

Invisibility and \mathcal{PT} symmetry

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For a general complex scattering potential defined on a real line, we show that the equations governing invisibility of the potential are invariant under the combined action of parity and time-reversal (\mathcal{PT}) transformation. We determine the \mathcal{PT} -symmetric as well as non- \mathcal{PT} -symmetric invisible configurations of an easily realizable exactly solvable model that consists of a two-layer planar slab consisting of optically active material. Our analysis shows that although \mathcal{PT} symmetry is neither necessary nor sufficient for the invisibility of a scattering potential, it plays an important role in the characterization of the invisible configurations. A byproduct of our investigation is the discovery of certain configurations of our model that are effectively reflectionless in a spectral range as wide as several hundred nanometers.

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I. INTRODUCTION

Consider a scattering problem for a real or complex potential v that is defined on the real line. The asymptotic solution of the time-independent Schrödinger equation has the form

$$\psi(z) \rightarrow A_{\pm} e^{ikz} + B_{\pm} e^{-ikz} \quad \text{as } z \rightarrow \pm\infty, \quad (1)$$

where k is the wave number that takes real and positive values, and A_{\pm} and B_{\pm} are possibly k -dependent complex coefficients. We can obtain the information about the scattering properties of v using its transfer matrix \mathbf{M} that is defined by the equation

$$\begin{bmatrix} A_+ \\ B_+ \end{bmatrix} = \mathbf{M} \begin{bmatrix} A_- \\ B_- \end{bmatrix}. \quad (2)$$

It is not difficult to show that \mathbf{M} has a unit determinant [1]. This in turn implies that for every real or complex scattering potential the left and right transmission amplitudes coincide while this is not generally true for the left and right reflection amplitudes [2]. Let T and $R^{r,l}$ respectively denote the (complex) transmission and right/left reflection amplitudes, so that the transmission and right/left reflection coefficients are given by $|T|^2$ and $|R^{r,l}|^2$. We can express \mathbf{M} in terms of T and $R^{r,l}$ according to [2]

$$\mathbf{M} = \begin{bmatrix} T - \frac{R^l R^r}{T} & \frac{R^r}{T} \\ -\frac{R^l}{T} & \frac{1}{T} \end{bmatrix}. \quad (3)$$

Alternatively, we can describe the scattering properties of the system using its S matrix,

$$\mathbf{S} := \begin{bmatrix} T & R^r \\ R^l & T \end{bmatrix}. \quad (4)$$

The potential is called *reflectionless from the left (right)*, if $R^l = 0$ and $R^r \neq 0$ ($R^r = 0$ and $R^l \neq 0$.) As seen from (3) and (4), unidirectional reflectionlessness implies the nondiagonalizability of both \mathbf{M} and \mathbf{S} . Therefore, the parameters of the

potential for which it becomes unidirectionally reflectionless correspond to exceptional points [3] of \mathbf{M} and \mathbf{S} .

The potential is called *invisible from the left (right)*, if it is reflectionless from left (right) and in addition $T = 1$. Therefore, in light of (3), we can express the condition for the invisibility of the potential from the left in terms of the entries of the transfer matrix according to

$$M_{12} \neq 0, \quad M_{21} = 0, \quad M_{11} = M_{22} = 1. \quad (5)$$

Similarly, for invisibility from the right we have

$$M_{12} = 0, \quad M_{21} \neq 0, \quad M_{11} = M_{22} = 1. \quad (6)$$

Because $\det(\mathbf{M}) = 1$, Eqs. (5) are not independent. The same holds for (6). We can reduce (5) and (6) to the following sets of equivalent independent equations respectively:

$$M_{12} \neq 0, \quad M_{11} = M_{22} = 1, \quad (7)$$

$$M_{21} \neq 0, \quad M_{11} = M_{22} = 1. \quad (8)$$

Recently Lin *et al.* [4] have reported the emergence of unidirectional invisibility for complex \mathcal{PT} -symmetric locally periodic potentials of the form

$$v(z) := \begin{cases} \alpha_0 + \alpha e^{2i\beta z} & \text{for } |z| \leq \frac{L}{2}, \\ 0 & \text{for } |z| > \frac{L}{2}, \end{cases} \quad (9)$$

where $\alpha_0, \alpha, \beta, L$ are real parameters, and β and L are positive. The spectral properties of the periodic potentials: $v(z) = \sum_{n=0}^{\infty} \alpha_n e^{i\beta_n z}$, have been studied by Gasyimov more than three decades ago [5]. The special case $v(z) = \alpha e^{i\beta z}$ has been reexamined more recently in Refs. [6–8] (see also [9]). The publication of Ref. [4] has motivated a more detailed study of the potential (9). In particular, the authors of Refs. [10,11] use the analytic solution of the wave equation [5,6] to improve the approximate results of Ref. [4] that rely on the rotating-wave approximation. For an earlier discussion of reflectionlessness and \mathcal{PT} symmetry, see Ref. [12].

In the present paper, we provide a thorough assessment of the role of \mathcal{PT} symmetry [13] in the phenomenon of invisibility of scattering potentials in one dimension, and explore the invisible configurations of an exactly solvable model that admits experimental realizations.

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II. \mathcal{PT} -SYMMETRIC NATURE OF INVISIBILITY

We begin our analysis by examining the combined effect of parity and time-reversal (\mathcal{PT}) transformation on the transfer matrix \mathbf{M} . It is easy to show that under this transformation the asymptotic solutions of the wave equation (1) transform according to

$$\psi(z) \xrightarrow{\mathcal{PT}} (\mathcal{PT}\psi)(z) := \psi(-z)^* \quad \text{for } z \rightarrow \mp\infty$$

$$= A_{\pm}^* e^{ikz} + B_{\pm}^* e^{-ikz}, \quad (10)$$

where we suppose that k is real and positive. The transformation rule (10) implies that the transfer matrix of the \mathcal{PT} -transformed system, which we denote by $\mathbf{M}^{(\mathcal{PT})}$, satisfies

$$\begin{bmatrix} A_-^* \\ B_-^* \end{bmatrix} = \mathbf{M}^{(\mathcal{PT})} \begin{bmatrix} A_+^* \\ B_+^* \end{bmatrix}. \quad (11)$$

