possible to establish a link between operators in SU(4) and real vectors: $X \sim \mathbf{k}$, where $\mathbf{k}$ is the *class vector* of X. The following operations are class-preserving: (1) *shift*—$\mathbf{k}$ can be shifted by $\pm\frac{\pi}{2}$ along one of its components; (2) *reverse*—the sign of two components of $\mathbf{k}$ can be reversed; and (3) *swap*—two components of $\mathbf{k}$ can be swapped.

By the use of these operations it is always possible to reduce any $\mathbf{k}$ into a bounded region K given by the following:

(1) $\frac{\pi}{2} > k_1 \geqslant k_2 \geqslant k_3 \geqslant 0$.

(2) $k_1 + k_2 \leqslant \frac{\pi}{2}$.

(3) If $k_3 = 0$, then $k_1 \leqslant \frac{\pi}{4}$.

$\mathbf{k} \in K$ are referred to as *canonical class vectors*.

The expression of the operator V may be simplified using a different set of parameters, e.g.,

$$k_1 = -\frac{\alpha_1 - \alpha_2}{4},$$
$$k_2 = -\frac{\alpha_1 + \alpha_2}{4},$$
$$k_3 = -\frac{\alpha_3}{2}.$$

Additionally, we introduce the operators $\Sigma_i = \frac{1}{2}\sigma_i \otimes \sigma_i$, which are normalized in the space of 4×4 operators, with the inner product $\langle A, B \rangle = \text{Tr}[B^\dagger A]$. Eventually, we obtain the following matrix V:

$$V = \exp\{-i[\tfrac{1}{2}(\alpha_1 - \alpha_2)\Sigma_1 + \tfrac{1}{2}(\alpha_1 + \alpha_2)\Sigma_2 + \alpha_3\Sigma_3]\}. \quad (3)$$

In terms of the α parameters the bounded region corresponding to the canonical class vectors is given by

$$-\pi \leqslant \alpha_1 \leqslant 0,$$
$$0 \leqslant \alpha_2 \leqslant -\alpha_1,$$
$$\alpha_1 + \alpha_2 \leqslant 2\alpha_3 \leqslant 0,$$
$$\text{If} \quad \alpha_3 = 0 \quad \text{then} \quad \alpha_1 - \alpha_2 \geqslant -\pi.$$

As we will see in the next section, the Cartan decomposition simplifies the problem of finding the canonical Naimark extension for the quantum roulette, since we will be able to neglect single-qubit operations, and thus reducing the 15-parameters operator U to the 3-parameters operator V. Furthermore, we will restrict the interval of the parameters of V to the bounded region defined above.

III. THE PAULI QUANTUM ROULETTE WHEEL

Let us consider K observables $\{X_k\}$ in a Hilbert space H_A with dimension $d = \dim(H_A)$. All the observables are nondegenerate and isospectral. Since the observables are nondegenerate and the Hilbert space is finite dimensional, each of them has d eigenvalues. We use a detector which chooses at random one of these observables and performs a measurement of that observable. Each observable has a probability z_k of being selected by the detector and $\sum_k z_k = 1$. This scheme,

denoted by the K-tuple $\{\{X_1, z_1\}, \{X_2, z_2\}, \ldots, \{X_K, z_K\}\}$, is referred to as *quantum roulette*.

If we have a system represented by the state $\rho \in L(H_A)$ and we send it to the detector, the probability distribution of the outcomes is given by

$$p_x = \sum_k z_k p_x^{(k)} = \sum_k z_k \text{Tr}\left[\rho P_x^{(k)}\right]$$

$$= \text{Tr}\left[\rho \sum_k z_k P_x^{(k)}\right] = \text{Tr}\left[\rho \Pi_x\right], \qquad (4)$$

where $p_x^{(k)}$ is the probability distribution of the outcome x for the observable X_k and $P_x^{(k)} = |x\rangle^{(k)(k)}\langle x|$ is the one-dimensional projector on the eigenspace of the eigenvalue x for the observable X_k. In the last equality of Eq. (4) we have introduced the POVM of the roulette, whose elements are given by

$$\Pi_x = \sum_k z_k P_x^{(k)}. \qquad (5)$$

Π_x are positive operators; indeed, given any $|\varphi\rangle \in H_A$, we have $\langle\varphi|\Pi_x|\varphi\rangle = \sum_k z_k |\langle\varphi|x\rangle^{(k)}|^2 \geqslant 0$; and they represent a decomposition of identity, since

$$\sum_x \Pi_x = \sum_x \sum_k z_k P_x^{(k)} = \sum_k z_k \sum_x P_x^{(k)} = \sum_k z_k \mathbb{I} = \mathbb{I}.$$

