Detecting mixedness of qutrit systems using the uncertainty relation

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(Received 29 May 2012; revised manuscript received 14 November 2012; published 7 January 2013)

We show that the uncertainty relation as expressed in the Robertson-Schrodinger generalized form can be used to detect the mixedness of three-level quantum systems in terms of measurable expectation values of suitably chosen observables when prior knowledge about the basis of the given state is known. In particular, we demonstrate the existence of observables for which the generalized uncertainty relation is satisfied as an equality for pure states and a strict inequality for mixed states corresponding to single as well as bipartite systems of qutrits. Examples of such observables are found for which the magnitude of uncertainty is proportional to the linear entropy of the system, thereby providing a method for measuring mixedness.

DOI: 10.1103/PhysRevA.87.012105 PACS number(s): 03.67.Mn, 03.65.Ta

Introduction. The uncertainty relation lies at the heart of quantum mechanics, providing one of the first and foremost points of departure from classical concepts. As originally formulated by Heisenberg [1], it prohibits certain properties of quantum systems from being simultaneously well defined. A generalized form of the uncertainty relation was proposed by Robertson [2] and Schrodinger [3], and since then, several other versions of the uncertainty principle have been suggested. A reformulation takes into account the inevitable noise and disturbance associated with measurements [4]. The consideration of state independence has led to the formulation of entropic versions of the uncertainty principle [5]. A modification of the entropic uncertainty relation occurs in the presence of quantum memory associated with quantum correlations [6]. Another version provides a fine-grained distinction between the uncertainties inherent in obtaining possible different outcomes of measurements [7].

In recent years certain important applications of uncertainty relations have been discovered in the realm of quantum information processing. The security of quantum key distribution protocols is based fundamentally on quantum uncertainty [8], and the amount of key extractable per state can be linked to the lower limit of entropic uncertainty [6,9]. The fine-grained uncertainty relation can be used to determine the nonlocality of the underlying physical system [7,10]. The uncertainty principle has been used for discrimination between separable and entangled quantum states [11], and the Robertson-Schrodinger generalized uncertainty relation (GUR) has also been applied in this context [12]. In the present work our motivation is to investigate the role of GUR in the context of another important property, viz., the purity of quantum systems.

At the practical level the ubiquitous interaction with the environment inevitably affects the purity of a quantum system. A relevant issue for an experimenter is to ascertain whether a prepared pure state has remained isolated from environmental interaction. It becomes important to test whether a given quantum state is pure, in order to use it effectively as a resource for quantum information processing [13,14]. The purity of a given state is also related to the entanglement of a larger multipartite system of which it may be a part [15]. The mixedness of states can be characterized by the property of linear entropy, which is a nonlinear functional of the quantum state. The linear entropy can be extracted from the given state by tomography which usually is expensive in terms of resources and measurements involved. Bypassing a classical evaluation process, estimation of purity of a system using quantum networks has been suggested [16]. Discrimination between pure and mixed states by positive operator valued measurements that amounts to a maximum confidence discrimination, has also been proposed [17].

In this work we connect the Robertson-Schrodinger GUR to the property of mixedness of quantum states of discrete variables. For the case of continuous variable systems there exist certain pure states for which the uncertainty as quantified by the GUR is minimized [18], and the connection of purity with observable quantities of the relevant states have been found [13]. Here we show that GUR can be used to distinguish between pure and mixed states of finite dimensional systems. To set the background we first briefly mention the essential results for two-level systems. Our focus here is on three-level systems which are not only of fundamental relevance in laser physics, but also the properties of which have generated much recent interest from the perspective of information processing [19–24]. We show using examples of single and bipartite class of qutrit states that the GUR can be satisfied as an equality for pure states while it remains an inequality for mixed states by the choice of suitable observables. We prescribe an observational scheme using GUR which can detect mixedness of qutrit systems unambiguously, requiring less resources compared to tomography, and is implementable through the measurement of Hermitian witnesslike operators.

GUR as a witness of mixedness. GUR for any pair of observables A,B and for any quantum state represented by the density operator ρ can be written as [2,3]

$$Q(A,B,\rho) \geqslant 0, \tag{1}$$

where

$$Q(A,B,\rho) = (\Delta A)^2(\Delta B)^2 - \left|\frac{\langle [A,B]\rangle}{2}\right|^2 - \left|\left(\frac{\langle\{A,B\}\rangle}{2} - \langle A\rangle\langle B\rangle\right)\right|^2, \tag{2}$$