Combining (2) and (11), we find the following transformation rule for the transfer matrix:

$$\mathbf{M} \xrightarrow{\mathcal{PT}} \mathbf{M}^{(\mathcal{PT})} := \mathbf{M}^{-1*}. \quad (12)$$

Because $\det \mathbf{M} = 1$, this means that

$$\begin{aligned} M_{11} &\xrightarrow{\mathcal{PT}} M_{22}^*, & M_{12} &\xrightarrow{\mathcal{PT}} -M_{12}^*, \\ M_{21} &\xrightarrow{\mathcal{PT}} -M_{21}^*, & M_{22} &\xrightarrow{\mathcal{PT}} M_{11}. \end{aligned} \quad (13)$$

In particular, as noted in Ref. [14], the transfer matrix of a \mathcal{PT} -symmetric potential satisfies $\mathbf{M}^{-1} = \mathbf{M}^*$.

A straightforward consequence of (13) is that in general, regardless of whether the potential is \mathcal{PT} symmetric or not, the equations governing its invisibility (both from left and right), namely (7) and (8), are invariant under the \mathcal{PT} transformation [15]. We can restate this fact as follows.

Invisibility Theorem. Consider a general real or complex scattering potential v that is defined on \mathbb{R} . Let $v^{(\mathcal{PT})}$ be the \mathcal{PT} transform of v that is given by $v^{(\mathcal{PT})}(z) := v(-z)^*$, and k_* be a positive real number. Then the following equivalent statements hold:

- (i) v is invisible from the left (or right) for $k = k_*$ if and only if so is $v^{(\mathcal{PT})}$.
- (ii) v is invisible from the left (right) for $k = k_*$ if and only if v^* is invisible from the right (left) for $k = k_*$.

For non- \mathcal{PT} -symmetric potentials this theorem implies a pairing of the left/right-invisible configurations that are related by the \mathcal{PT} transformation. We also have a pairing of all complex (nonreal) potentials that display unidirectional invisibility at a given wave number where each member of the pair is the time reversal (complex conjugate) of the other. Obviously, parity reflection transformation changes a left-(right-)invisible potential to a right- (left-)invisible potential. In particular, even (\mathcal{P} -symmetric) potentials do not support unidirectional invisibility. In view of the above theorem, the same holds for real (\mathcal{T} -symmetric) potentials. Unlike \mathcal{P} -symmetric and \mathcal{T} -symmetric potentials, \mathcal{PT} -symmetric potentials can display unidirectional invisibility.

We can use the argument leading to the proof of the invisibility theorem to show that its statement also holds, if we replace the term “invisible” with “reflectionless.” In particular, we have \mathcal{PT} -dual and \mathcal{T} -dual pairings of complex reflectionless potentials.

III. TWO-LAYER INFINITE SLAB MODEL

Consider an infinite planar two-layer slab of optically active material as depicted in Fig. 1. Suppose that the layers have equal thickness, $L/2$, and different gain or loss properties. Then the wave equation for a linearly polarized time-harmonic electromagnetic wave that propagates along the normal direction to the slab can be reduced to the Helmholtz equation [16],

$$\Psi''(z) + k^2 n(z)^2 \Psi(z) = 0, \quad (14)$$

where

$$n(z) := \begin{cases} n_1 & \text{for } -\frac{L}{2} \leq z < 0, \\ n_2 & \text{for } 0 \leq z \leq \frac{L}{2}, \\ 1 & \text{for } |z| > \frac{L}{2}, \end{cases} \quad (15)$$

and n_1 and n_2 are complex refractive indices of the layers. In Ref. [16], we have used this model to examine the importance of \mathcal{PT} symmetry for generating self-dual spectral singularities (i.e., spectral singularities [2] that coincide with their time-reversal dual). The concept of a self-dual spectral singularity provides the mathematical basis for an optical device that functions as a coherent perfect absorber (CPA) [14,17] whenever it is subject to incidence coherent waves with identical amplitude and phase from both sides and operators as a laser otherwise. It is therefore called a CPA laser [18]. In Ref. [16], we establish the existence of non- \mathcal{PT} -symmetric CPA lasers. Here we use the optical system defined by Eq. (15) to examine the role of \mathcal{PT} symmetry in achieving unidirectional invisibility.

We can calculate the transfer matrix \mathbf{M} of the above model using the method outlined in Ref. [19] and employed in Ref. [16]. Doing the necessary calculations this yields the following expressions for the entries of \mathbf{M} .

$$M_{11} = (n_+^2 - n_-^2)^{-1} \{ n_+^2 \cos \alpha_+ - n_-^2 \cos \alpha_- + i(\tilde{n}_+ n_+ \sin \alpha_+ + \tilde{n}_- n_- \sin \alpha_-) \} e^{-i\Re}, \quad (16)$$

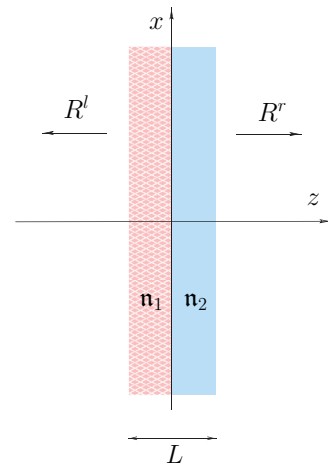


FIG. 1. (Color online) Schematic view of the cross section of a two-layer infinite planar slab of gain material of thickness L that is aligned in the x - y plane. The arrows marked by R^l and R^r represent the reflected waves associated with incident waves from the left and right, respectively.