On the other hand, Π_x are not orthogonal projectors since $\Pi_x \Pi_{x'} \neq \delta_{xx'} \Pi_x$. Indeed,

$$\Pi_x \Pi_{x'} = \sum_{k,k'} z_k z_{k'} P_x^{(k)} P_{x'}^{(k')} \neq 0.$$

In fact, while $^{(k)}\langle x|x'\rangle^{(k)}$ has to be equal to $\delta_{xx'}$ for a fixed value of k, the quantity $^{(k)}\langle x|x'\rangle^{(k')}$ (with $k \neq k'$) could be different from zero also when $x \neq x'$.

A. The noncanonical Naimark extension

The quantum roulette has a Naimark extension that is not the canonical one (i.e., the indirect measurement scheme described by the Naimark theorem), that can be obtained as follows: consider an additional Hilbert space H_B, describing the ancilla, with dimension equal to the number K of observables $X_k \in L(H_A)$ and a basis $\{|\theta_k\rangle\}$ in H_B. Then we introduce projectors $Q_x = \sum_k P_x^{(k)} \otimes |\theta_k\rangle\langle\theta_k|$ in the larger Hilbert space $H_A \otimes H_B$ and we prepare the initial state of the ancilla in $|\omega_B\rangle = \sum_k \sqrt{z_k}|\theta_k\rangle$, obtaining the probability distribution

$$p_x = \text{Tr}_{AB}[\rho \otimes |\omega_B\rangle\langle\omega_B| \, Q_x],$$

that gives us the POVM's elements:

$$\Pi_x = \text{Tr}_B[\mathbb{I} \otimes |\omega_B\rangle\langle\omega_B| \, Q_x] = \sum_k z_k P_x^{(k)}.$$

Moreover, the postmeasurement states will be given by

$$\rho_x = \frac{1}{p_x}\text{Tr}_B[Q_x \, \rho \otimes |\omega_B\rangle\langle\omega_B| \, Q_x]$$

$$= \frac{1}{p_x}\sum_k z_k \, P_x^{(k)}\rho P_x^{(k)}.$$

This measurement scheme does not involve an evolution operator $U \in L(H_A \otimes H_B)$ and the projective measure is performed on both system and ancilla, unlike the canonical extension that involves a projective measurement on the sole ancilla.

IV. THE CANONICAL NAIMARK EXTENSION OF THE PAULI QUANTUM ROULETTE WHEEL

We focus on quantum roulettes which work on qubit systems with Hilbert space $H_A \equiv \mathbb{C}^2$ and address quantum roulettes involving the measurement of Pauli operators. In order to obtain the canonical Naimark extension for these roulettes we have to add a probe system (the ancilla). We assume a two-dimensional ancilla and show that this is enough to realize the canonical extension using Eq. (1), where the elements of the POVM are given by Eq. (5).

Since H_B is a Hilbert space with dimension 2, we choose the following representations: the state $\rho_B = |\omega_B\rangle\langle\omega_B|$, and the projector $P_x = |x\rangle\langle x|$, where $|\omega_B\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$ and $|x\rangle = \cos\frac{\alpha}{2}|0\rangle + e^{i\beta}\sin\frac{\alpha}{2}|1\rangle$, with the parameters $\alpha,\theta \in [0;\pi]$ and $\beta,\varphi \in [0;2\pi)$. Notice that $|0\rangle,|1\rangle$ comprise a basis in H_B; we assume that $|0\rangle$ is the eigenvector of σ_3 related to the eigenvalue 1, while $|1\rangle$ is the eigenvector related to -1.