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with $(\Delta A)^2$ and $(\Delta B)^2$ representing the variances of the observables, A and B, respectively, given by $(\Delta A)^2 = (\langle A^2 \rangle) - (\langle A \rangle)^2$, $(\Delta B)^2 = (\langle B^2 \rangle) - (\langle B \rangle)^2$, and the square (curly) brackets representing the standard commutators (anticommutators) of the corresponding operators. The quantity $Q(A,B,\rho)$ involves the measurable quantities, that is, the expectation values and variances of the relevant observables in the state ρ. States of a d-level quantum system are in one-to-one correspondence with Hermitian, positive semidefinite, unit trace operators acting on a d-dimensional Hilbert space. The defining properties of these density operators ρ are (i) $\rho \dagger = \rho$, (ii) $\rho \geqslant 0$, and (iii) $\mathrm{tr}[\rho] = 1$. Pure states correspond to the further condition $\rho^2 = \rho$, which is equivalent to the scalar condition $\mathrm{tr}[\rho^2] = 1$. Hence, the complement of the trace condition can be taken as a measure of mixedness given by the linear entropy defined for a d-level system as

$$S_l(\rho) = \left(\frac{d}{d-1}\right)[1 - \mathrm{tr}(\rho^2)]. \tag{3}$$

We now investigate how the quantity $Q(A,B,\rho)$ can act as an experimentally realizable measure of mixedness of a system.

We first briefly describe the status of GUR with regard to the purity of qubit states. The density operator for two-level systems can be expressed in terms of the Pauli matrices. The state of a single qubit can be written as

$$\rho(\vec{n}) = \frac{(I + \vec{n} \cdot \vec{\sigma})}{2}, \quad \vec{n} \in \mathbb{R}^3. \tag{4}$$

The positivity of this Hermitian unit trace matrix demands $|\vec{n}|^2 \leqslant 1$. It follows that single qubit states are in one-to-one correspondence with the points on or inside the closed unit ball centered at the origin of $\mathbb{R}^3$. Points on the boundary correspond to pure states. We show that for a pair of suitably chosen spin observables, GUR is satisfied as an equality for the extremal states, that is, the pure states, and as an inequality for points other than extremals, that is, for the mixed states. The linear entropy of the state ρ can be written as $S_l(\rho) = (1 - \vec{n}^2)$. If we choose spin observables along two different directions, that is, $A = \hat{r} \cdot \vec{\sigma}$ and $B = \hat{t} \cdot \vec{\sigma}$, then Q becomes

$$Q(A,B,\rho) = [1 - (\Sigma r_i t_i)^2] S_l(\rho). \tag{5}$$

It thus follows that for $\hat{r} \cdot \hat{t} = 0$, Q coincides with the linear entropy. For orthogonal spin measurements, the uncertainty quantified by GUR, Q, and the linear entropy S_l are exactly same for single-qubit systems. Thus, it turns out that $Q = 0$ is both a necessary and sufficient condition for any single-qubit system to be pure when the pair of observables are qubit spins along two different directions.

For the treatment of composite systems the states considered are taken to be polarized along a specific known direction, say, the z axis forming the Schmidt decomposition basis. In order to enable $Q(A,B,\rho)$ to be a mixedness measure, A and B are chosen for the two-qubit case to be of the form

$$\begin{aligned} A &= (\hat{m} \cdot \vec{\sigma}^1) \otimes (\hat{n} \cdot \vec{\sigma}^2), \\ B &= (\hat{p} \cdot \vec{\sigma}^1) \otimes (\hat{q} \cdot \vec{\sigma}^2), \end{aligned} \tag{6}$$

where $\hat{m},\hat{n},\hat{p},\hat{q}$ are unit vectors. For enabling $Z(A,B,\rho)$ to be used for discerning the purity or mixedness of a given two-qubit state specified, say, the z axis, the appropriate choice of observables A and B is found to be that of lying on the two-dimensional x-y plane (i.e., $\hat{m},\hat{n},\hat{p},\hat{q}$ are all taken to be on the x-y plane), normal to the z axis pertaining to the relevant Schmidt decomposition basis. Then, $Q(A,B,\rho) = 0$ (i.e., GUR is satisfied as an equality) necessarily holds good for pure two-qubit states whose individual spin orientations are all along a given direction (say, the z axis) normal to which lies the plane on which the observables A and B are defined. On the other hand, $Q(A,B,\rho) > 0$ holds good for most settings of A and B for two-qubit isotropic states, for the Werner class of states given by $\rho_w = [(1-p)/4]I + p\rho_s$ (ρ_s is the two-qubit singlet state), as well for other types of one-parameter two-qubit states which comprise of pure states whose individual spin orientations are all along the same given direction normal to the plane on which the observables A and B are defined. For the case of multipartite systems, in our purpose the general form of n-qubit observables is given by

$$\begin{aligned} A &= \hat{r}_1 \cdot \vec{\sigma} \otimes \hat{r}_2 \cdot \vec{\sigma} \otimes \cdots \otimes \hat{r}_n \cdot \vec{\sigma}, \\ B &= \hat{t}_1 \cdot \vec{\sigma} \otimes \hat{t}_2 \cdot \vec{\sigma} \otimes \cdots \otimes \hat{t}_n \cdot \vec{\sigma}, \end{aligned} \tag{7}$$

where, $\hat{r}_i,\hat{t}_i$ are unit vectors in $\mathbb{R}^3$. GUR may be used to distinguish pure states from mixed ones with the choice of suitable observables for composite qubit systems, whose detailed implications will be presented in a separate work.