$$M_{12} = (n_+^2 - n_-^2)^{-1} \{n_- n_+ (\cos \alpha_+ - \cos \alpha_-) + i(\tilde{n}_+ n_- \sin \alpha_- + \tilde{n}_- n_+ \sin \alpha_+)\}, \quad (17)$$

$$M_{21} = (n_+^2 - n_-^2)^{-1} \{n_- n_+ (\cos \alpha_+ - \cos \alpha_-) - i(\tilde{n}_+ n_- \sin \alpha_- + \tilde{n}_- n_+ \sin \alpha_+)\}, \quad (18)$$

$$M_{22} = (n_+^2 - n_-^2)^{-1} \{n_+^2 \cos \alpha_+ - n_-^2 \cos \alpha_- - i(\tilde{n}_+ n_+ \sin \alpha_+ + \tilde{n}_- n_- \sin \alpha_-)\} e^{i\mathfrak{K}}, \quad (19)$$

where

$$n_{\pm} := n_1 \pm n_2, \quad \alpha_{\pm} := \frac{n_{\pm} \mathfrak{K}}{2}, \quad (20)$$

$$\mathfrak{K} := Lk, \quad \tilde{n}_{\pm} := n_1 n_2 \pm 1 = \frac{1}{4}(n_+^2 - n_-^2) \pm 1.$$

As a nontrivial check on the validity of Eqs. (16)–(19), we have shown by direct calculations that indeed $\det \mathbf{M} = 1$.

In view of Eqs. (17)–(19), the condition (5) for the invisibility of the system from the left takes the form

$$n_- n_+ (\cos \alpha_+ - \cos \alpha_-) + i(\tilde{n}_+ n_- \sin \alpha_- + \tilde{n}_- n_+ \sin \alpha_+) \neq 0, \quad (21)$$

$$n_- n_+ (\cos \alpha_+ - \cos \alpha_-) - i(\tilde{n}_+ n_- \sin \alpha_- + \tilde{n}_- n_+ \sin \alpha_+) = 0, \quad (22)$$

$$n_+^2 \cos \alpha_+ - n_-^2 \cos \alpha_- + i(\tilde{n}_+ n_+ \sin \alpha_+ + \tilde{n}_- n_- \sin \alpha_-) = (n_+^2 - n_-^2) e^{i\mathfrak{K}}, \quad (23)$$

$$n_+^2 \cos \alpha_+ - n_-^2 \cos \alpha_- - i(\tilde{n}_+ n_+ \sin \alpha_+ + \tilde{n}_- n_- \sin \alpha_-) = (n_+^2 - n_-^2) e^{-i\mathfrak{K}}. \quad (24)$$

Note that for $n_1 = n_2$, $n_- = 0$ and we can use Eqs. (22) and (24) to show that in this case $n_1 = 1$. This corresponds to an empty slab. Therefore, without loss of generality, we take $n_2 \neq n_1$. Moreover, by adding and subtracting Eqs. (24) and (23), we can reduce them to

$$n_+^2 \cos \alpha_+ - n_-^2 \cos \alpha_- = (n_+^2 - n_-^2) \cos \mathfrak{K}, \quad (25)$$

$$\tilde{n}_+ n_+ \sin \alpha_+ + \tilde{n}_- n_- \sin \alpha_- = (n_+^2 - n_-^2) \sin \mathfrak{K}. \quad (26)$$

Similarly using Eqs. (21) and (22), we have

$$n_- n_+ (\cos \alpha_+ - \cos \alpha_-) \neq 0, \quad (27)$$

$$\tilde{n}_+ n_- \sin \alpha_- + \tilde{n}_- n_+ \sin \alpha_+ \neq 0. \quad (28)$$

For nonexotic material, $\text{Re}(n_{1,2}) \geq 1$ and $n_+ \neq 0$. We also have $n_- \neq 0$. Therefore, (27) holds if and only if

$$\cos \alpha_+ \neq \cos \alpha_-. \quad (29)$$

We arrive at the same conclusion if we demand that the potential be invisible from the right. Therefore, a necessary and sufficient condition for the unidirectional invisibility of our system is that we satisfy (25), (26), and (29). To determine the direction of invisibility we need to check which of M_{12} and M_{21} vanishes.

IV. BIDIRECTIONAL INVISIBILITY

Suppose that $n_1 \neq n_2$ and the system is invisible from both directions [i.e., it satisfies (25), (26)] and $\cos \alpha_+ = \cos \alpha_-$. Substituting this relation in (22) and (25) and noting that $n_+^2 - n_-^2 \neq 0$ gives

$$\cos \alpha_{\pm} = \cos \mathfrak{K}, \quad (30)$$

$$\tilde{n}_+ n_- \sin \alpha_- + \tilde{n}_- n_+ \sin \alpha_+ = 0. \quad (31)$$

Equation (30) means that

$$\alpha_+ = 2\pi m_+ \pm \mathfrak{K}, \quad \alpha_- = 2\pi m_- \pm \mathfrak{K}, \quad (32)$$

where m_{\pm} are integers. In view of (20), this implies that n_{\pm} , \tilde{n}_{\pm} , n_1 , and n_2 must take real values. Another consequence of (30) is

$$\sin \alpha_+ = \pm \sin \alpha_-. \quad (33)$$

Inserting this relation in (22) and (24) and using the fact that n_{\pm} , \tilde{n}_{\pm} , and α_{\pm} are real, we find

$$(\tilde{n}_+ n_- \pm \tilde{n}_- n_+) \sin \alpha_+ = 0, \quad (\tilde{n}_+ n_+ \pm \tilde{n}_- n_-) \sin \alpha_+ = 0.$$

Adding these two equations gives $2n_1(\tilde{n}_+ \pm \tilde{n}_-) \sin \alpha_+ = 0$. Because $n_1(\tilde{n}_+ \pm \tilde{n}_-) \neq 0$ and (33) holds, this implies that $\sin \alpha_{\pm} = 0$. Equivalently, there are integers m'_{\pm} such that

$$\alpha_{\pm} = \pi m'_{\pm}. \quad (34)$$

Next, we combine this relation and (32) to obtain

$$\mathfrak{K} = \pi m, \quad (35)$$

where m is an integer. In light of (34) and (35), we also find that $m'_+ \pm m'_-$ are even, and

$$n_1 = \frac{2m_1}{m}, \quad n_2 = \frac{2m_2}{m}, \quad \lambda = \frac{2L}{m}, \quad (36)$$

where $m_1 := (m'_+ + m'_-)/2$, $m_2 := (m'_+ - m'_-)/2$, and $\lambda := 2\pi/k = 2\pi L/\mathfrak{K}$ is the wavelength.

