Measure of multipartite entanglement with computable lower bounds

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In this paper, we present a measure of multipartite entanglement (k-nonseparable), k-ME concurrence $C_{k-\mathrm{ME}}(\rho)$, that unambiguously detects all k-nonseparable states in arbitrary dimensions, where the special case 2-ME concurrence $C_{2-\mathrm{ME}}(\rho)$ is a measure of genuine multipartite entanglement. The measure k-ME concurrence satisfies important characteristics of an entanglement measure, including the entanglement monotone, vanishing on k-separable states, convexity, subadditivity, and being strictly greater than zero for all k-nonseparable states. Two powerful lower bounds on this measure are given. These lower bounds are experimentally implementable without quantum state tomography and are easily computable as no optimization or eigenvalue evaluation is needed. We illustrate detailed examples in which the given bounds perform better than other known detection criteria.

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I. INTRODUCTION

Entanglement as a physical resource plays an important role in quantum information, such as quantum communication [1–9] and quantum computing [10,11]. So it is significant work to quantify entanglement not only in theoretical research but also in practical application. One of the main goals of the theory of entanglement is to develop measures of entanglement. Several entanglement measures [12–14] have been introduced, such as entanglement distillation [15–17], entanglement cost [17,18], entanglement of formation [17,19], negativity [20,21], three-tangle [22], and localizable entanglement [9,23]. These measures, except localizable entanglement, are entanglement monotones [12–14] in that they cannot increase under local operations and classical communication (LOCC), whereas localizable entanglement can deterministically increase under LOCC operations between all parties [24]. In a bipartite setting, entanglement cost, entanglement of formation, and negativity are convex; moreover, entanglement cost and entanglement of formation are also subadditive. It is an open question whether entanglement distillation is convex [12]. The negativity fails to recognize entanglement in positive partial transpose states. In the multipartite setting, three-tangle is invariant under permutation of the three systems and is, in fact, an entanglement monotone for three-qubit systems. However, there are states with genuine three-party entanglement for which the three-tangle can be zero (the W state serves as an example [22]); i.e., the three-tangle has the disadvantageous property that it vanishes for some entangled states. Localizable entanglement [23] requires an underlying measure of bipartite entanglement to quantify the entanglement between the two singled-out parties. When concurrence was used as the underlying measure of bipartite entanglement, Gao *et al.* [9] derived an easily computable formula for localizable entanglement in the three-qubit case.

The concurrence is a very popular measure for the quantification of bipartite quantum correlations [12,13,25,26] and is also defined for bipartite high dimensional states [27], but it is not computable because of optimization for bipartite high-dimensional mixed states. For multipartite quantum systems, although there are some criteria [13,28–37] to detect genuine multipartite entanglement, there is no computable measure quantifying the amount of multipartite entanglement in general. Ma *et al.* [38] defined a generalized concurrence called the genuine multipartite entanglement (GME) concurrence which satisfies the necessary conditions for a genuine multipartite entanglement measure [39,40]. Although for general mixed states it is not computable owing to the optimization, they gave lower bounds [38,41]. What we are looking for is a multipartite entanglement (ME) measure whose values vanish with respect to k-separable states but are strictly positive for k-nonseparable states.

In this paper, we introduce a generalized concurrence (the k-ME concurrence) for finite-dimensional systems of arbitrarily many parties as an entanglement measure, which satisfies important characteristics of an entanglement measure, such as the entanglement monotone, vanishing on k-separable states, invariant under local unitary transformations, convexity, subadditivity, and being strictly greater than zero for all k-nonseparable states. This multipartite entanglement measure unambiguously detects all k-nonseparable states in arbitrary dimensions. The GME concurrence [38,41] is a special case of our k-ME concurrence when $k = 2$. We show that strong lower bounds on this measure can be derived by exploiting close analytic relations between this concurrence and recently introduced detection criteria for multipartite entanglement [32–34]. Then we provide examples in which the entanglement criteria based on our lower bounds have better performance with respect to the known methods, the lower bounds obtained by Refs. [38,41].

