

Qubit-qudit states with positive partial transpose

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We show that the length of a qubit-qudit separable state is equal to $\max(r,s)$, where r is the rank of the state and s the rank of its partial transpose. We refer to the ordered pair (r,s) as the birank of this state. We also construct examples of qubit-qudit separable states of any feasible birank (r,s) . We determine the closure of the set of normalized two-qudit entangled states having positive partial transpose (PPT) of rank 4. The boundary of this set consists of all separable states of length at most 4. We prove that the length of any qubit-qudit separable state of birank $(d+1, d+1)$ is equal to $d+1$. We also show that all qubit-qudit PPT entangled states of birank $(d+1, d+1)$ can be built in a simple way from edge states. If V is a subspace of dimension $k < d$ in a $2 \otimes d$ space such that V contains no product vectors, we show that the set of all product vectors in V^\perp is a vector bundle of rank $d-k$ over the projective line. Finally, we explicitly construct examples of qubit-qudit PPT states (both separable and entangled) of any feasible birank.

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I. INTRODUCTION

Bipartite quantum states are key ingredients in many fundamental applications and theoretical problems of quantum information. Bell states are pure bipartite states and are useful for teleportation [1] and dense coding [2]. It has been shown by experiment [3,4] that Bell states violate the Bell inequality, indicating nonlocality, which is an essential feature of quantum physics. Unfortunately, there is no pure state existing in nature, as it extremely quickly turns into a mixed state due to the decoherence from the environment. Extraction of Bell states, as an original quantum resource, from mixed states under local operations and classical communication (LOCC) is known as entanglement distillation. It is a central task in entanglement theory [5]. This task is also the key method for constructing the distillable key, which supports the security proof in quantum cryptography [6]. Entanglement distillation is possible only if the mixed state is entangled. An unentangled state, also known as a separable state, is by definition a convex sum of product states [7]. Such states can be prepared locally in experiments. It is natural to pose the separability problem, i.e., to ask whether a given state is separable. It is known in computational complexity theory [8] that this problem is nondeterministic polynomial-time hard. Actually, both the entanglement distillation and separability problem cannot be effectively solved, even for bipartite states (for recent progress in a particular case see [9]).

For a bipartite state ρ acting on the Hilbert space $\mathcal{H} := \mathcal{H}_A \otimes \mathcal{H}_B$, the partial transpose computed in an orthonormal (o.n.) basis $\{|a_i\rangle\}$ of system A is defined by $\rho^\Gamma = \sum_{ij} |a_j\rangle\langle a_i| \otimes \langle a_i|\rho|a_j\rangle$. The dimensions of \mathcal{H}_A and \mathcal{H}_B are denoted by M and N , respectively. We say that ρ is a $k \times l$ state if its local ranks are k and l , i.e., $\text{rank } \rho_A = k$ and $\text{rank } \rho_B = l$. We say that ρ is a PPT [NPT] state if $\rho^\Gamma \geq 0$ [ρ^Γ has at least one negative eigenvalue]. Evidently, a separable state must be PPT. The converse is true only if $MN \leq 6$ [10,11], in which case the separability problem is solved. The

first examples of two-qudit PPT entangled states (PPTES) were constructed in purely mathematical context by Choi and Størmer in the 1980s [12,13]. They were introduced into quantum information theory in 1997 [14]. The full description of two-qudit PPTES of rank 4 was constructed in 2011 in Refs. [15,16] (independently). The most intriguing feature of PPTES are that they are not distillable, i.e., they cannot be converted into Bell states under LOCC. So PPTES are not directly useful for entanglement distillation. Nevertheless, some PPTES can be used to construct a distillable key [17].

In the bipartite setting, $2 \times N$ states are related to many problems in quantum information and have received a lot of attention. First, one of the most known analytical formulas for entanglement measures is the entanglement of formation of two-qubit states [18]. Part of the derivation of this formula relies on the observation that the two-qubit separable states have *lengths* of at most 4. The length of a separable state ρ , denoted by $L(\rho)$, is defined as the minimal number of pure product states whose mixture is ρ [19]. So it represents the minimal physical efforts that realize ρ by the entanglement of formation. Two separable states with different lengths are not equivalent under stochastic local operations and classical communications (SLOCC) [20].

On the other hand, the purification of a $2 \times N$ separable state ρ of rank r is a $2 \times N \times r$ tripartite pure state $|\psi\rangle$. So the tensor rank of the latter is not larger than the length of ρ [21]. This connection is computationally operational, since the tensor rank of $|\psi\rangle$ can be computed by efficient programs [22,23].

Second, a first systematic study of $2 \times N$ PPT states ρ was published in 1999 [24]. Their main result is that ρ is separable when its rank is equal to N . Recently, 2×4 extremal PPTES for various biranks have been constructed in Ref. [25]. Such states are extreme points of the set of PPT states and have been studied in bipartite systems of arbitrary dimensions [26]. Entanglement witnesses for physically detecting entanglement of ρ have also been studied [27].

Third, all $2 \times N$ NPT states are distillable [28], while the distillability of 3×3 NPT states still remains as a major open problem in entanglement theory.

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TABLE I. Lengths of separable 2×2 states ρ of birank (r, s) , $2 \leq r \leq s \leq 4$. (All such pairs that actually occur are listed.) Here $|e\rangle = |0\rangle + |1\rangle$.

(r, s)	$L(\rho)$	Example	Reducibility
(2,2)	2 (see [18])	$ 00\rangle\langle 00 + 11\rangle\langle 11 $	A, B reducible
(3,3)	3 (see [18])	$ 00\rangle\langle 00 + 11\rangle\langle 11 + e, e\rangle\langle e, e $	Irreducible
(3,4)	4 (see [18])	Example 2	Irreducible
(4,4)	4 (see [18])	$I \otimes I$	A, B reducible

Fourth, it has been shown that the $2 \times N$ states contain quantum correlation measured by quantum discord [29].

Motivated by a desire for deeper understanding of these results and their possible applications to various quantum-information tasks and to computational complexity, we continue in this paper the investigation of $2 \times N$ separable states and PPTES. After a preliminary technical Lemma 4, we prove in Corollary 1 that given a $2 \times N$ separable state σ we can subtract from it a pure product state to obtain another PPT state of lower birank. This result is essential for the computation of the length of a 2×3 separable state ρ of given birank (r, s) . Namely, we show in Proposition 3 that $L(\rho) = \max\{r, s\}$. We give in Table II concrete examples of separable states ρ for all possible lengths and biranks. Similar results for two-qubit separable states are shown in Table I. By using these results and Lemmas 5, 6, and 8, we determine the closure of the set \mathcal{E} of normalized two-qutrit PPTES of rank 4 (see Theorem 2). It turns out that this closure is the union of \mathcal{E} and the set \mathcal{S}'_4 of separable states of length at most 4. In Example 3, we construct a two-qutrit separable state ρ of rank 5, such that whenever $\sigma = \rho - |e, f\rangle\langle e, f|$ is a PPT state of birank equal to $(r - 1, s)$, $(r, s - 1)$, or $(r - 1, s - 1)$, then σ is necessarily entangled. This fact can be regarded in physics as the loss of separability by subtraction of a pure product state. In Theorem 3, we show that the $2 \times N$ separable state of birank $(N + 1, N + 1)$ has length $N + 1$. In the same theorem we show that a $2 \times N$ PPTES ρ of birank $(N + 1, N + 1)$ must be the B-direct sum of several pure product states and an edge state σ [30]. So two $2 \times N$ PPTES ρ_1 and ρ_2 of birank $(N + 1, N + 1)$ are equivalent under SLOCC only if the edge states σ_1 and σ_2 , and the pure product states are simultaneously equivalent under SLOCC. This is a method to the hard problem of deciding equivalent mixed states. Furthermore, the entanglement witness detecting the entanglement of the edge state σ would be able to detect the entanglement of the PPTES ρ .

In Proposition 4 we study the set of all product vectors contained in the orthogonal complement V^\perp of a completely entangled space V of dimension $k < N$. We show that this set is a vector bundle of rank $N - k$ over the projective line. In the special case $k = N - 1$, its projectivization is a rational normal curve, a well-known object in classical algebraic geometry. In Propositions 5 and 6, we prove the existence of $2 \times N$ separable as well as PPT entangled states having birank (r, s) , where r and s are arbitrary integers in the range $N + 1, \dots, 2N$. The proofs are based on Proposition 4 and the recently constructed PPTES in Ref. [31]. Finally, in Example 5, for each $m \in \{1, \dots, N - 1\}$, we construct a $2 \times N$

NPT state whose partial transpose has exactly m negative eigenvalues.