In summary, our device is invisible from both directions provided that the refractive indices of both the layers are rational numbers, and the wavelength λ at which the device is invisible is such that L is a half integer multiple of λ , [20]. Notice that even bidirectional invisibility does not require \mathcal{PT} symmetry.

V. UNIDIRECTIONAL INVISIBILITY AND \mathcal{PT} SYMMETRY

Equations (25) and (26) are complex equations involving two complex variables, n_{\pm} , and a real variable \mathfrak{K} . Therefore, in principle, we can fix, say, the real part of n_+ , which is bounded from below by 2, and try to solve these equations for the imaginary part of n_+ , the real and imaginary parts of n_- , and \mathfrak{K} . For the physically relevant range of the values of these variables this turns out to be difficult to implement directly. This calls for a more systematic study of the structure of (25) and (26). An important clue is provided by the \mathcal{PT} symmetry of the invisibility equations that we established for general scattering potentials in Sec. II. Here we offer a direct and explicit verification for the presence of this symmetry.

First we note that for our system the parity transformation means swapping the labels of n_j (i.e., $n_1 \xleftrightarrow{\mathcal{P}} n_2$) and the time-reversal transformation corresponds to complex conjugation of the refractive indices, $n_j \xrightarrow{\mathcal{T}} n_j^*$. Therefore, $n_1 \xleftrightarrow{\mathcal{PT}} n_2^*$, and consequently

$$\begin{aligned} n_+ &\xrightarrow{\mathcal{PT}} n_+^*, & n_- &\xrightarrow{\mathcal{PT}} -n_-^*, \\ a_+ &\xrightarrow{\mathcal{PT}} a_+^*, & a_- &\xrightarrow{\mathcal{PT}} -a_-^*, & \tilde{n}_\pm &\xrightarrow{\mathcal{PT}} \tilde{n}_\pm^*. \end{aligned} \quad (37)$$

Performing these transformation on Eqs. (25) and (26) and noting that their left-hand sides are analytic functions of n_\pm, a_\pm , and \tilde{n}_\pm , we see that \mathcal{PT} transformation maps these equations to their complex conjugate. Therefore, the real and imaginary parts of the transformed equations are identical with those of (25) and (26). This confirms our general result concerning the \mathcal{PT} symmetry of the invisibility equations. The presence of this symmetry suggests that we first examine the \mathcal{PT} -symmetric solutions of (25) and (26).

A. \mathcal{PT} -symmetric invisible configurations

The optical system described by (15) is \mathcal{PT} symmetric provided that $n_1^* = n_2$. In this case, n_+ and n_- are respectively real and imaginary. The same is true about a_\pm . In particular, if we denote by η and κ the real and imaginary parts of n_1 , we have

$$\begin{aligned} n_+ &= 2\eta, & n_- &= 2i\kappa, & a_+ &= \mathfrak{R}\eta, \\ a_- &= i\mathfrak{R}\kappa, & \tilde{n}_\pm &= \eta^2 + \kappa^2 \pm 1 = |n_1|^2 \pm 1. \end{aligned} \quad (38)$$

Using these relations and the identities $\cos(iz) = \cosh z$ and $\sin(iz) = i \sinh z$, we can express (25) and (26) in the form

$$\left(\frac{\eta^2}{\eta^2 + \kappa^2} \right) \cos(\mathfrak{R}\eta) + \left(\frac{\kappa^2}{\eta^2 + \kappa^2} \right) \cosh(\mathfrak{R}\kappa) = \cos \mathfrak{R}, \quad (39)$$

$$\begin{aligned} \frac{1}{2} \left[\left(1 + \frac{1}{\eta^2 + \kappa^2} \right) \eta \sin(\mathfrak{R}\eta) \right. \\ \left. - \left(1 - \frac{1}{\eta^2 + \kappa^2} \right) \kappa \sinh(\mathfrak{R}\kappa) \right] = \sin \mathfrak{R}. \end{aligned} \quad (40)$$

It is easy to see that (29) holds automatically. Notice that (39) and (40) are real equations involving three real variables, η , κ , and \mathfrak{R} . This is an enormous simplification over the general non- \mathcal{PT} -symmetric case that amounts to solving four real equations for five real unknowns.

Next, we recall that for typical optically active material, $|\kappa| \lesssim 10^{-3}$ and $1 \lesssim \eta \lesssim 5$. Therefore $\kappa^2/\eta^2 \lesssim 10^{-6}$, and we can safely approximate $\eta^2 + \kappa^2$ by η^2 . This simplifies (39) and (40), and we find

$$\cos(\mathfrak{R}\eta) + \frac{\kappa^2 \cosh(\mathfrak{R}\kappa)}{\eta^2} \approx \cos \mathfrak{R}, \quad (41)$$

$$\frac{1}{2} [(1 + \eta^{-2})\eta \sin(\mathfrak{R}\eta) - (1 - \eta^{-2})\kappa \sinh(\mathfrak{R}\kappa)] \approx \sin \mathfrak{R}. \quad (42)$$

Here and in what follows we use the symbol “ \approx ” to mean that we ignore terms of order κ^2/η^2 .

Because $\mathfrak{R} = Lk = 2\pi L/\lambda$, where λ is the wavelength, \mathfrak{R} can take very large values. However, according to (39), $\cosh(\mathfrak{R}\kappa) \leq 2\eta^2/\kappa^2$. This in turn implies

$$\mathfrak{R} \leq \frac{2}{|\kappa|} \ln \left(\frac{2\eta}{|\kappa|} \right). \quad (43)$$

Therefore the smaller $|\kappa|$ (or the gain or loss coefficient) is, the larger the upper bound on \mathfrak{R} becomes. In particular, (43) does not impose any severe restriction on the wavelength at which the system is invisible from the left. For example, using the fact that usually $1 \lesssim \eta \lesssim 5$ and $|\kappa| \lesssim 10^{-3}$, we find from (43) that

$$\mathfrak{R} \lesssim 18420. \quad (44)$$

This is equivalent to $L \lesssim 2930\lambda$. In particular, for $\lambda \geq 100\text{nm}$, we have $L \lesssim 293\mu\text{m}$, which can be easily realized experimentally.