Three-level systems. The structure of the state space of the generalized Bloch sphere (Ω_d) is much richer for $d \geqslant 3$ [25,26]. Qutrit states can be expressed in terms of Gellmann matrices that are familiar generators of the unimodular unitary group SU(3) in its defining representation with eight Hermitian, traceless, and orthogonal matrices $\lambda_j, j = 1,....,8$ satisfying $\mathrm{tr}(\lambda_k \lambda_l) = 2\delta_{kl}$, and $\lambda_j \lambda_k = (2/3)\delta_{jk} + d_{jkl}\lambda_l + i f_{jkl}\lambda_l$. The expansion coefficients f_{jkl}, the structure constants of the Lie algebra of SU(3), are totally antisymmetric, while d_{jkl} are totally symmetric. Single-qutrit states can be expressed as

$$\rho(\vec{n}) = \frac{I + \sqrt{3}\vec{n} \cdot \vec{\lambda}}{3}, \quad \vec{n} \in \mathbb{R}^{8.} \tag{8}$$

The set of all extremals (pure states) of Ω_3 constitute also CP^2, and can be written as $\Omega_3^{\mathrm{ext}} = CP^2 = \{\vec{n} \in \mathbb{R}^8 | \vec{n} \cdot \vec{n} = 1, \vec{n} * \vec{n} = \vec{n}\}$, with $\vec{n} * \vec{n} = \sqrt{3} d_{jkl} n_k n_l \hat{e}_j$. Here $\hat{e}_j$ is the unit vector belongs to $\mathbb{R}^8$. Non-negativity of ρ demands that $\vec{n}$ should satisfy the additional inequality $|\vec{n}|^2 \leqslant 1$. The boundary $\partial\Omega_3$ of Ω_3 is characterized by $\partial\Omega_3 = \{\vec{n} \in \mathbb{R}^8 | 3\vec{n} \cdot \vec{n} - 2\vec{n} * \vec{n} \cdot \vec{n} = 1, \vec{n} \cdot \vec{n} \leqslant 1\}$, and the state space Ω_3 is given by $\Omega_3 = \{\vec{n} \in \mathbb{R}^8 | 3\vec{n} \cdot \vec{n} - 2\vec{n} * \vec{n} \cdot \vec{n} \leqslant 1, \vec{n} \cdot \vec{n} \leqslant 1\}$. For two-level systems the whole boundary of the state space represents pure states, that is, $\Omega_2^{\mathrm{ext}} = \partial\Omega_2$, while for three-level systems $\Omega_3^{\mathrm{ext}} \subset \partial\Omega_3$. The four-parameter family Ω_3^{ext} is sprinkled over the seven-parameter surface $\partial\Omega_3$ of Ω_3.

The most general type of observables can be written as $A = \hat{a} \cdot \vec{\lambda} = a_i \lambda_i$, $B = \hat{b} \cdot \vec{\lambda} = b_i \lambda_i$, where, $\Sigma a_i^2 = 1$ and $\Sigma b_i^2 = 1$. The measurement of qutrit observables composed of the various λ_i's, can be recast in terms of qutrit spin observables, given by [24], for example, $\lambda_1 = (1/\sqrt{2})(S_x + 2\{S_z, S_x\})$, and similarly for the other λ_i's. The qutrit spins are given by

$$\sqrt{2}S_x = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \sqrt{2}S_y = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$
$$S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (9)$$

Note that with the choice of $A = \hat{A} \cdot \lambda$ and $B = \hat{B} \cdot \lambda$, Q becomes