The paper is organized as follows. In Sec. II we state the known facts which we often use in this paper. In Sec. III we solve the length problem for 2×3 separable states. The main result is presented in Proposition 3. In Sec. IV we determine the closure of 3×3 PPTES of rank 4. The main result is stated in Theorem 2. In Sec. V we study the $2 \times N$ PPT states of prescribed rank. The main results are presented in Theorem 3, Proposition 4, and Propositions 5 and 6.

II. PRELIMINARIES

We shall write I_k for the identity $k \times k$ matrix. We denote by $\mathcal{R}(\rho)$ and $\ker \rho$ the range and kernel of a linear map ρ , respectively. From now on, unless stated otherwise, the states will not be normalized. We shall denote by $\{|i\rangle_A : i = 0, \dots, M - 1\}$ and $\{|j\rangle_B : j = 0, \dots, N - 1\}$ o.n. bases of \mathcal{H}_A and \mathcal{H}_B , respectively. The subscripts A and B will often be omitted. For any bipartite state ρ we have

$$(\rho^\Gamma)_B = \text{Tr}_A(\rho^\Gamma) = \text{Tr}_A \rho = \rho_B, \quad (1)$$

$$(\rho^\Gamma)_A = \text{Tr}_B(\rho^\Gamma) = (\text{Tr}_B \rho)^T = (\rho_A)^T. \quad (2)$$

Here the exponent T denotes transposition. Consequently,

$$\text{rank}(\rho^\Gamma)_{A,B} = \text{rank} \rho_{A,B}. \quad (3)$$

If ρ is an $M \times N$ PPT state, then ρ^Γ is too. If ρ is a PPTES, so is ρ^Γ , but they may have different ranks. An example is the two-qubit separable state of birank (3,4) (see Table I).

Let us now recall some basic results from quantum information regarding the separability and PPT properties of bipartite states. Let us start with the basic definition.

Definition 1. We say that two n -partite states ρ and σ are equivalent under stochastic local operations and classical communications (SLOCC equivalent or just equivalent) if there exists an invertible local operator (ILO) $A = \bigotimes_{i=1}^n A_i \in \text{GL} := \text{GL}_{d_1}(\mathbb{C}) \times \dots \times \text{GL}_{d_n}(\mathbb{C})$ such that $\rho = A\sigma A^\dagger$ [20].

In most cases of the present work, we will have $n = 2$. It is easy to see that any ILO transforms PPT, entangled, or separable state into the same kind of states. The length of a separable state is invariant under ILO and is nonincreasing under all local operations. We shall often use ILOs to simplify the density matrices of states. We say that a subspace of \mathcal{H} is *completely entangled* (CES) if it contains no product vectors. We require product vectors to be nonzero. For counting purposes we do not distinguish product vectors which are scalar multiples of each other.

We recall that $D = d_1 d_2 \dots d_n - \sum_{i=1}^n d_i + n - 1$ is the maximal dimension of CES in $d_1 \otimes \dots \otimes d_n$ [32]. It follows easily from [26, Theorem 60] that any CES is contained in one of dimension D .

The first assertion of the following theorem is ([9], Theorem 23). The second one follows from its proof where only parameter a was shown to be real and nonzero. The stronger claim that (like b, c, d) a can also be chosen to be positive has been proved in Ref. ([33], Theorem 7).

Proposition 1. ($M = N = 3$) Any 3×3 PPTES ρ of rank 4 is SLOCC equivalent to one which is invariant under partial transpose, i.e., there exist $A, B \in \text{GL}_3(\mathbb{C})$ such that

$\sigma := A \otimes B \rho A^\dagger \otimes B^\dagger$ satisfies the equality $\sigma^\Gamma = \sigma$. Moreover, we may assume that $\sigma = C^\dagger C$, where $C = [C_0 C_1 C_2]$ and

$$C_0 = \begin{bmatrix} 0 & a & b \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & 0 & 1 \\ 1 & 0 & -1/d \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 & -1/b & 0 \\ 0 & 1 & 0 \\ 1 & -c & 0 \\ d & 0 & 0 \end{bmatrix}; \quad a, b, c, d > 0. \quad (4)$$

This equation will be used to show that separable states of length at most 4 are in the closure of the set of non-normalized 3×3 PPTES of rank 4 in Lemma 5. To prove this lemma we will need the definition of the term ‘‘general position’’ ([26], Definition 7).

Definition 2. We say that a family of product vectors $\{|\psi_i\rangle = |\phi_i\rangle \otimes |\chi_i\rangle : i \in I\}$ is in general position (in \mathcal{H}) if for any $J \subseteq I$ with $|J| \leq M$ the vectors $|\phi_j\rangle, j \in J$ are linearly independent, and for any $K \subseteq I$ with $|K| \leq N$ the vectors $|\chi_k\rangle, k \in K$ are linearly independent.

The next result is from ([34], Theorem 1). It is useful in the characterization of the length of 2×3 separable states.

Theorem 1. The $M \times N$ states of rank less than M or N are 1 distillable, and consequently, they are NPT.

The next result follows from ([9], Theorem 10), Ref. [35], and Theorem 1, see also ([9], Proposition 6 (ii)).

Proposition 2. Let ρ be an $M \times N$ state of rank N .

(i) If ρ is PPT, then it is a sum of N pure product states. Consequently, $\text{rank } \rho > \max(\text{rank } \rho_A, \text{rank } \rho_B)$ for any PPTES ρ , and any bipartite PPT state of rank ≤ 3 is separable.

(ii) If ρ is NPT, then it is 1 distillable.

We shall apply Proposition 2 to the problems of computing the length of separable states to find the closure of the set of 3×3 PPTES of rank 4 and to characterize $2 \times N$ separable states studied in Secs. III–V. So it is an important fact which we use throughout this paper.

Another useful concept (based on ([9], Definition 11)) in this paper is that of reducible and irreducible states, which we are going to introduce now.

Definition 3. A linear operator $\rho: \mathcal{H} \rightarrow \mathcal{H}$ is an A -direct sum of linear operators $\rho_1: \mathcal{H} \rightarrow \mathcal{H}$ and $\rho_2: \mathcal{H} \rightarrow \mathcal{H}$, written as $\rho = \rho_1 \oplus_A \rho_2$, if $\mathcal{R}(\rho_A) = \mathcal{R}[(\rho_1)_A] \oplus \mathcal{R}[(\rho_2)_A]$. A bipartite state ρ is A reducible if it is an A -direct sum of two states; otherwise ρ is A irreducible. One defines similarly the B -direct sum $\rho = \rho_1 \oplus_B \rho_2$, the B -reducible and the B -irreducible states. A state ρ is reducible if it is either A or B reducible. A state ρ is irreducible if it is both A and B irreducible.

The next result is from ([26], Lemma 15).

Lemma 1. Let ρ_1 and ρ_2 be linear operators on \mathcal{H} .

(i) If $\rho = \rho_1 \oplus_B \rho_2$, then $\rho^\Gamma = \rho_1^\Gamma \oplus_B \rho_2^\Gamma$.

(ii) If ρ_1 and ρ_2 are Hermitian and $\rho = \rho_1 \oplus_A \rho_2$, then $\rho^\Gamma = \rho_1^\Gamma \oplus_A \rho_2^\Gamma$.

(iii) If a PPT state ρ is reducible, then so is ρ^Γ .

Let us recall a related result ([9], Corollary 16).

Lemma 2. Let $\rho = \sum_i \rho_i$ be an A - or B -direct sum of the states ρ_i . Then ρ is separable [PPT] if and only if each ρ_i is

separable [PPT]. Consequently, ρ is a PPTES if and only if each ρ_i is PPT and at least one of them is entangled.

III. LENGTHS OF SEPARABLE STATES IN $2 \otimes 3$

We shall need the following result from ([24], Corollary 1, Lemma 2). Their proof is based on their Lemma 1 and is valid for arbitrary M, N . If a Hermitian operator ρ is not invertible, then ρ^{-1} will denote its pseudoinverse. (If $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, p_i > 0$, is the spectral decomposition, then $\rho^{-1} = \sum_i p_i^{-1} |\psi_i\rangle\langle\psi_i|$.)