Another simple consequence of (41) is that $\eta = 1$ implies $\kappa = 0$. Again this corresponds to an empty slab. Therefore unlike for the \mathcal{PT} -symmetric periodic potential considered in Ref. [4], the unidirectional invisibility can be achieved in our system provided that we use an optically active medium with $\eta > 1$.

Next, we obtain an analytic solution of (41) and (42). To do this we introduce

$$X := (\mathfrak{R}\kappa)^2, \quad \alpha := \frac{\eta^2 + 1}{\eta^2 - 1}, \quad \beta := \frac{2\eta}{\eta^2 - 1}, \quad (45)$$

respectively solve (41) and (42) for $\cosh(\mathfrak{R}\kappa)$ and $\sinh(\mathfrak{R}\kappa)$, and employ the identity $\cosh^2(\mathfrak{R}\kappa) - \sinh^2(\mathfrak{R}\kappa) = 1$. This gives a quadratic equation in X with a single positive solution. We can use this solution to express κ in terms of η and \mathfrak{R} as follows:

$$\kappa \approx \pm \eta \sqrt{\frac{1}{4} [\alpha \sin(\mathfrak{R}\eta) - \beta \sin \mathfrak{R}]^4 + [\cos(\mathfrak{R}\eta) - \cos \mathfrak{R}]^2 - \frac{1}{2} [\alpha \sin(\mathfrak{R}\eta) - \beta \sin \mathfrak{R}]^2}. \quad (46)$$

Using this relation in (41) we find an equation for η and \mathfrak{R} . Because (41) involves \cosh with possibly large arguments, it is not suitable for numerical calculations. Therefore, we solve this equation for $\cosh(\mathfrak{R}\kappa)$ and employ the identity $\cosh^{-1}(z) = \ln(z \pm \sqrt{z^2 - 1})$ to express it as

$$\mathfrak{R}|\kappa| - \ln \left\{ \frac{\eta^2}{\kappa^2} \left(\cos \mathfrak{R} - \cos(\mathfrak{R}\eta) \pm \sqrt{[\cos \mathfrak{R} - \cos(\mathfrak{R}\eta)]^2 - \frac{\kappa^4}{\eta^4}} \right) \right\} \approx 0. \quad (47)$$

Now, we substitute (46) in the left-hand side of (47). The result is a function of η and \mathfrak{K} that we denote by $F_{\pm}(\eta, \mathfrak{K})$. In this way, we can express (47) in the form

$$F_{\pm}(\eta, \mathfrak{K}) \approx 0. \quad (48)$$

We can choose some typical values for η and solve (48) numerically to find the possible values of \mathfrak{K} . Once these are determined we use (46) to obtain the corresponding values of κ . In order for the latter to be experimentally meaningful, they should be sufficiently small. We expect this to be the case provided that the \mathfrak{K} values we find fulfill the condition (44).

To be specific, suppose that the left-hand (right-hand) layer of the slab is the one containing the gain (lossy) material, so that $\kappa < 0$. In this case, we find that $F_{-}(\eta, \mathfrak{K}) \neq 0$, while for each $\eta > 0$ the equation $F_{+}(\eta, \mathfrak{K}) = 0$ gives a (countably) infinite set of values for \mathfrak{K} . Using these values in (46) we can determine the corresponding values of κ . The \mathfrak{K} and κ values we obtain in this way satisfy (48), but they may not satisfy (41) and (42). This is because in deriving (48) we affected functions on our expressions that are not one-to-one. Therefore, although every solution of (41) and (42) is a solution of (46) and (48), the opposite is generally not true. In our numerical calculation, we found solutions of (46) and (48) and checked if they satisfy the exact equations (39) and (40). In this way we determined the acceptable approximate solutions of the problem that are valid whenever we can neglect κ^2/η^2 . We then obtained the exact numerical solutions of (39) and (40) in the neighborhood of the approximate solutions. For example for $\eta = 1.4, 2.0, 3.4$ with \mathfrak{K} ranging over (1995, 2005) we respectively found two, three, and five different values of λ/L (between 317.7 and 318.9) for which the system displayed unidirectional invisibility. Except for a single data point, the approximate and exact results agreed to eight significant figures. We also checked the values of M_{12} and M_{22} to determine the directional of invisibility. It turns out that for the case that $\kappa < 0$ ($\kappa > 0$) we have invisibility from the left (right).

Note also that because the equations we used to characterize invisibility of our system are even in κ , changing the sign of the values of κ that lead for invisibility from the left, we find the values that support invisibility from the right. This confirms the statement (ii) of the invisibility theorem.

Figure 2 shows plots of $|T| - 1$, $|R^l|$, and the argument (phase angle) of T as a function of \mathfrak{K} for a \mathcal{PT} -symmetric sample with $\eta = 3.4$ and $\kappa = -0.00342163$. The fact that the graphs have a common intersection point located on the \mathfrak{K} axis is a clear demonstration of the invisibility of the device from the left for the value of \mathfrak{K} at this point (i.e., $\mathfrak{K} = 2000.147552$). For these values of η, κ , and \mathfrak{K} , we have

$$|R^l|^2 < 10^{-10}, \quad |T|^2 - 1 < 10^{-5}, \quad |R^r|^2 > 0.89. \quad (49)$$

Taking the thickness of each layer of the slab to be $150 \mu\text{m}$, so that $L = 300 \mu\text{m}$, and using $\mathfrak{K} = 2000.147552$, we find that the device displays invisibility from the left for $\lambda = 942.408269 \text{ nm}$. If we keep only three decimal places in the above figures for κ, \mathfrak{K} and λ , the first two bounds given in (49) change to $|R^l|^2 < 10^{-5}$ and $|T|^2 - 1 < 3 \times 10^{-3}$, respectively.