$$\begin{aligned} Q = {} & (4/9)[1 - (\hat{A} \cdot \hat{B})^2] + (4/9)[\hat{A} \cdot \vec{n} + \hat{B} \cdot \vec{n} \\ & - 2(\hat{A} \cdot \hat{B})(\hat{A} * \hat{B}) \cdot \vec{n}] + (4/9)\{(\hat{A} \cdot \vec{n})(\hat{B} \cdot \vec{n}) \\ & - [(\hat{A} * \hat{B}) \cdot \vec{n}]^2 + 4(\hat{A} \cdot \hat{B})(\hat{A} \cdot \vec{n})(\hat{B} \cdot \vec{n}) - 2(\hat{A} \cdot \vec{n})^2 \\ & - 2(\hat{B} \cdot \vec{n})^2 - 3[(\hat{A} \wedge \hat{B}) \cdot \vec{n}]^2\} - (4/9)\{2(\hat{A} \cdot \vec{n})(\hat{B} \cdot \vec{n})^2 \\ & + 2(\hat{A} \cdot \vec{n})^2(\hat{B} \cdot \vec{n}) - 4[(\hat{A} * \hat{B}) \cdot \vec{n}](\hat{A} \cdot \vec{n})(\hat{B} \cdot \vec{n})\}, \quad (10) \end{aligned}$$

where $(\hat{A} * \hat{B})_k = \sqrt{3} d_{ijk} A_i B_j$ and $(\hat{A} \wedge \hat{B})_k = f_{ijk} A_i B_j$. From the expression of Q it is clear that it changes if ρ is changed by some unitary transformation. For such change of states the norm of $\vec{n}$ does not change. Purity/mixedness property of a state does not change under unitary operations on the state. Hence, it is desirable for any mixedness measure to remain invariant under unitary operation. This would be possible if Q becomes some function of only $|\vec{n}|^2$ for a suitable choice of observables. However, unlike the case of the single qubit, for the single qutrit Q becomes independent of the linear and cubic terms of $|\vec{n}|$ only for the trivial choice of observables, that is, $\hat{A} = \hat{B}$, in which case Q becomes zero, whatever be the state, pure or mixed. Here we employ suitably chosen observables and a sequence of measurements to turn Q to a detector of mixedness, that is, $Q = 0$ for pure states and $Q > 0$ for mixed states. Note further that, under a basis transformation $\lambda_i' = U\lambda_i U^\dagger$, the state becomes $\rho' = (1/3)(I + \sqrt{3}\vec{n}' \cdot \vec{\lambda}') = U(1/3)(I + \sqrt{3}\vec{n}' \cdot \vec{\lambda})U^\dagger$. Now, for any observable χ' in the prime basis, one has $\mathrm{Tr}[\chi'\rho'] = \mathrm{Tr}[\chi(1/3)(I + \sqrt{3}\vec{n}' \cdot \vec{\lambda})]$. Thus, any nonvanishing expectation value in the primed basis cannot vanish in the unprimed one, and vice versa. Hence, in order to measure in another basis one has to simply choose observables which are unitary conjugates to the observables written in terms of standard λ basis. Such observables would again yield $Q = 0$ for pure states and $Q > 0$ for mixed states in the new basis. Hence, though we have specified our scheme based on the single-qutrit state in terms of the standard λ basis [25,26], our scheme remains invariant with regard to the choice of the basis as long as the knowledge of the specific basis chosen is available to the experimenter. This means that the experiment shall involve not only the observables A and B but also a possibility for simultaneous unitary rotations of these observables.

In what follows we take up to a three-parameter family of states from Ω_3 [26] and find that there exist observable pairs which for pure states exhibit minimum uncertainty, viz., $Q = 0$. Our scheme runs as follows. Economizing on the number of measurements required, we take λ_3 as A and, sequentially, the members of any one of the pairs (λ_7,λ_6),(λ_5,λ_4),(λ_1,λ_2) as B. The significance of such pairing will be clear later. If two successive measurements taking B from any of the above pairs yield $Q = 0$, the state concerned is pure. In contrast, if

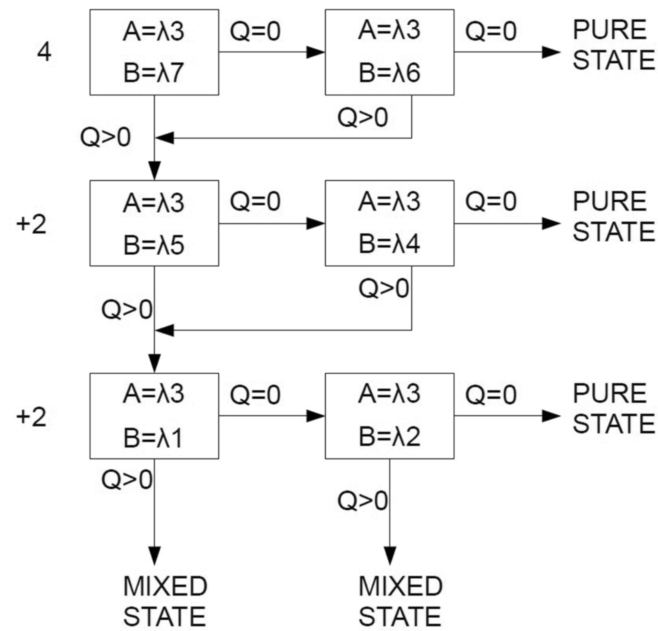

FIG. 1. Detection scheme for the purity of single-qutrit states of up to three parameters. The numbers to the left of the boxes indicate the number of measurements required corresponding to each of the horizontal levels.