Lemma 3. Let ρ be a (non-normalized) bipartite PPT state of birank (r, s) and let $\sigma = \rho - \lambda |e, f\rangle\langle e, f|$, where $|e, f\rangle$ is a product vector and λ is a real number. Set $\lambda_0 = (|e, f\rangle\langle e, f| \rho^{-1} |e, f\rangle)^{-1}$ and $\lambda_1 = [|e^*, f\rangle\langle e^*, f| (\rho^\Gamma)^{-1} |e^*, f\rangle]^{-1}$. Then σ is a PPT state if and only if $|e, f\rangle \in \mathcal{R}(\rho), |e^*, f\rangle \in \mathcal{R}(\rho^\Gamma)$, and $\lambda \leq \min(\lambda_0, \lambda_1)$. Moreover, if σ is a PPT state, then its birank is $(r, s), (r - 1, s), (r, s - 1)$, or $(r - 1, s - 1)$, according to whether $\lambda < \min(\lambda_0, \lambda_1), \lambda = \lambda_0 < \lambda_1, \lambda = \lambda_1 < \lambda_0$, or $\lambda = \lambda_0 = \lambda_1$.

Alternatively, this lemma follows from the following simple fact: If $\rho \geq 0$ acts on \mathcal{H} , $\lambda \in \mathbf{R}$, and $|\phi\rangle \in \mathcal{R}(\rho)$ is a nonzero vector, then $\rho - \lambda |\phi\rangle\langle\phi| \geq 0$ if and only if $\lambda \langle\phi| \rho^{-1} |\phi\rangle \leq 1$. Indeed, $\rho - \lambda |\phi\rangle\langle\phi| \geq 0$ is equivalent to $\text{id} - \lambda \rho^{-1/2} |\phi\rangle\langle\phi| \rho^{-1/2} \geq 0$. It remains to observe that $\rho^{-1/2} |\phi\rangle\langle\phi| \rho^{-1/2}$ is a Hermitian operator of rank 1 with the nonzero eigenvalue $\langle\phi| \rho^{-1} |\phi\rangle$.

Next we strengthen part (i) of ([24], Lemma 11).

Lemma 4. ($N \geq M = 2$) Let $V[W]$ be a subspace of the $2 \otimes N$ Hilbert space \mathcal{H} of dimension $k[l]$ with $k + l > 3N$. Then for each unit vector $|a\rangle \in \mathcal{H}_A$ there exist infinitely many pairwise nonparallel unit vectors $|y\rangle \in \mathcal{H}_B$ such that $|a, y\rangle \in V$ and $|a^*, y\rangle \in W$. Moreover, the set S of all such pairs $(|a\rangle, |y\rangle)$ is connected.

Proof. For the first assertion we essentially follow the proof of ([24], Lemma 11). Let $f_i (i = 1, \dots, 2N - k)$ and $g_j (j = 1, \dots, 2N - l)$ be linear functions $\mathcal{H} \rightarrow \mathbf{C}$ such that $V = \cap_i \ker f_i$ and $W = \cap_j \ker g_j$. Let $S_A [S_B]$ denote the unit sphere of $\mathcal{H}_A [\mathcal{H}_B]$. Let us fix $|a\rangle \in S_A$. We have $|a, y\rangle \in V$ if and only if $f_i(|a, y\rangle) = 0$ for all i , and $|a^*, y\rangle \in W$ if and only if $g_j(|a^*, y\rangle) = 0$ for all j . Since $k + l > 3N$, we have $(2N - k) + (2N - l) < N$ and so the space of solutions of the system of these $4N - k - l$ homogeneous linear equations for the unknown vector $|y\rangle$ has (complex) dimension $d_a \geq k + l - 3N \geq 1$. Hence, the set S_a of all $|y\rangle \in S_B$ such that $|a, y\rangle \in V$ and $|a^*, y\rangle \in W$ is the unit sphere in some complex subspace of \mathcal{H} of dimension d_a . In particular, S_a is connected.

Note that S is a closed subset of the product $S_A \times S_B$ and so it is compact. Let $p_1: S \rightarrow S_A$ be the restriction of the first projection map $S_A \times S_B \rightarrow S_A$. We have just shown that p_1 is onto and that all of its fibers are connected. This implies that S itself is connected. ■

We remark that in fact S is a real algebraic subset of $S_A \times S_B$ and that $\text{Dim } S \geq 2(k + l - 3N) - 1$.

From the lemma we deduce an important corollary.

Corollary 1. Let ρ be a $2 \times N$ separable state of birank (r, s) with $r \leq s$.

(i) If $r = s$ and $2r > 3N$, then there is a product vector $|e, f\rangle$ such that $\sigma := \rho - |e, f\rangle\langle e, f|$ is a PPT state of birank $(r - 1, r - 1)$.

TABLE II. Lengths of separable 2×3 states ρ of birank (r, s) with $3 \leq r \leq s \leq 6$. (All such pairs that actually occur are listed.) In the example of birank $(4, 6)$, we have $|f\rangle = |0\rangle - |1\rangle$, $|g\rangle = |0\rangle + |1\rangle + |2\rangle$, $|a_0, b_0\rangle = F|i\rangle$, $|a_1, b_1\rangle = F[-i]$, where $F[x] := ((1+x)/(x-1), 1)^T \otimes (-1, (x-1)/(x+1), x-1)^T$. Another example of birank $(4, 6)$ is constructed in Example 4.

(r, s)	$L(\rho)$	Example	Reducibility
$(3, 3)$	3 (see Proposition 2)	$ 00\rangle\langle 00 + 11\rangle\langle 11 + 12\rangle\langle 12 $	A, B reducible
$(4, 4)$	4 (see Proposition 3)	$ 00\rangle\langle 00 + 01\rangle\langle 01 + 11\rangle\langle 11 + 12\rangle\langle 12 $	A, B reducible
$(4, 5)$	5 (see Proposition 3)	Example 1	B reducible
$(4, 6)$	6 (see Proposition 3)	$ 00\rangle\langle 00 + 11\rangle\langle 11 + e, 2\rangle\langle e, 2 + f, g\rangle\langle f, g + a_0, b_0\rangle\langle a_0, b_0 + a_1, b_1\rangle\langle a_1, b_1 $	Irreducible
$(5, 5)$	5 (see Proposition 3)	$ 00\rangle\langle 00 + 01\rangle\langle 01 + 02\rangle\langle 02 + 11\rangle\langle 11 + 12\rangle\langle 12 $	A, B reducible
$(5, 6)$	6 (see Proposition 3)	Example 1	B reducible
$(6, 6)$	6 (see Proposition 3)	$I \otimes I$	A, B reducible

(ii) If $r < s$ then there is a product vector $|e, f\rangle$ such that $\sigma := \rho - |e, f\rangle\langle e, f|$ is a PPT state of birank $(r, s - 1)$.

Proof. We have $\rho = \sum_{i=1}^k |a_i, b_i\rangle\langle a_i, b_i|$ where $k = L(\rho)$. The real-valued function g defined on the set of product vectors by $g(|e, f\rangle) = \langle e, f | \rho^{-1} | e, f \rangle - \langle e^*, f | (\rho^\Gamma)^{-1} | e^*, f \rangle$ is continuous. Note that $\sum_i g(|a_i, b_i\rangle) = \text{Tr}(\rho\rho^{-1}) - \text{Tr}[\rho^\Gamma(\rho^\Gamma)^{-1}] = r - s$.

In case (i) we have $\sum_i g(|a_i, b_i\rangle) = 0$, and so $g(|a_i, b_i\rangle) \geq 0 \geq g(|a_j, b_j\rangle)$ for some i and j . By Lemma 4, the set S of normalized product vectors $|e, f\rangle \in \mathcal{R}(\rho)$ such that $|e^*, f\rangle \in \mathcal{R}(\rho^\Gamma)$ is connected. Consequently, we have $g(|e, f\rangle) = 0$ for some product vector $|e, f\rangle$. The assertion now follows from Lemma 3 by using this vector and setting $\lambda = (\langle e, f | \rho^{-1} | e, f \rangle)^{-1}$.

In case (ii) we have $\sum_i g(|a_i, b_i\rangle) < 0$ and so there exists an index i such that $g(|a_i, b_i\rangle) < 0$, i.e., $(\langle a_i, b_i | \rho^{-1} | a_i, b_i \rangle)^{-1} > [\langle a_i^*, b_i | (\rho^\Gamma)^{-1} | a_i^*, b_i \rangle]^{-1}$. Hence the assertion follows from Lemma 3. ■

Proposition 3. If ρ is a 2×3 separable state of birank (r, s) , then $L(\rho) = \max(r, s)$.