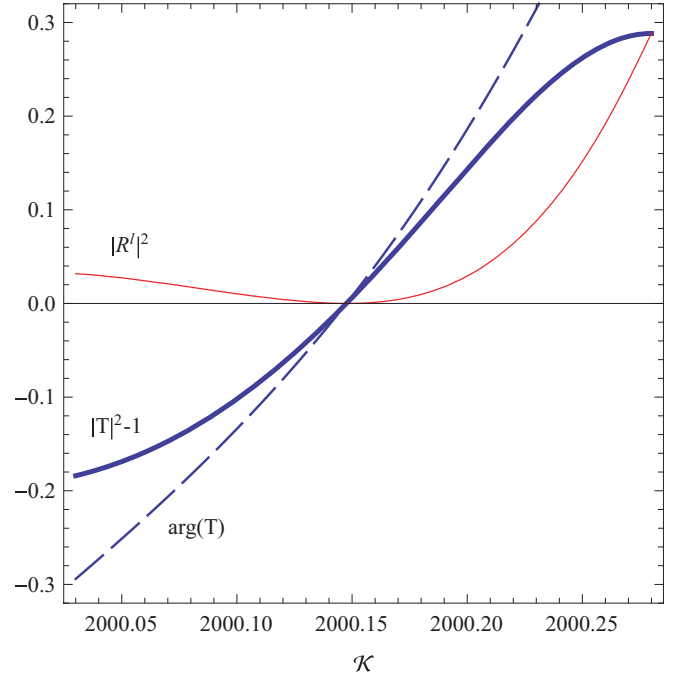


FIG. 2. (Color online) Graphs of $|T|^2 - 1$ (thick solid curve), argument (phase angle) of T (dashed curve), and $|R^l|^2$ (thin solid red curve) as functions of $\mathfrak{K} = Lk = 2\pi L/\lambda$ for a \mathcal{PT} -symmetric two-layer slab with $n_1 = n_2^* = 3.4 - 0.003422i$. The intersection points of these curves and the \mathfrak{K} axis corresponds to invisibility from the left for $\mathfrak{K} = 2000.148$. For a sample consisting of two $150 \mu\text{m}$ thick layers, this gives the value 942.408 nm for the wavelength.

B. Non- \mathcal{PT} -symmetric invisible configurations

In the absence of \mathcal{PT} symmetry, the equations governing unidirectional invisibility, (25) and (26), are complex equations involving two complex and one real unknowns, n_+ , n_- , and \mathfrak{K} , respectively. It is more convenient to parametrize them in terms of a_+ , a_- , and \mathfrak{K} . This leads to

$$a_+^2(\cos a_+ - \cos \mathfrak{K}) - a_-^2(\cos a_- - \cos \mathfrak{K}) = 0, \quad (50)$$

$$(a_+^2 - a_-^2 - \mathfrak{K}^2)a_+ \sin a_+ + (a_+^2 - a_-^2 + \mathfrak{K}^2)a_- \sin a_- - 2(a_+^2 - a_-^2)\mathfrak{K} \sin \mathfrak{K} = 0. \quad (51)$$

Equating the real and imaginary parts of the left-hand side of these equations to zero, we find a set of extremely complicated real equations for \mathfrak{K} and

$$x_{\pm} := \text{Re}(a_{\pm}), \quad y_{\pm} := \text{Im}(a_{\pm}). \quad (52)$$

A direct MATHEMATICA-based numerical treatment of these equations proves to be intractable for the physically relevant ranges of the values of \mathfrak{K} , x_{\pm} , and y_{\pm} . A careful study of the reasons for the failure of such a treatment guided us to a particular change of variables that allowed for a reliable perturbative method of solving these equations. The following is a brief outline of this method.

(i) Because $|x_{\pm}| = |\text{Re}(\mathfrak{K}n_{\pm})|/2$ and \mathfrak{K} are typically large numbers, we express them as

$$x_{\pm} = 2\pi m_{\pm} + \frac{\gamma_{\pm}}{2\pi m_{\pm}}, \quad \mathfrak{K} = 2\pi m_0 + \frac{\gamma_0}{2\pi m_0}, \quad (53)$$

where m_{\pm} and m_0 are integers, m_+ and m_0 are positive, and γ_{\pm} and γ_0 are real parameters that are of the same order of magnitude as y_{\pm} . In particular, $|\gamma_{\pm}|, |\gamma_0|$ and $|y_{\pm}|$ are at least two orders of magnitude smaller than $m_+, |m_-|$, and m_0 . In view of (53) and the fact that

$$\alpha_{\pm} = x_{\pm} + iy_{\pm}, \quad (54)$$

this justifies the following approximations

$$\cos \alpha_{\pm} \approx \cosh y_{\pm} - \frac{i\gamma_{\pm} \sinh y_{\pm}}{2\pi m_{\pm}}, \quad (55)$$

$$\alpha_{\pm} \sin \alpha_{\pm} \approx \gamma_{\pm} \cosh y_{\pm} - y_{\pm} \sinh y_{\pm} + 2\pi i m_{\pm} \sinh y_{\pm}, \quad (56)$$

$$\alpha_{\pm}^2 \approx 4\pi m_{\pm}(\pi m_{\pm} + iy_{\pm}), \quad \cos \mathfrak{K} \approx 1, \quad \mathfrak{K} \sin \mathfrak{K} \approx \gamma_0. \quad (57)$$

Note also that because $\alpha_{\pm} = \mathfrak{K}n_{\pm}/2 = \mathfrak{K}(n_1 \pm n_-)/2$ and $\text{Re}(n_1) \geq 1$ and $\text{Re}(n_2) \geq 1$, Equations (53) and (54) imply that

$$|m_-| < m_+, \quad m_0 \leq m_+. \quad (58)$$

(ii) Substituting Eqs. (55)–(57) in (50) and (51) and keeping only the dominant terms, we respectively find from the real part of (50) and the imaginary part of (51) the following approximate equations:

$$\cosh y_- \approx \mu^2(\cosh y_+ - 1) + 1, \quad (59)$$

$$\sinh y_- \approx -\mu v \sinh y_+, \quad (60)$$

where

$$\mu := \frac{m_+}{m_-}, \quad v := \frac{m_+^2 - m_-^2 + m_0^2}{m_+^2 - m_-^2 - m_0^2}. \quad (61)$$