B taken from all the above pairs sequentially yields $Q > 0$, the state is found to be mixed. (see, Fig. 1 for an illustration of the scheme).

Let us first consider the one-parameter family of single-qutrit states for which only one of the eight parameters $(n_i, i = 1,\ldots,8)$ is nonzero while the remaining seven vanish. The linear entropy of this class of states is given by

$$S_l(\rho) = 1 - n_i^2. \quad (11)$$

There exist many pairs of observables which can detect the mixedness of this class of states unambiguously. For example, when $i = 8$, the only pure state of this class is given by $n_8 = -1$ [26]. Here

$$Q(\lambda_3,\lambda_7) = Q(\lambda_3,\lambda_6) = (4/9)(2 - n_8)(1 + n_8). \quad (12)$$

Hence, $Q = 0$ only for $n_8 = -1$, but $Q > 0$ otherwise. Next, for example when $i = 1$, one has

$$Q(\lambda_3,\lambda_7) = Q(\lambda_3,\lambda_6) = Q(\lambda_3,\lambda_5) = Q(\lambda_3,\lambda_4) = 4/9. \quad (13)$$

It turns out that there is no choice of B from both the sequential pairs (as depicted in Fig. 1) for which $Q = 0$. Similar considerations are valid also for other single-parameter qutrit states, enabling the detection scheme as given in Fig. 1.

Moving to the two-parameter family of density matrices (two of the eight parameters $n_1,\ldots,n_8$ are nonzero, while the remaining six vanish), note that in this case there are 28 combinations of different pairs of nonzero parameters, and these classes belong to one of the four different types of unitary equivalence classes, viz., circular, parabolic, elliptical, and triangular [26]. In this case, for example, for states belonging to the parabolic class, by choosing n_3 and n_4 to be nonvanishing,

Q takes the forms

$$Q(\lambda_3,\lambda_5) = (2/9)(2+\sqrt{3}n_3)\big(1-2n_3^2\big) - n_4^2/4,$$
$$Q(\lambda_3,\lambda_4) = (1/9)\big[4-8n_3^2-4\sqrt{3}n_3^3-11n_4^2 + 2\sqrt{3}n_3\big(1+4n_4^2\big)\big]. \quad (14)$$

Here pure states occur for $(n_3,n_4) = (1/\sqrt{3}, \pm\sqrt{2/3})$, leading to $Q = 0$, while $Q > 0$ corresponding to all mixed states, as is also evident from the expression for the linear entropy given by

$$S_l(\rho) = \big(1-n_3^2-n_4^2\big). \quad (15)$$

Similar considerations apply to other single-qutrit states of the two-parameter family, enabling the detection of pure states when two successive measurements with B taken from sequential pairs (Fig. 1) lead to $Q = 0$.

Next consider the three-parameter family of qutrit states where there are seven geometrically distinct and ten unitary equivalent types of three sections out of 56 standard three sections. Considering an example of states belonging to the parabolic geometric shape, Q has the forms

$$Q(\lambda_3,\lambda_5) = (1/9)\big[4-8n_3^2-4\sqrt{3}n_3^3-3n_4^2-11n_5^2 + 2\sqrt{3}n_3\big(1+4n_5^2\big)\big],$$
$$Q(\lambda_3,\lambda_4) = (1/9)\big[4-8n_3^2-4\sqrt{3}n_3^3-3n_5^2-11n_4^2 + 2\sqrt{3}n_3\big(1+4n_4^2\big)\big]. \quad (16)$$

The linear entropy of this class of states is given by

$$S_l(\rho) = 1-n_3^2-n_4^2-n_5^2. \quad (17)$$

When B is chosen from the (λ_4,λ_5) pair as above, Q turns out to be zero for pure states given by $n_3 = 1/\sqrt{3}$ and $n_4^2+n_5^2 = 2/3$, and Q is greater than zero for all mixed states. It can be checked that the purity of all three parameter families of single-qutrit states can be determined by the scheme depicted in Fig. 1.