Proof. Without any loss of generality, we may assume that $r \leq s$. We recall that $L(\rho) \geq s$ always holds, and that any PPT state in $2 \otimes 3$ is separable. By Theorem 1, we have $r \geq 3$.

If $r = 3$ then Proposition 2 shows that also $s = 3$ and that $L(\rho) = 3$.

Let $r = 4$. If $s = 4$ then $L(\rho) = 4$ by Theorem 3. If $s = 5$ or 6 we can apply Corollary 1 (ii) once or twice, respectively, to reduce these cases to $s = 4$.

Let $r = 5$. If also $s = 5$ then we can apply Corollary 1 (i) to obtain that $\rho = \sigma + |e, f\rangle\langle e, f|$, where σ is a separable state of birank $(4, 4)$. Hence, $L(\sigma) = 4$ and so $L(\rho) = 5$. If $s = 6$ we can apply Corollary 1 (ii) to reduce it to the case $s = 5$. ■

In Table I, we recall the well-known facts concerning the lengths of separable 2×2 states [18,36] (see also ([37], Sec. III)). Our results concerning the lengths of separable 2×3 states are summarized in Table II. In particular, note that we have proved that $L(\rho) \leq 6$ for all separable states on $2 \otimes 3$. Thus ([38], Conjecture 10) is valid in this case. By inspecting these two tables, it appears that there exist separable states ρ of birank (r, s) when $\text{rank } \rho > \max(\text{rank } \rho_A, \text{rank } \rho_B)$. In Proposition 5 below, we shall prove that this is indeed the case for $2 \times N$ separable states. However, it is false for separable states in general, see Proposition 6.

IV. CLOSURE OF 3×3 PPTES OF RANK FOUR

The equivalence classes of states are just the orbits under the action of the group $G = \text{GL}_3(\mathbf{C}) \times \text{GL}_3(\mathbf{C})$. The set \mathcal{E}' of non-normalized 3×3 PPTES of rank 4 is G -invariant and the quotient space \mathcal{E}'/G parametrizes the set of equivalence classes of 3×3 PPTES of rank 4. We equip \mathcal{E}'/G with the quotient topology and let $\pi : \mathcal{E}' \rightarrow \mathcal{E}'/G$ be the projection map. In this section we shall determine the closure $\overline{\mathcal{E}'}$ of the set \mathcal{E}' in the ordinary (Euclidean) topology. Note that the closure $\overline{\mathcal{E}}$ of \mathcal{E} is the intersection $\overline{\mathcal{E}'} \cap H$, where H is the space of normalized Hermitian matrices.

A quantum state ρ belongs to the closure $\overline{\mathcal{E}'}$ of the set \mathcal{E}' if and only if there exists an infinite series of states $\rho_1, \rho_2, \dots \in \mathcal{E}'$ such that $\lim_{i \rightarrow \infty} \|\rho_i - \rho\| = 0$. So this closure is a set of states attached to the set of two-qutrit PPTES of rank 4. The former can be investigated by using the properties of the latter. We observe that if $\sigma \in \overline{\mathcal{E}'} \setminus \mathcal{E}'$, then σ must be separable and both σ and σ^Γ must have rank at most 4. This observation can be used to show that there exist separable states of rank 4 which are not in $\overline{\mathcal{E}'}$. We give an example by modifying ([26], Example 40).

Example 1. The separable 2×3 state $\sigma = |00\rangle\langle 00| + |02\rangle\langle 02| + 2|11\rangle\langle 11| + (|01\rangle + |10\rangle)(\langle 01| + \langle 10|)$ has birank $(4, 5)$. We have $L(\sigma) = L(\sigma^\Gamma) \geq \text{rank } \sigma^\Gamma = 5$. Since $\sigma - |02\rangle\langle 02|$ is a two-qubit separable state, its length is at most 4 [18,36]. Hence, $L(\sigma)$ must be 5. As σ^Γ has rank 5, $\sigma \notin \overline{\mathcal{E}'}$.

Similarly, the separable 2×3 state $\sigma + |12\rangle\langle 12|$ has birank $(5, 6)$ and length 6. ■

On the other hand, we have the following result.

Lemma 5. ($M = N = 3$) We have $S'_4 \subseteq \overline{\mathcal{E}}$.

Proof. For convenience, we shall work with non-normalized states. It suffices to prove that if $\sigma = \sum_{i=0}^3 |a_i, b_i\rangle\langle a_i, b_i|$, where the four product vectors $|a_i, b_i\rangle$ are in general position, then $\sigma \in \overline{\mathcal{E}'}$. Since \mathcal{E}' and $\overline{\mathcal{E}'}$ are G -invariant, we may assume that

$$\sigma = \sum_{i=0}^2 p_i |ii\rangle\langle ii| + |e_A, e_B\rangle\langle e_A, e_B|, \tag{5}$$

where $|e\rangle_A = \sum_i |i\rangle_A$, $|e\rangle_B = \sum_i |i\rangle_B$, and p_i are the positive scalars.

We consider the states $\rho = \rho(a, b, d) = C^\dagger C$, where $C = [C_0 C_1 C_2]$ and the blocks C_i are 4×3 matrices in Eq. (4) with $c = 0$. Clearly, ρ belongs to the closure of \mathcal{E}' . It is easy to

verify that $\rho = \sum_{i=0}^3 p_i |v_i\rangle\langle v_i|$, where

$$p_0 = \frac{1}{1+b^2}, \quad p_1 = \frac{1}{1+d^2}, \tag{6}$$

$$p_2 = \frac{1}{d^2(1+d^2)}, \quad p_3 = \frac{1}{b^2(1+b^2)};$$

$$|v_0\rangle = |0\rangle \otimes [ab|1\rangle + (1+b^2)|2\rangle], \tag{7}$$

$$|v_1\rangle = [d|1\rangle + (1+d^2)|2\rangle] \otimes |0\rangle, \tag{8}$$

$$|v_2\rangle = |1\rangle \otimes [d|0\rangle - (1+d^2)|2\rangle], \tag{9}$$

$$|v_3\rangle = [ab|0\rangle - (1+b^2)|2\rangle] \otimes |1\rangle. \tag{10}$$

Let $V = b(1+b^2)^{-3/2} V_A \otimes V_B$, where

$$V_A = \begin{bmatrix} (1+b^2)/ab & 0 & 0 \\ 0 & 0 & -1 \\ 0 & (1+d^2)/d & -1 \end{bmatrix}, \tag{11}$$

$$V_B = \begin{bmatrix} 0 & 1+b^2 & 0 \\ -ab(1+d^2)/d & 1+b^2 & -ab \\ 0 & 1+b^2 & -ab \end{bmatrix}.$$

A computation shows that $V\rho V^\dagger = \sigma$, provided we choose the positive parameters a, b, d such that

$$b^2 = p_0, \quad \frac{a^2 b^4}{d^2} \left(\frac{1+d^2}{1+b^2} \right)^3 = p_1, \quad d^2 = \frac{p_1}{p_2}. \tag{12}$$

We can now show that $\overline{\mathcal{E}^r}$ contains many separable states.

Lemma 6. Separable states of rank at most 3 have length at most 4.

Proof. Let ρ be a separable $k \times l$ state of rank $r \leq 3$. We may assume that $k \leq l$. By ([34], Theorem 1), we have $l \leq r$. The assertion is trivial if $l = 1$, it follows from [18,36] if $l = 2$, and from ([26], Proposition 9) if $l = 3$. ■

Lemma 7. The maximum length of 3×3 separable states of rank 4 is 5.