Similarly implementing the approximations (55)–(57) in the imaginary part of (50) and the real part of (51) and using (59) and (60) to simplify the resulting equations, we obtain

$$\mu \gamma_+ \sinh y_+ - \gamma_- \sinh y_- \approx 2\mu(y_+ - \mu y_-)(\cosh y_+ - 1), \quad (62)$$

$$\begin{aligned} &v \gamma_+ \cosh y_+ + \gamma_- \cosh y_- \\ &\approx [(v + v_+)y_+ - v_0 y_-] \sinh y_+ \\ &+ [v_0 y_+ + (1 - v_-)y_-] \sinh y_- + (v_+ - v_-)\gamma_0, \end{aligned} \quad (63)$$

where

$$v_{\pm} := \frac{2m_{\pm}^2}{m_+^2 - m_-^2 - m_0^2}, \quad v_0 := \frac{2m_- m_+}{m_+^2 - m_-^2 - m_0^2}.$$

(iii) Inserting (59) and (60) in the identity $\cosh^2 y_- - \sinh^2 y_- = 1$ and using $\cosh^2 y_+ - \sinh^2 y_+ = 1$ to simplify the result, we find a quadratic equation in $\cosh y_+$ with solutions

$$\cosh y_+ \approx 1, \quad (64)$$

$$\cosh y_+ \approx \frac{\mu^2 + v^2 - 2}{\mu^2 - v^2} = \frac{1 + \frac{v^2 - 1}{\mu^2 - 1}}{1 - \frac{v^2 - 1}{\mu^2 - 1}}. \quad (65)$$

In view of (59), (64) implies $y_{\pm} \approx 0$. This corresponds to the bidirectional invisibility that we examined in Sec. IV. (65) is acceptable provided that its right-hand side is greater than 1.

This leads to the condition: $v^2 < \mu^2$ which, in light of (58) and (61), is equivalent to

$$m_0 < m_+ - |m_-|. \quad (66)$$

Supposing that this condition is fulfilled, we can invert (65) to obtain

$$y_+ \approx \pm \text{arccosh} \left(\frac{\mu^2 + v^2 - 2}{\mu^2 - v^2} \right). \quad (67)$$

Furthermore, using (60), (65), and the identity: $\sinh y_+ = \pm \sqrt{1 - \cosh^2 y_+}$, we find [22]

$$y_- \approx \mp \text{arcsinh} \left(\frac{2\mu v \sqrt{(\mu^2 - 1)(v^2 - 1)}}{\mu^2 - v^2} \right). \quad (68)$$

Finally, we substitute (67) and (68) in (62) and (63) to obtain a pair of linear equations for γ_{\pm} and γ_0 . We can easily solve any two of these variables in terms of the third.

Our method includes four free parameters, namely m_{\pm}, m_0 , and any one of γ_{\pm} and γ_0 . We can easily relate m_{\pm} to the real part of n_1 and n_2 , that we respectively denote by η_1 and η_2 . More specifically,

$$m_{\pm} \approx \frac{(\eta_1 \pm \eta_2)m_0}{2}. \quad (69)$$

The integer m_0 is related to $\mathfrak{K} = Lk = 2\pi L/\lambda$ according to

$$m_0 \approx \frac{\mathfrak{K}}{2\pi} = \frac{L}{\lambda}. \quad (70)$$

Therefore, for a given two-layer slab and any wavelength λ we can use the real part of the refractive index to determine m_{\pm} and m_0 . The choice of the last free parameter (i.e., one of γ_{\pm} and γ_0) is arbitrary. We choose γ_0 to be a real number of order 1 so that the corresponding γ_{\pm} are also of order 1, [23]. It turns out that the choice of the + or - sign in (67) corresponds to invisibility from the right and left, respectively.

The following is the result of the application of our method for a sample with $\eta_1 \approx 3.4$, $\eta_2 \approx 1.4$, $\mathfrak{K} \approx 2000$, and $\gamma_0 = -6$.

$$m_- = m_0 = 318, \quad m_+ = 764, \quad \mathfrak{K} = 1998.049925, \quad (71)$$

$$y_- = 2.392197, \quad y_+ = -1.180878, \quad (72)$$

$$\gamma_- = -1.387129, \quad \gamma_+ = 1.062178, \quad (73)$$

$$n_1 = 3.402510 + i(6.062508 \times 10^{-4}), \quad (74)$$

$$n_2 = 1.402514 - i(1.788281 \times 10^{-3}).$$

For $L = 300\mu\text{m}$, this sample displays invisibility from the left at $\lambda = 943.397644\text{nm}$. Inserting the above numerical values in the exact expressions for the reflection and transmission coefficients gives

$$\begin{aligned} |T|^2 - 1 &< 2.1 \times 10^{-5}, \quad |\arg(T)| < 3.2 \times 10^{-3}, \\ |R'|^2 &< 2.8 \times 10^{-6}, \quad |R^r|^2 > 14.1. \end{aligned}$$

Therefore our device is invisible from the left. The large value of $|R^r|^2$ indicates that it reflects the incident waves of this wavelength from the right after a fourteenfold amplification. Furthermore, choosing the opposite sign for y_{\pm} as in (72) gives the time-reversed configuration

$$\begin{aligned} n_1 &= 3.402510 - i(6.062508 \times 10^{-4}), \\ n_2 &= 1.402514 + i(1.788281 \times 10^{-3}), \end{aligned} \quad (75)$$