Let us now discuss the case of two-qutrit state discrimination. Here we assume that the states considered are taken to be polarized along a specific known direction, say, the z axis forming the Schmidt decomposition basis. A two-qutrit pure state in the Schmidt form can be written as $|\psi\rangle = k_1|11\rangle + k_2|22\rangle + k_3|33\rangle$, where, k_1,k_2,k_3 are real with $k_1^2+k_2^2+k_3^2 = 1$, and $|1\rangle$, $|2\rangle$, and $|3\rangle$ are orthonormal unit vectors in $\mathbb{C}^3$. For our purpose a general form of observables acting on the two-qutrit system is given by $A = \hat{r}_1\cdot\vec{\lambda}\otimes\hat{r}_2\cdot\vec{\lambda}$ and $B = \hat{t}_1\cdot\vec{\lambda}\otimes\hat{t}_2\cdot\vec{\lambda}$, where $\hat{r}_1,\hat{t}_1,\hat{r}_2,\hat{t}_2$ are unit vectors in $\mathbb{R}^8$. For our purpose it is sufficient to take observables of the form

$$A = \lambda_i\otimes(\cos\theta_2\lambda_i+\sin\theta_2\lambda_j),$$
$$B = (\cos\theta_3\lambda_i+\sin\theta_3\lambda_j)\otimes(\cos\theta_4\lambda_i+\sin\theta_4\lambda_j), \quad (18)$$

where (i,j) are taken from the pair (1,2),(3,8),(4,5),(6,7), and $\theta_2,\theta_3,\theta_4$ are angles between $\hat{r}_1$ and $\hat{r}_2,\hat{t}_1,\hat{t}_2$, respectively. With the choice of observables $(i=1, j=2)$, the uncertainty becomes $Q(A,B,\rho_{\rm pure}) = 4k_1^2k_2^2k_3^2\sin(\theta_2-\theta_3-\theta_4)$. Hence, choosing $\theta_2-\theta_3 = \theta_4$, we can make $Q = 0$ for every pure state.

Now consider a one-parameter class of two-qutrit mixed states expressed as

$$\rho_m = p\rho_1 + (1-p)\rho_2, \quad (19)$$

where ρ_1 and ρ_2 are arbitrary pure states parametrized as $\rho_1 = |\psi_1\rangle\langle\psi_1|$ with $|\psi_1\rangle = k_1|11\rangle + k_2|22\rangle + k_3|33\rangle$ and $\rho_2 = |\psi_2\rangle\langle\psi_2|$ with $|\psi_2\rangle = k_4|11\rangle + k_5|22\rangle + k_6|33\rangle$. For such states the linear entropy is given by

$$S_l(\rho_m) = \tfrac{3}{2}p(1-p). \quad (20)$$

The expression for Q under the condition $\theta_2-\theta_3 = \theta_4$ is given by $Q(A,B,\rho_m) = 4k_1^2p(1-p)[1-k_6^2-4k_4^2k_5^2(1-p)\cos^2(\theta_3+\theta_4)]\sin^2(\theta_3)$, which, when maximized over all observables in the selected region $(i=1, j=2)$, leads to

$$Q = 4k_1^2\big(1-k_6^2\big)p(1-p). \quad (21)$$

We observe that the expression for the uncertainty may coincide with the value of linear entropy for certain choices of the state parameters. In general, Q always vanishes for pure states, and remains positive for mixed ones, for $k_1 \neq 0$, and $k_6 \neq 1$.

As another example of two-qutrit states, we consider the popular class of isotropic states that are invariant under the action of local unitary operations of the form $U\otimes U^*$. Two-qutrit isotropic states can be written as

$$\rho = p\rho_i + \frac{1-p}{9}I\otimes I, \quad (22)$$

where $0 \leqslant p \leqslant 1$ and $\rho_i = |\phi\rangle\langle\phi|$, with $|\phi_i\rangle = (1/\sqrt{3})(|11\rangle + |22\rangle + |33\rangle)$. The linear entropy of this state is given by

$$S_l(\rho) = \tfrac{2}{3}(1-p^2) \quad (23)$$

and our choice of observables leads to $Q = (8/81)(-1+p)\{-3-3p+2p^2+(-1+p)\cos(2\theta_3)+2p^2\cos[2(\theta_3+\theta_4)]\}^2\sin\theta_3$. Maximizing over all observables in the selected region we get

$$Q = \tfrac{16}{81}(1-p)(1+2p), \quad (24)$$

which is quadratic in the parameter p similar to the linear entropy and is able to distinguish mixed states from the pure state ($p = 1$). It may be noted that for the Werner class of states that are invariant under the local unitary operations of the form $U\otimes U$, and which differ from the Isotropic class for qutrits, there exists no pure state for qutrits, a fact that is reflected in the corresponding expression for Q that turns out to be $Q > 0$ always.

Measurement prescription. We now outline our suggested scheme for using the uncertainty relation to determine whether a given state is pure or mixed, provided the prior knowledge of the basis is available. The GUR through the scheme discussed here is able to distinguish between pure and mixed states for a broad category of two- and three-level systems (see Fig. 2). For single-party systems, the scheme works for all qubits and up to a three-parameter family of qutrit states for which the classification into unitary equivalence classes is available in the literature [25,26]. For bipartite systems, the scheme has been shown to work for the mixture of two arbitrary pure states, the isotropic class, and the Werner class of states as well. There may, of course, exist other classes of states within the above categories, for which we are yet to ascertain the viability of this scheme.