Proof. Separable 3×3 states of rank 4 and length 5 exist, see, e.g., ([26], Example 40). Let ρ be any 3×3 separable state of rank 4 and length $r > 4$. Thus we have $\rho = \sum_{i=0}^{r-1} |a_i, b_i\rangle\langle a_i, b_i|$. We may assume that the $|a_i, b_i\rangle$ with $i < 4$ are linearly independent. By ([26], Lemma 29), these four product vectors are not in general position. Consequently, we may assume that $|b_0\rangle = |0\rangle$, $|b_1\rangle = |1\rangle$, $|b_3\rangle = |2\rangle$, and $\langle b_2|2\rangle = 0$. Moreover, we may assume that $|a_i\rangle = |a_3\rangle$ for $3 \leq i \leq s < r$, while for $i > s$ the vectors $|a_i\rangle$ are not parallel to $|a_3\rangle$. It is not hard to show that we can rewrite $\sum_{i=3}^s |b_i\rangle\langle b_i|$ as $|b'_3\rangle\langle b'_3| + \sigma$, where σ is a state on \mathcal{H}_B such that $\sigma|2\rangle = 0$. Clearly, we have $\langle b'_3|2\rangle \neq 0$ and so $\mathcal{R}(\rho)$ is spanned by $|a_i, b_i\rangle$, $i = 0, 1, 2$ and $|a_3, b'_3\rangle$. Since $|a_i, b_i\rangle \in \mathcal{R}(\rho)$, it follows that for $i > s$ we must have $\langle b_i|2\rangle = 0$. Consequently, we have a B -direct decomposition $\rho = \rho' \oplus_B |a_3, b'_3\rangle\langle a_3, b'_3|$. Since ρ' is separable of rank 3, its length is at most 4 by Lemma 6. Hence ρ has length 5. ■

From the lemma we obtain

Corollary 2. A 3×3 separable state ρ of rank 4 has length 5 if and only if it is an A - or B -direct sum of a pure product state and a separable state σ of rank 3 and length 4.

Proof. Necessity. See the proof of Lemma 7.

Sufficiency. Suppose that $\rho = \sigma \oplus_B |a, b\rangle\langle a, b|$, with σ a separable state of rank 3 and length 4. As length does not increase under local operations, we have $L(\rho) \geq L(\sigma) = 4$. Assume that $L(\rho) = 4$ and so $\rho = \sum_{i=0}^3 |a_i, b_i\rangle\langle a_i, b_i| = \sigma \oplus_B |a, b\rangle\langle a, b|$. Suppose $\langle b|b_i\rangle \neq 0$ for $i = 0, \dots, s$. Then for these subscripts $|a_i\rangle$ are pairwise parallel, and we may assume $\langle b|b_i\rangle \neq 0$ for only $i = 0$. Thus $|b_0\rangle$ is proportional to $|b\rangle$. The equality $\sum_{i=0}^3 |a_i, b_i\rangle\langle a_i, b_i| = \sigma \oplus_B |a, b\rangle\langle a, b|$ indicates rank $\sigma = 3$, which gives us a contradiction. This completes the proof. ■

Lemma 8. A 3×3 separable state has birank (4,4) if and only if it has length 4.

Proof. Necessity. Suppose ρ is a 3×3 separable state of birank (4,4). By Lemma 7, $L(\rho) \leq 5$. Assume that $L(\rho) = 5$. By using Corollary 2 we obtain that, say, $\rho = \sigma \oplus_A |a, b\rangle\langle a, b|$, where σ is a 2×2 or 2×3 separable state of rank 3 and length 4. It follows from Proposition 2 (a) that σ must be 2 state. From Table I, we see that rank $\sigma^\Gamma = 4$. By Lemma 1 (ii), we have $\rho^\Gamma = \sigma^\Gamma \oplus_A |a^*, b\rangle\langle a^*, b|$. Therefore rank $\rho^\Gamma = 5$, which gives a contradiction. So ρ must have length 4.

Sufficiency. Suppose ρ is a 3×3 separable state of length 4. Suppose its birank is (r, s) , then $4 \geq r, s \geq 3$. If either of r, s is equal to 3, then $L(\rho) = 3$ by using Proposition 2. It gives us a contradiction, so $r = s = 4$. ■

We can now prove the main result of this section.

Theorem 2. ($M = N = 3$) We have $\overline{\mathcal{E}} = \mathcal{E} \cup \mathcal{S}'_4$.

Proof. Let $\rho \in \overline{\mathcal{E}}$ be separable. Then ρ is a $k \times l$ state of birank (r, s) with $\max(r, s) \leq 4$. In view of Lemma 5, it suffices to prove that $L(\rho) \leq 4$. Recall that $L(\rho) = L(\rho^\Gamma)$. If $r < 4$ or $s < 4$ then $L(\rho) \leq 4$ by Lemma 6. Assume now that $r = s = 4$. If $k = l = 3$ then $L(\rho) \leq 4$ by Lemma 8. If (k, l) is equal to (2,3) or (3,2), then $L(\rho) = 4$ by Proposition 3. Otherwise, $k = l = 2$ and $L(\rho) \leq 4$ by [19]. Hence, the proof is completed. ■

Recall that any $\rho \in \mathcal{E}'$ is equivalent to ρ^Γ ([15], Theorem 23). The following example shows that this property does not extend to $\overline{\mathcal{E}^r}$.

Example 2. The separable 2×2 state $\sigma = 2|00\rangle\langle 00| + |11\rangle\langle 11| + (|01\rangle + |10\rangle)(\langle 01| + \langle 10|)$ has birank (3,4), and so σ is not equivalent to σ^Γ . On the other hand, since $L(\sigma) = 4$, we have $\sigma \in \overline{\mathcal{E}^r}$ by Lemma 5. Explicitly, we have

$$\sigma = |00\rangle\langle 00| + \frac{1}{3} (|\psi_0\rangle\langle \psi_0| + |\psi_1\rangle\langle \psi_1| + |\psi_2\rangle\langle \psi_2|), \tag{13}$$

$$|\psi_k\rangle = (|0\rangle + \zeta^k |1\rangle) \otimes (|0\rangle + \zeta^k |1\rangle), \quad k = 0, 1, 2, \tag{14}$$

where $\zeta := (-1 + i\sqrt{3})/2$ is a primitive cube root of unity.

We can now show that the quotient space $\overline{\mathcal{E}^r}/G$ is not Hausdorff. Indeed, let (ρ_i) be a sequence in \mathcal{E}' converging to σ . Then the sequence (ρ_i^Γ) converges to σ^Γ . Consequently, the sequence $(G \cdot \rho_i)$ converges to $G \cdot \sigma$ and the sequence $(G \cdot \rho_i^\Gamma)$ converges to $G \cdot \sigma^\Gamma$ in the space $\overline{\mathcal{E}^r}/G$. But these two sequences coincide because ρ_i^Γ is equivalent to ρ_i for each i . On the other hand, the points $G \cdot \sigma$ and $G \cdot \sigma^\Gamma$ are distinct because the states σ^Γ and σ are not equivalent (they have different ranks). Hence, the sequence $(G \cdot \rho_i)$ converges to two different points and we conclude that the space $\overline{\mathcal{E}^r}/G$ is not Hausdorff. ■

Finally, we propose an application of two-qutrit PPTES of rank 4. Consider a separable state ρ of birank (r, s) , and the

set S of product vectors $|e, f\rangle \in \mathcal{R}(\rho)$ and $|e^*, f\rangle \in \mathcal{R}(\rho^\Gamma)$, such that $\sigma = \rho - |e, f\rangle\langle e, f|$ is a PPT state of birank equal to $(r - 1, s), (r, s - 1)$, or $(r - 1, s - 1)$. We are going to construct a family of ρ such that any σ is PPTES.

Example 3. ($M = N = 3$) Let ρ be a 3×3 PPTES of rank 4. Then $\ker \rho$ contains exactly six product vectors (up to a scalar factor) $|\psi_i\rangle$, $i = 1, \dots, 6$, and moreover, any five of these vectors are linearly independent (see Ref. [15]). Consequently, the six rank-1 operators $|\psi_i\rangle\langle\psi_i|$ are linearly independent. Since ρ^Γ is also a 3×3 PPTES of rank 4, the partial conjugates of the $|\psi_i\rangle$ have similar properties.

We consider the separable state

$$\sigma = \sum_{i=1}^6 |\psi_i\rangle\langle\psi_i| \tag{15}$$

of birank $(5, 5)$. Let $|e, f\rangle$ be a product vector such that $\sigma' := \sigma - |e, f\rangle\langle e, f|$ is a PPT state of birank (r, s) with $r \leq 5, s \leq 5$ and $r + s < 10$. (By Lemma 3, we know that such a product vector exists.) By the same lemma, $|e, f\rangle\langle e, f|$ must be a scalar multiple of some $|\psi_i\rangle\langle\psi_i|$, say $|e, f\rangle\langle e, f| = c|\psi_1\rangle\langle\psi_1|$. Clearly, we must have $c > 1$.

We claim that σ' must be entangled. Indeed, if σ' is separable, then it can be written as $\sigma' = \sum_i c_i |\psi_i\rangle\langle\psi_i|$ with $c_i \geq 0$. Since the $|\psi_i\rangle\langle\psi_i|$ are linearly independent, it follows that $c_1 = 1 - c$. Hence, $c = 1 - c_1 \leq 1$, which gives a contradiction. ■

We do not know that whether there is a similar example in $2 \otimes 4$. The following lemma is evident. It implies that the length of the state (15) is 6.