II. MULTIPARTITE ENTANGLEMENT

Before we state the definition of the multipartite entanglement measure, k-ME concurrence, and its lower bounds, an

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introduction of the concepts and notation that will be used in the subsequent sections of our article is necessary. Throughout the paper, we consider a multiparticle quantum system $\mathcal{H} = \otimes_{i=1}^{n}\mathcal{H}_i = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$, with n parts of respective dimension d_i, $i = 1,2,\ldots,n$. A k-partition $A_1|A_2|\cdots|A_k$ (of $\{1,2,\ldots,n\}$) means that the set $\{A_1,A_2,\ldots,A_k\}$ is a collection of pairwise disjoint sets, and the union of all sets in $\{A_1,A_2,\ldots,A_k\}$ is $\{1,2,\ldots,n\}$ (disjoint union $\bigcup_{i=1}^{k} A_i = \{1,2,\ldots,n\}$). An pure state $|\psi\rangle$ of an n-partite quantum system $\mathcal{H}$ is called k-separable if there is a k-partition $A_1|A_2|\cdots|A_k = j_1^1\cdots j_{n_1}^1|j_1^2\cdots j_{n_2}^2|\cdots|j_1^k\cdots j_{n_k}^k$ such that

$$|\psi\rangle = |\psi_1\rangle_{A_1}|\psi_2\rangle_{A_2}\cdots|\psi_k\rangle_{A_k}, \tag{1}$$

where $|\psi_i\rangle_{A_i}$ is the state of subsystem A_i and disjoint union $\bigcup_{t=1}^{k} A_t = \bigcup_{t=1}^{k}\{j_1^t,j_2^t,\ldots,j_{n_t}^t\} = \{1,2,\ldots,n\}$. An n-partite mixed state ρ is k-separable if it can be written as a convex combination of k-separable pure states,

$$\rho = \sum_m p_m|\psi_m\rangle\langle\psi_m|, \tag{2}$$

where $\{|\psi_m\rangle\}$ might be k-separable with respect to different partitions. Thus, a mixed k-separable state does not need to be separable under any particular k-partition. In general, k-separable mixed states are not separable with regard to any specific partition. If an n-partite state is not two-separable (biseparable), then it is called genuinely n-partite entangled. It is called fully separable, iff it is n-separable.

Note that whenever a state is k-separable, it is automatically also k'-separable for all $1 < k' < k$. If we denote the set of all k-separable states by S_k ($k = 2,3,\ldots,n$) and the set of all states by S_1, then each set S_k is convex and embedded within the next set, $S_n \subset S_{n-1} \subset \cdots \subset S_2 \subset S_1$, and the complement $S_1\backslash S_k$ of S_k in S_1 is the set of all k-nonseparable states. In particular, the complement $S_1\backslash S_2$ is the set of all genuine n-partite entangled (2-nonseparable) states. We illustrate the convex nested structure of multipartite entanglement in Fig. 1.

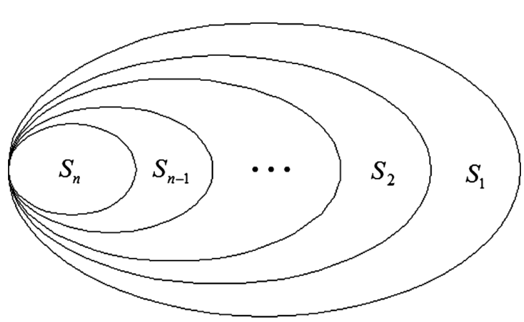

FIG. 1. Illustration of the convex nested structure of the sets S_k of all k-separable states. Each set is convexly embedded within the next set: $S_n \subset S_{n-1} \subset \cdots \subset S_2 \subset S_1$, and the complement $S_1\backslash S_k$ of S_k in S_1 is the set of all k-nonseparable states.