It may be noted here that the limitation of instrumental precision could make the observed value of Q for pure states

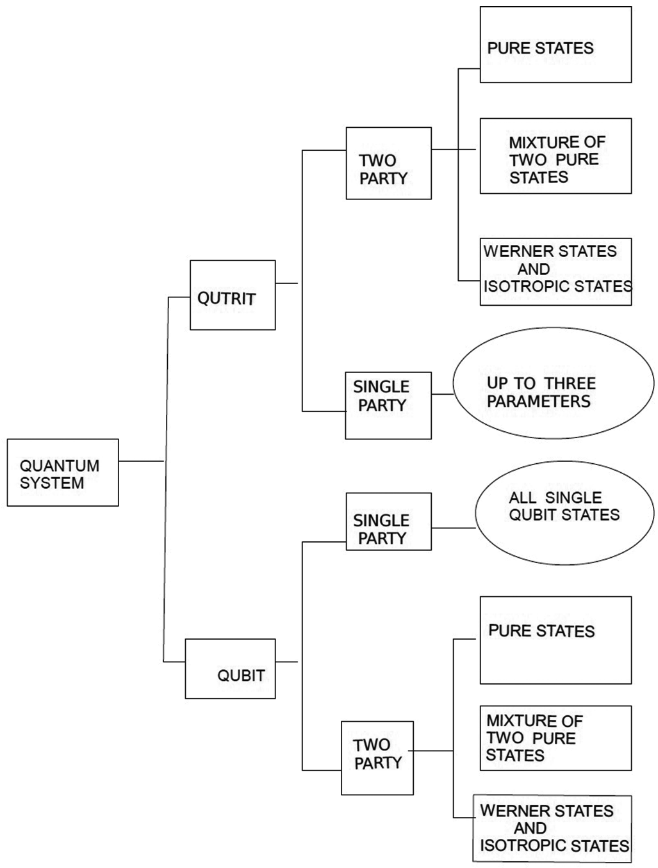

FIG. 2. Family of states that can be distinguished using the uncertainty relation.

to be a small number instead of exactly zero. In order to take into account the experimental inaccuracy, a parameter ε may be introduced in the analysis. For a single-qubit system, by choosing the measurement settings for A and B as qubit spins along the z and x directions, respectively, the measured value of the uncertainty obtained as $Q \geqslant \varepsilon$ leads to the conclusion that the given state is mixed. This prescription of determining mixedness holds for all single-qubit states $\rho(\vec{n}) = \frac{(I+\vec{n}\cdot\vec{\sigma})}{2}$, except those lying in the narrow range $1 \geqslant n \geqslant \sqrt{1-2\varepsilon/3}$, as determined by putting $Q < \varepsilon$ in Eq. (5).

A somewhat more elaborate procedure is required for qutrits, as may be expected from the richer structure of their state space. For the case of single qutrits belonging to the one-, two-, or three-parameter family of states, one has to find Q taking $A = \lambda_3$ and B from the $(\lambda_6,\lambda_7),(\lambda_4,\lambda_5),(\lambda_1,\lambda_2)$ pairs in succession, as depicted in the Fig. 1. If $Q < \varepsilon$ for the settings B corresponding to both members of a same pair measured in succession, then the state is pure within the limitations of experimental accuracy. Whenever $Q \geqslant \varepsilon$, B is chosen from the pair vertically below. If there exists no such pair for which $Q < \varepsilon$, then the state is mixed. In order to maximize the uncertainty measured by the variable Q, such that $Q \geqslant \varepsilon$ for the maximum number of mixed states, the observables need to be chosen so as to avoid $|a_i/b_i| \approx 1$. Our scheme pictorially represented in Fig. 1 is able to detect mixedness of single-qutrit states of up to three parameters.

For the case of two-qutrit states, the measurement of the observables given by Eq. (18) with the λ_i's chosen from the regions spanned by (λ_1,λ_2), together with the restriction on the angle $\theta_3 \neq \pi$, suffices to distinguish pure and mixed states. Such a procedure is able to detect all mixed states within the margin of experimental accuracy. For example, for the case of the two-qutrit isotropic states the method would fail only for states lying in the parameter range $\sqrt{1-3\varepsilon/2} < p < 1$.