The last tool to individuate the canonical extension is the evolution operator $U \in L(H_A \otimes H_B)$ which works on the overall state of the composite system. The operator $U \in \mathrm{SU}(4)$, then it is defined by 15 parameters. Therefore, the total number of parameters that defines the Naimark extension is 19 (4 parameters from $|\omega_B\rangle$ and $|x\rangle$ plus 15 from U). As we will see, this number can be greatly reduced by employing the Cartan decomposition.

A. Application of the Cartan decomposition to the Naimark extension

The Naimark theorem provides a practical connection between the generalized measurement given by the quantum roulette and the indirect measurement described by the extension. Indeed, both the probability distribution of the outcomes p_x and the postmeasurement states ρ_{Ax} have to be equal for these two schemes. That is,

$$p_x = \mathrm{Tr}_{AB}[U\rho_A \otimes \rho_B U^\dagger \mathbb{I} \otimes P_x] = \mathrm{Tr}_A[\rho_A \Pi_x], \quad (6)$$

$$\rho_{Ax} = \frac{1}{p_x}\mathrm{Tr}_B[U\rho_A \otimes \rho_B U^\dagger \mathbb{I} \otimes P_x] = \frac{1}{p_x}M_x\rho_A M_x^\dagger, \quad (7)$$

where the distribution p_x and the state ρ_{Ax} in the first equality belong to the projective measurement, while those in the last equality belong to the generalized measurement.

We focus now on the Born rule Eq. (6) in order to evaluate the operators Π_x; after straightforward calculation, we obtain the elements of the POVM:

$$\Pi_x = S_1^\dagger\,\mathrm{Tr}_B[(\mathbb{I} \otimes S_0\rho_B S_0^\dagger)V^\dagger(\mathbb{I} \otimes R_0^\dagger P_x R_0)V]\,S_1.$$

Consider now $S_0\rho_B S_0^\dagger$ and $R_0^\dagger P_x R_0$; the operators $R_0,S_0 \in L(H_B)$ represent a rotation in the qubit Hilbert space H_B. Since both ρ_B and P_x are not yet defined and depend on some parameters, we can combine the rotation to them, and we are

left with a transformation from $L(H_B)$ to $L(H_B)$:

$$\rho_B \to \rho_B' = S_0\rho_B S_0^\dagger,$$
$$P_x \to P_x' = R_0^\dagger P_x R_0;$$

i.e., we can neglect this transformation by a suitable reparametrization of ρ_B' and P_x'. Furthermore, we assume the operator S_1 to be the identity ($S_1 = \mathbb{I}$). We make this assumption in order to simplify the research of the canonical extension. This ansatz will be justified *a posteriori*: once we find the Naimark extension, if the probability distribution p_x obtained from the extension is equal to the one obtained from the POVM, then the extension is correct and $S_1 = \mathbb{I}$.

Now we have the POVM's elements obtained by the canonical extension:

$$\Pi_x = \mathrm{Tr}_B[(\mathbb{I} \otimes \rho_B)V^\dagger(\mathbb{I} \otimes P_x)V], \quad (8)$$

and these elements have to be equal to those evaluated for the quantum roulette in exam. Using the Cartan decomposition on the canonical Naimark extension reduces the number of parameters to 7 (four from ρ_B and P_x and three from V).

B. Detection operators for the quantum roulette

The operator R_1 is not involved in the definition of the elements of the POVM, but it is necessary for the evaluation of the postmeasurement state ρ_{Ax}. Since the operators R_0 and S_0 were absorbed into, respectively, P_x and ρ_B and $S_1 = \mathbb{I}$, then the decomposition of U is $U = (R_1 \otimes \mathbb{I})V$ and the left part of Eq. (7) becomes

$$\rho_{Ax} = \frac{1}{p_x}\mathrm{Tr}_B[(R_1 \otimes \mathbb{I})V\rho_A \otimes \rho_B V^\dagger(R_1^\dagger \otimes P_x)]$$
$$= \frac{1}{p_x}R_1\mathrm{Tr}_B[V\rho_A \otimes \rho_B V^\dagger(\mathbb{I} \otimes P_x)]R_1^\dagger.$$

Therefore, the operator R_1 describes a residual degree of freedom in the design of possible postmeasurement states. This freedom was expected; since when we define a POVM Π_x, the postmeasurement states can be evaluated using the detection operators M_x. These operators are defined as $M_x = U_x\sqrt{\Pi_x}$, where U_x is a unitary operator. The operator U_x provides the same freedom given by R_1 to the postmeasurement states.