III. A MEASURE OF MULTIPARTITE ENTANGLEMENT AND ITS LOWER BOUNDS

Let us now introduce a measure of multipartite entanglement (k-nonseparable) that unambiguously detects all k-nonseparable states in arbitrary dimensions. For n-partite pure state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$, where $\dim\mathcal{H}_l = d_l$, $l = 1,2,\ldots,n$, we define the k-ME concurrence as

$$C_{k-\mathrm{ME}}(|\psi\rangle) = \min_A \sqrt{2\left(1 - \frac{\sum_{t=1}^{k}\mathrm{Tr}\left(\rho_{A_t}^2\right)}{k}\right)} = \min_A \sqrt{\frac{2\sum_{t=1}^{k}\left[1 - \mathrm{Tr}\left(\rho_{A_t}^2\right)\right]}{k}}, \tag{3}$$

where $\rho_{A_t} = \mathrm{Tr}_{\bar{A}_t}(|\psi\rangle\langle\psi|)$ is the reduced density matrix of subsystem A_t ($\bar{A}_t$ is the complement of A_t in $\{1,2,\ldots,n\}$) and the minimum is taken over all possible k-partitions $A = A_1|\cdots|A_k$ of $\{1,2,\ldots,n\}$. Obviously, $C_{k-\mathrm{ME}}(|\psi\rangle)$ depends not only on $|\psi\rangle$ but also on the number k. However, it is independent of k-partitions. It should be pointed out that $C_{k-\mathrm{ME}}(|\psi\rangle)$ is nonvanishing if and only if $|\psi\rangle$ is k-nonseparable; that is, $C_{k-\mathrm{ME}}(|\psi\rangle)$ equals to zero if and only if $|\psi\rangle$ is k-separable.

For the n-partite mixed state ρ, we define the k-ME concurrence as

$$C_{k-\mathrm{ME}}(\rho) = \inf_{\{p_m,|\psi_m\rangle\}}\sum_m p_m C_{k-\mathrm{ME}}(|\psi_m\rangle), \tag{4}$$

where the infimum is taken over all possible pure state decompositions $\rho = \sum_m p_m|\psi_m\rangle\langle\psi_m|$. Specially, when $k = 2$, $C_{2-\mathrm{ME}}(\rho)$ is a measure of genuine multipartite entanglement. Note that the GME concurrence [38] is our special case $C_{2-\mathrm{ME}}(\rho)$, and the GME concurrence C_{GME} is equal to $\frac{1}{\sqrt{2}}C_{2-\mathrm{ME}}(\rho)$.

The k-ME concurrence $C_{k-\mathrm{ME}}(\rho)$, a measure of multipartite entanglement, satisfies the following useful properties: (1) $C_{k-\mathrm{ME}}(\rho) = 0$ for any $\rho \in S_k$ (vanishing on all k-separable states). (2) $C_{k-\mathrm{ME}}(\rho) > 0$ for any $\rho \in S_1\backslash S_k$ (strictly greater than zero for all k-nonseparable states). (3) $C_{k-\mathrm{ME}}(U_{\mathrm{Local}}^{\dagger}\rho U_{\mathrm{Local}}) = C_{k-\mathrm{ME}}(\rho)$ (invariant under local unitary transformations). (4) $C_{k-\mathrm{ME}}(\Lambda_{\mathrm{LOCC}}(\rho)) \leqslant C_{k-\mathrm{ME}}(\rho)$ (entanglement monotone: nonincreasing under LOCC). (5) $C_{k-\mathrm{ME}}(\sum_i p_i\rho_i) \leqslant \sum_i p_i C_{k-\mathrm{ME}}(\rho_i)$ (convexity). (6) $C_{k-\mathrm{ME}}(\rho\otimes\sigma) \leqslant C_{k-\mathrm{ME}}(\rho) + C_{k-\mathrm{ME}}(\sigma)$ (subadditivity).