The determination of mixedness using GUR may require in certain cases a considerably lesser number of measurements compared to tomography. In the case of single-qutrit states, full tomography involves the estimation of eight parameters, while in our prescription sometimes four measurements may suffice for detecting the purity of a single-qutrit state. In Fig. 1, the numbers beside the boxes indicate the numbers of measurements required to find the various expectation values including those of (anti-)commutators required to determine Q using Eq. (2). For instance, the number 4 beside the top box, means that the four measurements $(\langle\lambda_3\rangle,\langle\lambda_7\rangle,\langle\lambda_8\rangle$, and$\langle\lambda_6\rangle)$ are all that is required for the first horizontal level. This follows from the algebra

$$\begin{aligned}
&\langle\{\lambda_3,\lambda_7\}\rangle = -\langle\lambda_7\rangle/2, \quad \langle[\lambda_3,\lambda_7]\rangle = \langle\lambda_6\rangle/2,\\
&\quad\langle\lambda_3^2\rangle = \frac{2}{3}I + \frac{1}{\sqrt{3}}\langle\lambda_8\rangle, \quad \langle\lambda_7^2\rangle = \frac{2}{3}I - \frac{1}{2\sqrt{3}}\langle\lambda_8\rangle - \frac{1}{2}\langle\lambda_3\rangle,\\
&\langle[\lambda_3,\lambda_6]\rangle = -\langle\lambda_7\rangle/2, \quad \langle\{\lambda_3,\lambda_6\}\rangle = -\langle\lambda_6\rangle/2,\\
&\quad\langle\lambda_6^2\rangle = \frac{2}{3}I - \frac{1}{2\sqrt{3}}\langle\lambda_8\rangle - \frac{1}{2}\langle\lambda_3\rangle.
\end{aligned} \tag{25}$$

To proceed vertically down to the next level in Fig. 1, the number of extra measurements are indicated beside the boxes. It may be mentioned that in our scheme it does not matter if any horizontal pair of boxes is interchanged with another pair at a different level. A maximum of eight measurements thus suffices to distinguish between pure and mixed states of single qutrit up to three-parameter families. The maximum number of measurements required in particular cases may not provide a significant advantage over tomography, but would still form an independent check of states with prior knowledge of basis. The difference in the number of required measurements is substantially enhanced for composite states. For two qubits, GUR requires up to 5 measurements compared to 15 required by tomography for the class of states considered. For the case of two qutrits the measurement of at most 8 expectation values, viz., $\langle\lambda_1\otimes\lambda_1\rangle$, $\langle\lambda_1\otimes\lambda_2\rangle$, $\langle\lambda_2\otimes\lambda_1\rangle$, $\langle\lambda_2\otimes\lambda_2\rangle$, $\langle\lambda_3\otimes\lambda_3\rangle$, $\langle\lambda_3\otimes\lambda_8\rangle$, $\langle\lambda_8\otimes\lambda_3\rangle$, and $\langle\lambda_8\otimes\lambda_8\rangle$, suffices, using GUR for the observables defined by Eq. (18). A comparison of the number of measurements required using GUR with that needed in tomography is provided in Table I.

Conclusions. We have shown that the Robertson-Schrodinger uncertainty relation [2,3] is connected to the property of mixedness of single and bipartite three-level quantum systems. The generalized uncertainty corresponding to the measurement of suitable observables vanishes for pure states and is positive definite for mixed states. Using this feature we have proposed a scheme to distinguish pure and mixed states belonging to the classes of all single-qutrit states up to three parameters, as well as several classes of two-qutrit states, when prior knowledge of the basis is available. Since the class of all pure states is not convex, the witnesses

TABLE I. A comparison between the number of measurements required in tomography and those required through our method is shown for the categories of states considered. Using GUR considerably reduces the number of measurements required for bipartite systems and with increasing dimension.

System	In tomography	Using GUR
Single qubit	3	3
Two qubit	15	3–5
Single qutrit	8	4–8
Two Qutrit	80	4–8

proposed here for detecting mixedness do not arise from the separability criterion that holds for the widely studied entanglement witnesses [27], as well as the recently proposed teleportation witnesses [28]. Nonetheless, the same principle of distinction of categories of quantum states based on the measurement of expectation values of Hermitian operators is followed.

A possible implementation of the witnesses proposed here could be through techniques involving measurement of two-photon polarization-entangled modes for qutrits [20,22]. The procedure suggested here could be helpful also for the detection of entanglement, since purity of subsystems is related to the entanglement of the joint system. The method of detecting mixedness using the uncertainty relation is advantageous over tomography in terms of the number of measurements required, significantly for bipartite qutrit systems, which may have applications in information processing protocols such as distributed computing [19] and security enhancement of quantum cryptography [21,22].

Acknowledgments. We would like to thank Dipankar Home and Peter Holland for discussions and suggestions which lead to this work. A.S.M. and S.M. acknowledge support from DST Project No. SR/S2/PU-16/2007.

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