C. The general solution

The problem of finding the canonical Naimark extension for a given quantum roulette is now basically reduced to the solution of four equations dependent on seven parameters. Indeed, the considered roulettes are always in qubit spaces; hence, the elements of the POVM Π_x are self-adjoint 2×2 operators on the field $\mathbb{C}$ and the relation given by Eq. (8) provides four equations: one from the element Π_{x11} that is real, two from the element Π_{x12} (the real part and the imaginary part), and one from the element Π_{x22}.

D. Exchange of the parameters

One may wonder if it is possible to look for the canonical extension when the parameters α_i get values from all $\mathbb{R}$. The Cartan decomposition does not impose restriction on the range of the components of the class vectors (that is, the parameters

α_i), but we know that each operator in SU(4) is related [via the equivalence relation Eq. (2)] to a canonical class vector, whose components lie on the bounded region $K \in \mathbb{R}^3$. On the other hand, if we find an extension with α_i outside of K, it is possible to use the three class-preserving operations (shift, reverse, and swap) to bring back the parameters to K.

Can the parameters be brought back to K after we have found the canonical extension? This is not possible, unless we also modify the other objects of the extension. Indeed, if we have found the extension, then we have defined both α_i and $\theta,\varphi,\alpha,\beta$. But if the α_i are modified by one of the three class-preserving operations, then also the other parameters are modified and the state $|\omega_B\rangle$ and the operator P_x change. In fact, the operations are class preserving, so they transform the operator V into

$$V \rightarrow (R_1 \otimes R_0)V'(S_1 \otimes S_0),$$

and the operators $R_0, S_0 \in$ SU(2) modify both the initial state of H_B and the orthogonal projector:

$$\rho_B(\theta',\varphi') = S_0\,\rho_B(\theta,\varphi)\,S_0^\dagger,$$
$$P_x(\alpha',\beta') = R_0^\dagger\,P_x(\alpha,\beta)\,R_0.$$

Hence, to transform the α_i and keep the correct extension is necessary to modify ρ_B and P_x.

E. The canonical extension

We now focus on quantum roulettes given by Pauli operators σ_i and on their canonical Naimark extension. The most general quantum roulette of this kind is $\{\sigma_i,z_i\}_{i=1,2,3}$, and its canonical extension depends on two undefined parameters (e.g., z_1 and z_2). Finding the extension for the general roulette is analytically challenging, and thus we focus on roulettes involving two Pauli operators.

Let us consider the roulette $\{\{\sigma_1,z\},\{\sigma_3,1-z\}\}$, where z gets values from the interval $(0;1)$. The POVM's elements are given by

$$\Pi_1 = \frac{1}{2}\begin{pmatrix} 2-z & z \\ z & z \end{pmatrix}, \quad \Pi_{-1} = \frac{1}{2}\begin{pmatrix} z & -z \\ -z & 2-z \end{pmatrix}, \quad (9)$$

and the detection operators are given by $M_x = U_x\sqrt{\Pi_x}$, $x = \pm 1$. Upon expanding them on the Pauli basis, i.e., $M_x = a_0\,\mathbb{I} + a_1\,\sigma_1 + a_2\,\sigma_2 + a_3\,\sigma_3$, the coefficients a_i are evaluated using the inner product $\langle X,Y\rangle = \text{Tr}[XY^\dagger]$. For the roulette $\{\{\sigma_1,z\},\{\sigma_3,1-z\}\}$, we obtain $a_i = a_i(z)$ (for $i = 0,1,3$) and $a_2 = 0$. In other words, the detection operators of a roulette involving the Pauli operators σ_1 and σ_3 have no component on the missing one, i.e., σ_2. This result also holds for the other roulettes depending on two σs; e.g., for $\{\{\sigma_2,z\},\{\sigma_3,1-z\}\}$, the detection operators M_x have no component by σ_1.