Lemma 9. Let ρ be a separable state with $\text{rank } \rho = L(\rho) = r$. Then there is a product vector $|a, b\rangle$, such that $\sigma := \rho - |a, b\rangle\langle a, b|$ is a separable state with $\text{rank } \sigma = L(\sigma) = r - 1$.

V. QUBIT-QUDIT PPT STATES WITH PRESCRIBED BIRANK

So far we have mainly focused on 2×3 and 3×3 PPT states. In this section we investigate some typical types of $2 \times N$ PPT states ρ for arbitrary N . In Theorem 3 we characterize both separable and PPT entangled states ρ of birank $(N + 1, N + 1)$. This case is different from those discussed in Corollary 1. In Proposition 4 we study the properties of the set of product vectors contained in V^\perp , where V is a CES of dimension $k < N$ in $2 \otimes N$. It turns out that this set (with zero vectors included) is a vector bundle of rank $N - k$ over the projective line \mathcal{P}^1 . In the special case $k = N - 1$, the projectivization of this set is a rational normal curve. In Propositions 5 and 6 we construct separable states and PPTES of any birank (r, s) with $r, s > N$. The constructions are based on Proposition 4 and the recently constructed PPTES in Ref. [31]. Finally we obtain a result on NPT states. In Example 5, for each $m = 1, \dots, N - 1$, we construct a $2 \times N$ NPT state whose partial transpose has exactly m negative eigenvalues.

A PPT state ρ is an *edge state* if there is no product vector $|a, b\rangle \in \mathcal{R}(\rho)$ such that $|a^*, b\rangle \in \mathcal{R}(\rho^\Gamma)$. Any edge state is necessarily entangled. Any bipartite PPTES is the sum of a separable state and an edge state [39]. So, in the bipartite case, edge states play the role of “extreme points” in the set of

PPTES. It is useful to describe the structure of states in the following family.

Theorem 3. Let ρ be a $2 \times N$ PPT state of birank $(N + 1, N + 1)$.

(i) If ρ is separable then $L(\rho) = N + 1$.

(ii) If ρ is entangled then $\rho = \sigma \oplus_B |a_1, b_1\rangle\langle a_1, b_1| \oplus_B \dots \oplus_B |a_r, b_r\rangle\langle a_r, b_r|$, where σ is an edge state of birank $(N + 1 - r, N + 1 - r)$.

Proof. (i) First note that $L(\rho) \geq \text{rank } \rho = N + 1$. Table I shows that the assertion is true for $N = 2$. We proceed by induction on N . Now let $N > 2$. Since ρ is separable, by Lemma 3 we have $\rho = \sigma + |e, f\rangle\langle e, f|$, where σ is a PPT state of birank $(N, N + 1), (N + 1, N)$ or (N, N) , and $|e, f\rangle$ is a product vector. If $\text{rank } \sigma_A = 1$, the assertion clearly holds, and so we may assume that $\text{rank } \sigma_A = 2$. Since $\rho_B = \sigma_B + \|e\|^2 |f\rangle\langle f|$, we have $\text{rank } \sigma_B = N$ or $N - 1$. If $\text{rank } \sigma_B = N$, the assertion follows from Proposition 2. Otherwise, $\text{rank } \sigma_B = N - 1$ and Lemma 2 shows that σ is separable of birank (N, N) . By the induction hypothesis, $L(\sigma) = N$ and consequently, $L(\rho) = N + 1$.

(ii) If ρ is an edge state, then the assertion holds with $r = 0$. Otherwise, by Lemma 3, we have $\rho = \sigma + |e, f\rangle\langle e, f|$, where σ is a PPT state of birank $(N, N + 1), (N + 1, N)$ or (N, N) , and $|e, f\rangle$ is a product vector. As ρ is entangled, we must have $\text{rank } \sigma_A = 2$. We also have $\text{rank } \sigma_B = N$ or $N - 1$. Proposition 2 implies that $\text{rank } \sigma_B = N - 1$, and so $\rho = \sigma \oplus_B |e, f\rangle\langle e, f|$. By Lemma 1 (i), σ has birank (N, N) . We can continue to apply this procedure of splitting off a pure product state as long as the entangled summand is not an edge state. Eventually, this summand must become an edge state. This completes the proof. ■

We point out that part (i) generalizes the $2 \otimes 3$ case in Table II, and that part (ii) was also discussed in Ref. ([24], Sec. IV B). We further point out that $M \times N$ PPT states ρ , with $N \geq M \geq 3$, of rank $N + 1$ have been investigated in Ref. ([26], Theorems 44, 45). In particular, the first of these theorems implies that $\rho = \rho_1 \oplus_B \dots \oplus_B \rho_k \oplus_B \sigma$, where ρ_i are pure product states and σ is a B -irreducible state. Note that this decomposition is similar to one in Theorem 3 (ii). In physics, such a decomposition means that the entanglement of ρ is “absolutely” robust to the noise of separable states $\alpha = |a_1, b_1\rangle\langle a_1, b_1| \oplus_B \dots \oplus_B |a_r, b_r\rangle\langle a_r, b_r|$ in the following sense: the normalized state $\rho = (1 - p)\sigma + p\alpha$ is always entangled no matter how big the weight $p < 1$ is. This phenomenon usually does not occur for other $2 \times N$ entangled states, which would become separable by adding a separable state.

It was proved recently ([40], Theorem 5) that in $2 \otimes N$ the PPT states of birank $(2N, k)$ exist if and only if $N < k \leq 2N$. We shall obtain another existence result which, in particular, shows that there exist $2 \times N$ separable states of birank $(N + j, N + k)$ for any $j, k = 1, \dots, N$. For that we need two lemmas proved in Ref. ([27], Lemmas 1,2). In the next proposition we give a proof of the strengthened version of the combination of these two lemmas. For the definition and basic properties of the rational normal curves used in this lemma, see ([41], pp. 10–14).

Proposition 4. We consider the bipartite system $2 \otimes N$ with Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ of dimension $2N$. Let $V \subseteq \mathcal{H}$ be a CES of dimension $k < N$ and let Y be the set of all product vectors in V^\perp .

(i) The set Y (with zero vectors included) is an algebraic vector bundle of rank $N - k$ over the projective line.

(ii) V^\perp is spanned by Y .

(iii) The partial conjugates of members of Y span the whole space \mathcal{H} .

(iv) If $k = N - 1$ the projectivization of Y is a rational normal curve.

Proof. Let $|\psi_i\rangle = |0\rangle \otimes |a_i\rangle + |1\rangle \otimes |b_i\rangle$, $i = 1, \dots, k$, be a basis of V . We introduce the $2 \times N$ matrices

$$R_i = \begin{bmatrix} \alpha_{i0} & \alpha_{i1} & \cdots & \alpha_{i,N-1} \\ \beta_{i0} & \beta_{i1} & \cdots & \beta_{i,N-1} \end{bmatrix}^*, \quad i = 1, \dots, k, \quad (16)$$

where $\sum_j \alpha_{ij}|j\rangle = |a_i\rangle$ and $\sum_j \beta_{ij}|j\rangle = |b_i\rangle$. Since V is a CES, if the scalars ξ_i , $i = 1, \dots, k$ are not all zero then

$$\text{rank} \sum_{i=1}^k \xi_i R_i = 2. \quad (17)$$

The projectivization of \mathcal{H}_A is a projective line \mathcal{P}^1 . The point of \mathcal{P}^1 corresponding to the nonzero vector $z|0\rangle + w|1\rangle \in \mathcal{H}_A$ will be denoted by $[z : w]$. We claim that for each point $[z : w] \in \mathcal{P}^1$, the set of all vectors $|f\rangle \in \mathcal{H}_B$ such that $(z|0\rangle + w|1\rangle) \otimes |f\rangle \in V^\perp$ is a vector subspace of dimension $N - k$. We shall use the expansion $|f\rangle = \sum_j f_j|j\rangle \in \mathbf{C}^N$, $f_j \in \mathbf{C}$. To find the coefficients f_j we have to solve the system of k linear homogeneous equations $\langle \psi_i | (z|0\rangle + w|1\rangle) \otimes |f\rangle = 0$, i.e.,

$$\sum_{j=0}^{N-1} (\alpha_{ij}^* z + \beta_{ij}^* w) f_j = 0, \quad i = 1, \dots, k, \quad (18)$$

with matrix C of size $k \times N$. Suppose that for some $x = (\xi_1, \dots, \xi_k) \in \mathbf{C}^k$ we have $x C = 0$. We can rewrite this equation as $(z, w) \cdot \sum_i \xi_i R_i = 0$. Equation (17) implies that $x = 0$, and so $\text{rank } C = k$. Consequently, the set of solutions of the system (18) is a vector space of dimension $N - k$, and the claim is proved. Thus the fibers of the projection map $p : Y \rightarrow \mathcal{P}^1$ are vector spaces of dimension $N - k$, and (i) follows.