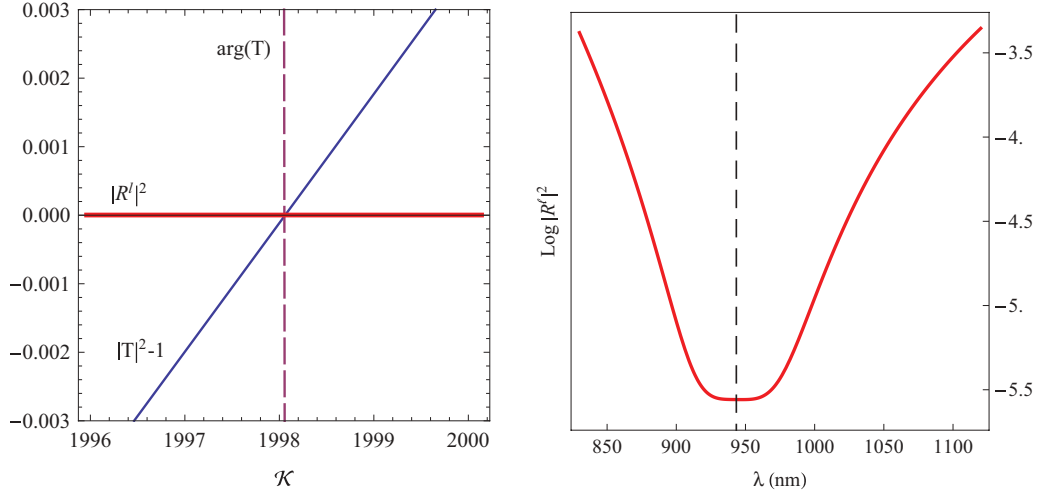


FIG. 3. (Color online) On the left are graphs of $|T|^2 - 1$ (solid blue curve), $\arg(T)$ (dashed purple curve), and $|R^l|^2$ (thick solid red curve) as functions of \mathfrak{K} for a non- \mathcal{PT} -symmetric two-layer slab with $n_1 = 3.403 + i(6.063 \times 10^{-4})$ and $n_2 = 1.403 - i(1.788 \times 10^{-3})$. The intersection points of these curves and the \mathfrak{K} axis corresponds to invisibility from the left for $\mathfrak{K} = 1998.050$. For $L = 300 \mu\text{m}$, this gives $\lambda = \lambda_* := 943.398 \text{ nm}$. On the right is the graph of $\log |R^l|^2$ as a function of the wavelength for $L = 300 \mu\text{m}$. The dashed line represents $\lambda = \lambda_*$ at which the system is invisible from the left.

that is invisible from the right and has a large reflection coefficient from the left (that is also greater than 14.). This is a manifestation of the statement (ii) of the invisibility theorem.

Figure 3 shows the plots of the $|R^l|^2$, $|T|^2 - 1$, and $\arg(T)$ as functions of \mathfrak{K} and $\log |R^l|^2$ as a function of λ for the non- \mathcal{PT} -symmetric sample with specifications (74). An interesting outcome of our findings is that the reflection coefficient from the left remains negligibly small for a very wide spectral

range. Although the invisibility from the left is lost for wavelengths slightly different from $\lambda = 943.397644 \text{ nm}$, the system remains essentially reflectionless from the left (with $|R^l|^2 < 10^{-4}$) for wavelengths ranging from about 860 nm to 1060 nm.

Next, we give the results for a sample with a different value of η_2 . We take $\eta_1 \approx 3.4$, $\eta_2 \approx 2.0$, $\mathfrak{K} \approx 2000$, and $\gamma_0 = -2$. These lead to

$$\begin{aligned} m_0 &= 318, \quad m_- = 223, \quad m_+ = 859, \quad \mathfrak{K} = 1998.051927, \\ y_- &= 1.704272, \quad y_+ = -0.492952, \quad \gamma_- = 0.612879, \quad \gamma_+ = -1.307241, \\ n_1 &= 3.402514 + i(6.062502 \times 10^{-4}), \quad n_2 = 1.999999 - i(1.099683 \times 10^{-3}), \\ ||T|^2 - 1| &< 2.4 \times 10^{-6}, \quad |\arg(T)| < 4.0 \times 10^{-4}, \quad |R^l|^2 < 7.0 \times 10^{-7}, \quad |R^r|^2 > 0.9. \end{aligned} \quad (76)$$

For $L = 300 \mu\text{m}$ this configuration is invisible from the left at $\lambda = 943.396699 \text{ nm}$. Again changing the sign of the imaginary parts of n_1 and n_2 , we find the dual configuration that is invisible from the right. Furthermore, for the original system (76), the reflection coefficient from the left remains negligibly small (less than 10^{-4}) for wavelengths between 780 nm and 1165 nm. Therefore, the device is essentially reflectionless in a spectral range that is 385 nm wide.

Finally, we should point out that if we keep only three decimal places in the numerical values we use in our calculations, the upper bounds we obtain for $||T|^2 - 1|$, $|\arg(T)|$, and $|R^l|^2$ are increased by about two orders of magnitude.

VI. CONCLUSION

The search for optical realizations of \mathcal{PT} -symmetric potentials have unraveled unusual and interesting physical properties [2,21]. Among these is the unidirectional invisibility

achieved for the \mathcal{PT} -symmetric locally periodic potentials given by (9), [4]. In this paper, we showed that the phenomenon of the invisibility of a complex scattering potential is \mathcal{PT} symmetric in nature. By this we mean that irrespective of whether the potential possesses \mathcal{PT} symmetry or not the equations characterizing invisible potentials are \mathcal{PT} invariant. This does not mean that non- \mathcal{PT} -symmetric configurations cannot display unidirectional invisibility. It only means that the \mathcal{PT} -symmetric invisible configurations are quite special, for they possess the same symmetry as the equations. A simple consequence of the presence of this symmetry is that the non- \mathcal{PT} -symmetric configurations come in pairs that are related by the \mathcal{PT} transformation (i.e., they are \mathcal{PT} dual of one another). Furthermore, if a one-dimensional scattering system is invisible from the left for some value of the wave number k , then its time reversal will be invisible from the right for the same value of k . These \mathcal{PT} - and T -dual pairings of invisible potentials also apply to reflectionless potentials.

As a concrete implementation of our general results we have examined a simple two-layer planar slab of optically active material. We have studied in great detail the structure of the \mathcal{PT} -symmetric as well as non- \mathcal{PT} -symmetric invisible configurations of this model using essentially analytic techniques. In particular, we have shown by explicit calculations how \mathcal{PT} symmetry simplifies the characterization of the invisibility of our model. We have also confirmed the above-noted \mathcal{T} -dual pairing of invisible configurations by showing that the time reversal of various left-invisible configurations are right-invisible. Furthermore, we found

invisible configurations that remain reflectionless within a very wide spectral range. This phenomenon is a subject for future research.

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