The solution for the canonical extension corresponds to the parameters

$$\alpha_1 = -\pi, \quad \alpha_2 = 0, \quad \alpha_3 = \arcsin\left(-\sqrt{\frac{1}{1-\frac{z}{2}}-1}\right),$$

$$\alpha = \arccos(z-1), \quad \beta = \pi, \quad \theta = \frac{\pi}{2}, \quad \forall\varphi,$$

where $\alpha_1,\alpha_2,\theta,\varphi$, and β are in the correct range and we are left to check whether also α_3 and α lie in the correct range. First,

$\cos\alpha = z - 1$, i.e., $\cos\alpha \in (-1;0)$; then, $\alpha \in (\frac{\pi}{2};\pi)$. Finally, $\sin\alpha_3 = -\sqrt{\frac{1}{1-\frac{z}{2}}-1}$, that is, $\sin\alpha_3 \in (-1;0)$; then $\alpha_3 \in (-\frac{\pi}{2};0)$. The parameter α_3 has to be in $[\frac{\alpha_1+\alpha_2}{2};0]$, and, since $\alpha_1 = -\pi$ and $\alpha_2 = 0$, its greatest range is $[-\frac{\pi}{2};0]$.

The ingredients of the canonical extension are thus the state $|\omega_B\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\varphi}}{\sqrt{2}}|1\rangle$, the projectors

$$P_1 = \frac{1}{2}\begin{pmatrix} 2-z & \sqrt{z(2-z)} \\ \sqrt{z(2-z)} & z \end{pmatrix},$$
$$P_{-1} = \mathbb{I} - P_1, \quad (10)$$

and the unitary

$$V = \begin{pmatrix} f(z) & 0 & 0 & 0 \\ 0 & 0 & i\,f^*(z) & 0 \\ 0 & i\,f^*(z) & 0 & 0 \\ 0 & 0 & 0 & f(z) \end{pmatrix}, \quad (11)$$

with $f(z) = \sqrt{\sqrt{\frac{2-2z}{2-z}} + \frac{i}{\sqrt{\frac{2}{z}-1}}}$.

In order to obtain the canonical Naimark extension for the roulettes $\{\{\sigma_1,z\},\{\sigma_2,1-z\}\}$ and $\{\{\sigma_2,z\},\{\sigma_3,1-z\}\}$, we have to remove our previous assumption $S_1 = \mathbb{I}$. In fact, to rotate a Pauli operator σ_i by an angle θ we have to use a rotation operator $W = e^{-i(\mathbf{n}\cdot\sigma)\theta}$, where $\mathbf{n}$ is the versor of the direction around which the rotation is made. Then, to move from a two-Pauli-operators roulette to another, we need to apply the correct rotation in order to modify the σ_i. For example, to move from $\{\{\sigma_1,z\},\{\sigma_3,1-z\}\}$ to $\{\{\sigma_2,z\},\{\sigma_3,1-z\}\}$ we have to apply the operator $W = e^{-i\frac{\pi}{4}\sigma_3}$, which changes σ_1 into σ_2 and leaves σ_3 unchanged.

Therefore, the extensions for the other roulettes depending on two Pauli operators are defined by the same parameters of the extension for $\{\{\sigma_1,z\},\{\sigma_3,1-z\}\}$, but the elements Π_x are rotated by the operator W, that is,

$$\Pi_x \rightarrow W\Pi_x W^\dagger.$$

This means that, while for the first found extension the operator S_1 can be assumed equal to $\mathbb{I}$, for the extensions of the other roulettes the operator S_1 has to be equal to the conjugate transpose of the rotation operator W. We find that, for the roulette given by σ_2 and σ_3, $S_1 = e^{i\frac{\pi}{4}\sigma_3}$, while for the one given by σ_1 and σ_2, $S_1 = e^{-i\frac{\pi}{4}\sigma_1}$.