The matrix C is in fact a matrix pencil $C = Az + Bw$, where $A = [\alpha_{ij}^*]$ and $B = [\beta_{ij}^*]$ are $k \times N$ complex matrices. We shall use the Kronecker's theory of matrix pencils as presented in the well-known book of Gantmacher [42]. He writes a matrix pencil in nonhomogeneous form as $A + \lambda B$, where λ is an indeterminate. We homogenize the notation by setting $\lambda = w/z$ and multiplying the pencil by z . The canonical form for matrix pencils is a direct sum of blocks of several types: L_m , their transposes L_m^T , $N^{(u)}$, and $wJ + zI_s$, where I_s is the identity matrix of order s and J a Jordan block. As we shall see below, it turns out that we have to deal only with the blocks of type

$$L_m = \begin{bmatrix} z & -w & 0 & \cdots & 0 & 0 \\ 0 & z & -w & & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & & z & -w \end{bmatrix}, \quad (19)$$

of size $m \times (m + 1)$. To simplify notation in some formulas below, we have replaced w with $-w$, which we can obviously

do. For instance, we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} z & w & 0 & 0 \\ 0 & z & w & 0 \\ 0 & 0 & z & w \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \\ = \begin{bmatrix} z & -w & 0 & 0 \\ 0 & z & -w & 0 \\ 0 & 0 & z & -w \end{bmatrix}. \quad (20)$$

Contrary to Gantmacher, we allow the index m of the block L_m to be 0, in which case L_m has size 0×1 . There exist invertible matrices P and Q (whose entries are complex constants independent of z and w) such that $C' := PCQ$ has the canonical form given by ([42], Eq. (34), p. 39). By changing the basis of \mathcal{H}_A , we may assume that $Q = I_N$ is the identity matrix. Any row of C' has the form $(z, w) \cdot \sum_i \xi_i R_i$, where $\xi_i \in \mathbf{C}$ are some constants, not all 0. Hence, the rank condition (17) implies that each row of C' must have at least two nonzero entries. This is a very strong condition; it implies that C' consists only of blocks of type L_m . Since L_m has size $m \times (m + 1)$, there are exactly $N - k$ blocks, i.e., we have

$$C' = L_{m_1} \oplus \cdots \oplus L_{m_{N-k}}, \quad (21)$$

where $m_1 + \cdots + m_{N-k} = k$. Consequently, the system (18) breaks up into $N - k$ simple independent subsystems of linear homogeneous equations $L_{m_i} f^{(i)} = 0$, $i = 1, \dots, N - k$. For instance, the first subsystem comprises only the unknowns f_1, \dots, f_{m_1+1} which are the components of the column vector $f^{(1)}$, etc. Since L_{m_i} has rank m_i , the i th subsystem has a unique solution (up to a scalar factor) when viewed as a system of equations in its own portion of the unknowns f_j . There is a unique solution whose unknowns are just monomials in z and w of total degree m_i . We refer to this solution as the *basic solution*. For instance, for the first subsystem the basic solution is given by

$$f_1 = w^{m_1}, \quad f_2 = zw^{m_1-1}, \dots, f_{m_1+1} = z^{m_1}. \quad (22)$$

Note that if $m_1 = 0$ then the first subsystem has only one unknown, namely, f_1 , but it has no equations. The basic solution in that case is just $f_1 = 1$. For convenience, we shall identify this basic solution with the vector $|g^{(1)}\rangle = \sum_{i=0}^{m_1} z^i w^{m_1-i} |i\rangle \in \mathcal{H}_B$. The other subsystems can be solved in the same manner. Their basic solutions are given explicitly by

$$|g^{(i)}\rangle = \sum_{j=0}^{m_i} z^j w^{m_i-j} |m'_{i-1} + j\rangle, \quad i = 1, \dots, N - k, \quad (23)$$

where $m'_{i-1} = m_1 + \cdots + m_{i-1} + i - 1$ (with $m'_0 = 0$). The general solution is given by an arbitrary linear combination of the basic solutions $|g^{(i)}\rangle$, $i = 1, \dots, N - k$. We shall form a special solution in which the coefficients of this linear combination are suitably chosen monomials in z and w . Thus we shall multiply $g^{(i)}$ with some monomial $z^{u_i} w^{v_i}$. After expanding the tensor product $(z|0\rangle + w|1\rangle) \otimes z^{u_i} w^{v_i} \sum_{j=0}^{m_i} z^j w^{m_i-j} |j\rangle$, we obtain a linear combination of the basis vectors with $m_1 + 2$ different monomial coefficients $z^{u_i+j+1} w^{v_i+m_i-j}$ with $j = -1, 0, 1, \dots, m_1$. We can choose the exponents u_i, v_i so that the monomials arising from different subsystems are all

different and moreover, the total degree $\delta := m_i + u_i + v_i$ is independent of the index i . Then the total number of different monomials that occur in the expansion of

$$(z|0\rangle + w|1\rangle) \otimes \sum_{i=1}^{N-k} z^{u_i} w^{v_i} |g^{(i)}\rangle \quad (24)$$

is $\sum_{i=1}^{N-k} (m_i + 2) = 2N - k$. Since these $2N - k$ monomials are linearly independent, we conclude that the product vectors (24) span a subspace of dimension $2N - k$. Since all of them belong to V^\perp , the assertion (ii) is proved.

The assertion (iii) follows by using a similar argument as above after replacing $z|0\rangle + w|1\rangle$ with $z^*|0\rangle + w^*|1\rangle$ and observing that the $2N$ “monomials” $z^* z^{u_i+j} w^{v_i+m_i-j}, w^* z^{u_i+j} w^{v_i+m_i-j}$, where $i = 1, \dots, N - k$ and for fixed i the index j takes the values $0, 1, \dots, m_i$, are linearly independent. Indeed, any nontrivial linear dependence relation among these “monomials” would give an identity $z^* p(z, w) + w^* q(z, w) = 0$, where $p(z, w)$ and $q(z, w)$ are nonzero homogeneous polynomials in z and w of degree δ . By dehomogenizing, i.e., dividing this identity by $z^* z^\delta$, we obtain that $(w/z)^*$ is an analytic function of w/z , which is a contradiction.

In the case $k = N - 1$, we have $C' = L_{N-1}$ and so all product vectors in V^\perp have the form $(z|0\rangle + w|1\rangle) \otimes \sum_{i=0}^{N-1} z^{N-1-i} w^i |i\rangle$. The assertion (iv) follows. ■

Note that Theorem 1 implies that if (r, s) is a birank of a $2 \times N$ PPT state, then $r, s \geq N$, and Proposition 2 shows that $r = N$ if and only if $s = N$. Now we can show that for any $r, s \in \{N + 1, \dots, 2N\}$, there exist $2 \times N$ separable states of birank (r, s) .

Proposition 5. There exist $2 \times N$ separable states of birank $(N + j, N + k)$ for any $j, k \in \{1, \dots, N\}$.

Proof. The identity operator on \mathcal{H} is a separable state of birank $(2N, 2N)$. Thus we may assume that $j \leq k \leq N$ and $j < N$. Let V be a CES of dimension $N - j$. By Proposition 4 (ii), V^\perp has a basis consisting of product vectors, say $|e_i, f_i\rangle$, $i = 1, \dots, N + j$. The space W spanned by their partial conjugates has dimension at most $N + j$. By Proposition 4 (iii), there exist product vectors $|e'_s, f'_s\rangle \in V^\perp$, $s = 1, \dots, m$, such that the partial conjugates of the $|e_i, f_i\rangle$ and the $|e'_s, f'_s\rangle$ together span a space $W' \supseteq W$ of dimension $N + k$. Then the sum of all $|e_i, f_i\rangle\langle e_i, f_i|$ and all $|e'_s, f'_s\rangle\langle e'_s, f'_s|$ is a separable state of birank $(N + j, N + k)$. ■

(According to the authors of [43], this proposition is contained in Sec. III of their paper.)

Let us give an *ad hoc* example for the case $N = 3$ with $(r, s) = (4, 6)$.

Example 4. We have constructed an explicit separable 2×3 state ρ of birank $(4, 6)$ and length 6. It can be written as $\rho = \sum_{i=1}^4 |\psi_i\rangle\langle\psi_i|$, where

$$|\psi_1\rangle = 2|00\rangle, \quad (25)$$

$$|\psi_2\rangle = |1\rangle(|0\rangle + 2|1\rangle), \quad (26)$$

$$|\psi_3\rangle = 2|01\rangle + (|0\rangle + |1\rangle)|2\rangle, \quad (27)$$

$$|\psi_4\rangle = |02\rangle + |1\rangle(|0\rangle - |1\rangle - |2\rangle). \quad (28)$$

Since the characteristic polynomial of ρ^Γ is $t^6 - 19t^5 + 133t^4 - 413t^3 + 520t^2 - 148t + 4$, we have $\rho^\Gamma > 0$. Consequently, ρ is separable of birank $(4, 6)$. By Proposition 3, ρ has length 6. ■

Let us now show that there exist $2 \times N$ PPTES of birank $(N + 1, N + k)$ for $k = 1, \dots, N$. We shall do that by using a recently constructed family [31, Eq. (5), Appendix B of $2 \times N$ PPTES of birank $(N + 1, N + 1)$. By dropping the normalization and setting the parameter $b = 1/2$, we obtain the $2 \times N$ PPTES:

$$\begin{aligned} \rho := \sum_{i=0}^{N-2} (|0, i\rangle + |1, i + 1\rangle)(\langle 0, i| + \langle 1, i + 1|) + |10\rangle\langle 10| \\ + \frac{1}{2}|0\rangle(|0\rangle + \sqrt{3}|N - 1\rangle)(\langle 0| + \sqrt{3}\langle N - 1|). \end{aligned} \quad (29)$$

Its partial transpose is

$$\begin{aligned} \rho^\Gamma = \sum_{i=0}^{N-2} (|0, i + 1\rangle + |1, i\rangle)(\langle 0, i + 1| \\ + \langle 1, i|) + |1, N - 1\rangle\langle 1, N - 1| + \frac{1}{2}|0\rangle(\sqrt{3}|0\rangle \\ + |N - 1\rangle)(\langle 0| + \sqrt{3}\langle N - 1|). \end{aligned} \quad (30)$$

One can verify that

$$\begin{aligned} |\varphi(a)\rangle := (|0\rangle + a|1\rangle) \left[\left(a^{N-1} + \frac{1}{\sqrt{3}} \right) |0\rangle + a^{N-2} |1\rangle \right. \\ \left. + \dots + a|N - 2\rangle + |N - 1\rangle \right] \in \mathcal{R}(\rho) \end{aligned} \quad (31)$$

for all $a \in \mathbf{C}$, and that the $|\varphi(a)\rangle$ with $a \in \mathbf{R}$ span $\mathcal{R}(\rho)$.

Lemma 10. For sufficiently small $\epsilon > 0$ and $k \in \{1, \dots, N - 1\}$, the state $\rho_k := \rho + \epsilon \sum_{i=1}^k |\varphi(a_i)\rangle\langle\varphi(a_i)|$ is a $2 \times N$ PPTES of birank $(N + 1, N + 1 + k)$.

Proof. Since $\epsilon > 0$ is small and ρ is a $2 \times N$ PPTES, so is ρ_k . Since $|\varphi(a)\rangle \in \mathcal{R}(\rho)$, it follows that $\text{rank } \rho_k = N + 1$. One can verify that $\mathcal{R}(\rho) + \mathcal{R}(\rho^\Gamma) = \mathcal{H}$. Hence, there are distinct real numbers a_i , $i = 1, \dots, N - 1$, such that the vectors $|\varphi(a_i)\rangle$ are linearly independent modulo $\mathcal{R}(\rho^\Gamma)$. Since the a_i are real, each product vector $|\varphi(a_i)\rangle$ is equal to its partial conjugate. It follows that $\text{rank } \rho_k^\Gamma = N + 1 + k$. ■

More generally, we have the following result.

Proposition 6. For any $r, s \in \{N + 1, \dots, 2N\}$, there exist $2 \times N$ PPTES of birank (r, s) .

Proof. Let $k, p \in \{0, \dots, N - 1\}$ and let ρ_k be the state constructed in Lemma 10. For the state ρ defined by Eq. (29), we have $\rho^\Gamma = (I \otimes V)\rho(I \otimes V^\dagger)$, where V is the antidiagonal matrix. So $\mathcal{R}(\rho^\Gamma)$ is spanned by the product vectors $|\psi(a)\rangle = (I \otimes V)|\varphi(a)\rangle$ with $a \in \mathbf{R}$. Since $\mathcal{R}(\rho) + \mathcal{R}(\rho^\Gamma) = \mathcal{H}$, there are distinct real numbers a'_j , $j = 1, \dots, N - 1$, such that the product vectors $|\psi(a'_j)\rangle$ are linearly independent modulo $\mathcal{R}(\rho)$. Note that $|\psi(a'_j)\rangle\langle\psi(a'_j)|^\Gamma = |\psi(a'_j)\rangle\langle\psi(a'_j)|$ for each i . It follows that for sufficiently small $\epsilon' > 0$, the state $\rho_k + \epsilon' \sum_{i=1}^p |\psi(a'_i)\rangle\langle\psi(a'_i)|$ is a $2 \times N$ PPTES of birank $(N + 1 + p, N + 1 + k)$. ■

One may expect that Propositions 5 and 6 generalize to arbitrary $M \otimes N$ space, i.e., that $M \times N$ separable states as well as PPTES of birank (r,s) exist for all $r,s > \max(M,N)$. However, this is false. For the former, we observe that there is no separable 3×3 state of birank (4,6). Indeed, let ρ be any 3×3 separable state of rank 4. By Lemma 7, $\text{rank } \rho^\Gamma \leq 5$. Then Proposition 2 (i) implies that $\text{rank } \rho^\Gamma < 6$. For the latter, we observe that there is no two-qutrit PPTES of birank (4,5) or (4,6) (see ([9], Theorem 23)).

We give a result on NPT states as the concluding remark of this section. It has been shown that for any NPT $2 \times N$ state, its partial transpose has at most $N - 1$ negative eigenvalues ([44], Theorem 1). This upper bound is sharp. More precisely, for each $m \in \{1, \dots, N - 1\}$, we shall construct $2 \times N$ NPT states whose partial transpose has exactly m negative eigenvalues.

Example 5. First observe that the partial transpose of the $2 \times N$ state $\rho = (|00\rangle + |11\rangle)(\langle 00| + \langle 11|) + |0\rangle\langle 0| \otimes I_N$ has exactly one negative eigenvalue. Next we consider the following family of $2 \times N$ states,

$$\rho = \sum_{i=0}^{N-2} (|0,i\rangle + c_{i+1}|1,i+1\rangle)(\langle 0,i| + c_{i+1}\langle 1,i+1|), \quad (32)$$

where $0 < c_1 = \dots = c_k < \dots < c_{N-2}$, $1 \leq k < N - 1$, and $c_{N-1} = 1$. Then $\rho^\Gamma = \sum_{i=1}^N M_i$,

where

$$M_i = |0,i+1\rangle\langle 0,i+1| + c_i^2|1,i\rangle\langle 1,i| + c_{i+1}|0,i+1\rangle\langle 1,i| + c_{i+1}|1,i\rangle\langle 0,i+1|, \quad (33)$$

$$i < N - 2,$$

$$M_{N-2} = |0,1\rangle\langle 0,1| + c_1|0,1\rangle\langle 1,0| + c_1|1,0\rangle\langle 0,1|, \quad (34)$$

$$M_{N-1} = c_{N-2}^2|1,N-2\rangle\langle 1,N-2| + |0,N-1\rangle\langle 1,N-2| + |1,N-2\rangle\langle 0,N-1|, \quad (35)$$

$$M_N = |0,0\rangle\langle 0,0| + |1,N-1\rangle\langle 1,N-1| \quad (36)$$

are Hermitian matrices such that $M_i M_j = 0$ for $i \neq j$. For $k \leq i < N$, each M_i has exactly one negative eigenvalue, while for all other indexes i the matrix $M_i \geq 0$. Hence, ρ^Γ has exactly $N - k$ negative eigenvalues. ■

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