V. THE STERN-GERLACH APPARATUS AS A QUANTUM ROULETTE

The so called Stern-Gerlach apparatus allows one to measure a component (e.g., the component along the z-axis S_z) of the quantum observable *spin*, i.e., the intrinsic angular momentum of a particle. The measurement is usually performed on a collimated beam of particles (e.g., neutral atoms) sent with thermal speed into a region of inhomogeneous magnetic field. Here the particles are deflected by the field in some beams which, after propagating into the vacuum, are collected by a screen. The magnetic field is usually assumed to be of the form $\mathbf{B} = (B - bz)\,\mathbf{e}_3$, where z is the coordinate along the z axis, B is the field in the origin, and b is a constant. Actually, this is an artificial model, since a field like this one does not respect the Maxwell equations, as $\nabla \cdot \mathbf{B} = -b \neq 0$. On the other hand,

we may assume that $b \ll B$ so that we can neglect the other components of $\mathbf{B}$. The interaction Hamiltonian is given by $H = (B - bz)\sigma_3$ (neglecting the vacuum permittivity), and the corresponding evolution operator is given by $U = e^{-i\tau(B-bz)\sigma_3}$, where τ is an effective interaction time. Starting from an initial state which is factorized into a spin and a spatial part, i.e., $|\Psi\rangle\!\rangle = (c_0|0\rangle + c_1|1\rangle) \otimes |\psi(\mathbf{q})\rangle$, the evolved state is given by

$$U|\Psi\rangle\!\rangle = c_0|0\rangle \otimes |\psi_-(\mathbf{q})\rangle + c_1|1\rangle \otimes |\psi_+(\mathbf{q})\rangle,$$

where $|\psi_\pm(\mathbf{q})\rangle = e^{\pm i\tau(B-bz)}|\psi(\mathbf{q})\rangle$. The evolution is thus coupling the spin and the spatial part. Moving to the momentum representation

$$\begin{aligned} |\widetilde{\psi}_\pm(\mathbf{p})\rangle &= \int d^3\mathbf{q}\, e^{-i\mathbf{q}\cdot\mathbf{p}} |\psi_\pm(\mathbf{q})\rangle \\ &= |\widetilde{\psi}(\mathbf{p} \pm \tau b\mathbf{e}_3)\rangle, \end{aligned}$$

and tracing out the spin part after the interaction, we have that the motional degree of freedom after the interaction is described by the density operator:

$$\varrho_{\mathbf{p}} = |c_1|^2 |\widetilde{\psi}_+(\mathbf{p})\rangle\langle\widetilde{\psi}_+(\mathbf{p})| + |c_0|^2 |\widetilde{\psi}_-(\mathbf{p})\rangle\langle\widetilde{\psi}_-(\mathbf{p})|.$$

As a consequence the beam is divided in two parts, and it is possible to perform measurements on a screen placed at a given distance from the magnetic field, where we can see the beams as two different spots.

If for some reason the direction of the magnetic field is tilted we have $\mathbf{B} = (B - bt)\mathbf{e}_\alpha$, where t is a coordinate along the new direction and $\mathbf{e}_\alpha = \cos\alpha\,\mathbf{e}_3 + \sin\alpha\,\mathbf{e}_\perp$, $\mathbf{e}_\perp$ denoting any direction perpendicular to the z axis, say $\mathbf{e}_1$. The above analysis is still valid if we perform the substitution $H \to (B - bt)\sigma_\theta$, where σ_θ is a Pauli matrix describing a spin component along a tilted axis. Assuming a rotation along the x axis we have that σ_θ corresponds to the rotated operator:

$$\sigma_\theta = U_\theta \sigma_3 U_\theta^\dagger,$$

where $U_\theta = e^{-i\sigma_1\theta}$ and $\theta = \alpha/2$.

A. The continuous quantum roulette

Usually, the magnetic field of the apparatus is assumed to be a stable classical quantity. However, in any practical situation the magnetic field unavoidably fluctuates. In particular, we focus on Stern-Gerlach apparatuses in which the magnetic field fluctuates in one dimension around a pre-established direction and provide a more detailed analysis of nonideal setups [23,24]. As mentioned above, a measurement of spin made with a tilted magnetic field corresponds to measure the operator σ_θ. If the magnetic field is fluctuating, then we may describe this situation using a continuous quantum roulette where θ is randomly fluctuating around the z axis according to a given probability distribution.

In principle, the magnetic field may fluctuate in any direction on the z-y plane; i.e., the angle θ takes values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. On the other hand, in a realistic situation, the magnetic field moves away from the pre-established direction (the z axis, in this case) by a small angle. We thus introduce a Gaussian probability distribution $z(\theta)$ for the fluctuating values of θ:

$$z(\theta) = \frac{1}{A}\exp\left\{-\frac{\theta^2}{2\Delta^2}\right\},$$

where the normalization A is

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp\left\{-\frac{\theta^2}{2\Delta^2}\right\} d\theta = \sqrt{2\pi}\Delta\,\mathrm{erf}\left(\frac{\pi}{2\sqrt{2}\Delta}\right).$$

In order to evaluate the elements of the POVM, which describes this continuous quantum roulette, we need the projectors on the eigenspaces of σ_θ, i.e.,

$$\begin{aligned} P_1(\theta) &= \begin{pmatrix} \cos^2\theta & i\cos\theta\sin\theta \\ -i\cos\theta\sin\theta & \sin^2\theta \end{pmatrix}, \\ P_{-1}(\theta) &= \mathbb{I} - P_1(\theta). \end{aligned} \tag{12}$$

Therefore, the elements of the POVM are given by

$$\Pi_x = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} z(\theta) P_x(\theta)\, d\theta, \tag{13}$$

that is the equation equivalent to Eq. (5) in the continuous case. We can evaluate the elements of the POVM using the distribution $z(\theta)$ and the projectors $P_x(\theta)$ given above, and we obtain

$$\begin{aligned} \Pi_1 &= \begin{pmatrix} \frac{1}{2} + f(\Delta) & 0 \\ 0 & \frac{1}{2} - f(\Delta) \end{pmatrix}, \\ \Pi_{-1} &= \mathbb{I} - \Pi_1, \end{aligned} \tag{14}$$

where the function $f(\Delta) : [0; +\infty) \to [\frac{1}{2}; 0)$ is given by

$$f(\Delta) = \frac{\mathrm{erf}\left(\frac{\pi - i4\Delta^2}{2\sqrt{2}\Delta}\right) + \mathrm{erf}\left(\frac{\pi + i4\Delta^2}{2\sqrt{2}\Delta}\right)}{4e^{2\Delta^2}\mathrm{erf}\left(\frac{\pi}{2\sqrt{2}\Delta}\right)}.$$

We have $f(\Delta) \simeq 1/2 - \Delta^2$ for vanishing Δ and $f(\Delta) \simeq 1/8\Delta^2$ for $\Delta \to \infty$.

B. The canonical extension for the continuous roulette

We look for the canonical extension for this roulette in order to obtain a practicable measurement scheme with the same behavior (same probability distribution and postmeasurement states) as the Stern-Gerlach experiment with fluctuating magnetic field. A canonical extension may be found, corresponding to the parameters

$$\begin{aligned} \alpha_1 &= \arccos[-2f(\Delta)], \quad \alpha_2 = \arccos[2f(\Delta)], \quad \alpha_3 = 0, \\ \alpha &= \pi, \quad \beta = 0, \quad \theta = 0, \quad \varphi = 0. \end{aligned}$$

Let consider the parameters α_1 and α_2; the codomain of the function $f(\Delta)$ is $(0; \frac{1}{2}]$. Therefore, if $\cos\alpha_1 = -2f(\Delta)$, then $\cos\alpha_1 \in (0; -1]$ and $\alpha_1 \in (-\frac{\pi}{2}; -\pi]$. Instead, $\cos\alpha_2 = 2f(\Delta)$, i.e., $\cos\alpha_2 \in (0; 1]$ and $\alpha_2 \in (\frac{\pi}{2}; 0]$. Both α_1 and α_2 depend on the function f, so when we choose a value for f the two parameters are fixed. For example, when $f(\Delta) \to 0$, then $\alpha_1 \to -\frac{\pi}{2}$ and $\alpha_2 \to \frac{\pi}{2}$; on the other hand, if $f(\Delta) = \frac{1}{2}$ then $\alpha_1 = -\pi$ and $\alpha_2 = 0$. The ranges of these two parameters

are correct, and we have $\alpha_2 \leqslant -\alpha_1 \ \forall \ f(\Delta)$. Finally, since we have fixed $\alpha_3 = 0$, we have to check whether $\alpha_1 - \alpha_2 \geqslant -\pi$, and this is the case: as it can be easily checked $\alpha_1 - \alpha_2 = -\pi$ for all $\Delta \in [0; +\infty)$. The canonical extension is thus given by the state $|\omega_B\rangle = |0\rangle$, the observable σ_3 (measured on the ancilla), and the unitary $V \in L(H_A \otimes H_B)$:

$$V = \frac{1}{4} \sum_{k=0}^{3} v_k \, \sigma_k \otimes \sigma_k, \tag{15}$$

$$v_{0/3} = \sqrt{\frac{1}{2} + f(\Delta)} \pm \sqrt{\frac{1}{2} - f(\Delta)}, \tag{16}$$

$$\stackrel{\Delta \to 1}{\simeq} 1 \pm \Delta,$$

$$v_{1/2} = i \, v_{0/3}. \tag{17}$$

For a particle with spin up, represented by the pure state $|0\rangle$, the probability distribution of the outcomes is given by $p_1 = \frac{1}{2} + f(\Delta)$, $p_{-1} = 1 - p_1$ and, thus, when such a particle is measured, there is always a probability that the apparatus measures the spin down $|1\rangle$.

VI. CONCLUSIONS

We have addressed Pauli quantum roulettes and found their canonical Naimark extensions. The extensions are minimal, i.e., they involve a single ancilla qubit, and they provide a concrete model to realize the roulettes without destroying the signal state, which can be measured again after the measurement or can be transmitted. Our results provide a natural framework to describe a measurement scheme in which the measured observable depends on an external parameter, which cannot be fully controlled and may fluctuate according to a given probability distribution. As an illustrative example, we have applied our results to the description of Stern-Gerlach-like experiments on a two-level system, taking into account possible uncertainties in the splitting force.

[1] C. W. Helstrom, Int. J. Theor. Phys. **8**, 361 (1973).

[2] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).

[3] A. S. Holevo, in *Statistical Structure of Quantum Theory*, Lecture Notes in Physics, Vol. 61 (Springer, Berlin, 2001).

[4] B. He, J. A. Bergou, and Z. Wang, Phys. Rev. A **76**, 042326 (2007).

[5] M. A. Naimark, Iza. Akad. Nauk USSR Ser. Mat. **4**, 277 (1940); C. R. Acad. Sci. USSR **41**, 359 (1943).

[6] A. Peres, Found. Phys. **20**, 1441 (1990).

[7] J. Bergou, J. Mod. Opt. **57**, 160 (2010).

[8] M. G. A. Paris, Eur. Phys. J. S.T. **203**, 61 (2012).

[9] K. Banaszek, Phys. Rev. Lett. **86**, 1366 (2001).

[10] M. G. Genoni and M. G. A. Paris, Phys. Rev. A **71**, 052307 (2005).

[11] L. Mišta and R. Filip, Phys. Rev. A **72**, 034307 (2005).

[12] M. G. Genoni and M. G. A. Paris, Phys. Rev. A **74**, 012301 (2006).

[13] A. Mandilara and J. W. Clark, Phys. Rev. A **71**, 013406 (2005).

[14] M. Hannout, S. Hoyt, A. Kryowonos, and A. Widom, Am. J. Phys. **66**, 377 (1998).

[15] G. Potel, F. Barranco, S. Cruz-Barrios, and J. Gomez-Camacho, Phys. Rev. A **71**, 052106 (2005).

[16] B. M. Garraway and S. Stenholm, Phys. Rev. A **60**, 63 (1999).

[17] T. Qureshi and Z. Rahman, Prog. Theor. Phys. **127**, 71 (2012).

[18] G. M. D'Ariano and M. G. A. Paris, Acta Phys. Slov. **47**, 281 (1997).

[19] M. Munroe, D. Boggavarapu, M. E. Anderson, and M. G. Raymer, Phys. Rev. A **52**, R924 (1995).

[20] B. Kraus and J. I. Cirac, Phys. Rev. A **63**, 062309 (2001).

[21] J. Zhang, J. Vala, S. Sastry, and K. B. Whaley, Phys. Rev. A **67**, 042313 (2003).

[22] R. R. Tucci, arXiv:quant-ph/0507171.

[23] D. Home, A. K. Pan, M. M. Ali, and A. S. Majumdar, J. Phys. A **40**, 13975 (2007).

[24] X. Xu and Z. Xiao-Ji, Chin. Phys. Lett. **27**, 010309 (2010).