IV. LOWER BOUNDS

A. Statement of results

Let $|\phi(x)\rangle = \otimes_{i=1}^{n}|x_i\rangle = |x_1x_2\cdots x_n\rangle$ be a fully separable state on Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$ and $|\Phi_{ij}(x)\rangle = |\phi_i(x)\rangle|\phi_j(x)\rangle$ be a product state in $\mathcal{H}^{\otimes 2}$, where $|\phi_i(x)\rangle = |x_1x_2\cdots x_{i-1}x_i'x_{i+1}\cdots x_n\rangle$ and $|\phi_j(x)\rangle = |x_1x_2\cdots x_{j-1}x_j'x_{j+1}\cdots x_n\rangle$ are the fully separable states obtained from $|\phi(x)\rangle$ by applying (independently) local unitary transformations to $|x_i\rangle \in \mathcal{H}_i$ and $|x_j\rangle \in \mathcal{H}_j$, respectively. Let P_{tot} denote the operator that performs a simultaneous local permutation on all subsystems in $\mathcal{H}^{\otimes 2} = (\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n)^{\otimes 2}$, while P_i just performs a permutation on $\mathcal{H}_i^{\otimes 2}$ and leaves all other subsystems unchanged. That is, $P_{\mathrm{tot}} = P_1 \circ P_2 \circ \cdots \circ P_n$, where P_i is the operator swapping the two copies of $\mathcal{H}_i$ in $\mathcal{H}^{\otimes 2}$. For instance, $P_{\mathrm{tot}}|x_1x_2\cdots x_n\rangle|y_1y_2\cdots y_n\rangle = |y_1y_2\cdots y_n\rangle|x_1x_2\cdots x_n\rangle$, while

$P_i|x_1\cdots x_{i-1}x_ix_{i+1}\cdots x_n\rangle|y_1\cdots y_{i-1}y_iy_{i+1}\cdots y_n\rangle = |x_1\cdots x_{i-1}y_ix_{i+1}\cdots x_n\rangle|y_1\cdots y_{i-1}x_iy_{i+1}\cdots y_n\rangle$. Let

$$\begin{aligned} I_k(\rho,\phi(x)) = &\sum_{i\neq j}\sqrt{\langle\Phi_{ij}(x)|\rho^{\otimes 2}P_{\rm tot}|\Phi_{ij}(x)\rangle} \\ &-\sum_{i\neq j}\sqrt{\langle\Phi_{ij}(x)|P_i^{+}\rho^{\otimes 2}P_i|\Phi_{ij}(x)\rangle} \\ &-(n-k)\sum_{i}\sqrt{\langle\Phi_{ii}(x)|P_i^{+}\rho^{\otimes 2}P_i|\Phi_{ii}(x)\rangle}, \end{aligned} \tag{5}$$

then we have the following bounds.

For bound 1,

$$C_{k-\rm ME}(\rho)\geqslant H_kI_k(\rho,\phi(x)), \tag{6}$$

where

$$H_k=\min_A\frac{\sqrt{k}}{\sqrt{\sum_{t=1}^k n_t(n-n_t)}}=\min_{\sum_{t=1}^k n_t=n}\frac{\sqrt{k}}{\sqrt{n^2-\sum_{t=1}^k n_t^2}}. \tag{7}$$

Here the minimum is taken over all possible k-partitions $A=A_1|\cdots|A_k$ of $\{1,2,\ldots,n\}$, and n_t is the number of elements in A_t.

Particularly, when $k=2$,

$$H_2=\begin{cases}\frac{2}{n}, & n \text{ is even},\\ \frac{2}{\sqrt{n^2-1}}, & n\text{ is odd}.\end{cases} \tag{8}$$

Therefore,

$$C_{2-\rm ME}(\rho)\geqslant\begin{cases}\frac{2}{n}I_2(\rho,\phi(x)), & n\text{ is even},\\ \frac{2}{\sqrt{n^2-1}}I_2(\rho,\phi(x)), & n\text{ is odd}.\end{cases} \tag{9}$$

Our bound 1 is stronger than lower bound 1 in [41] since H_2 is greater than $\frac{1}{\sqrt{2(n-1)}}$. That is, our lower bound 1 is more powerful than that in [41].

For bound 2,

$$C_{k-\rm ME}(\rho)\geqslant\max_{\{\phi(x),\phi(y)\}}\bar{H}_k[I_k(\rho,\phi(x))+I_k(\rho,\phi(y))], \tag{10}$$

where

$$\bar{H}_k=\min_A\frac{\sqrt{k}}{\sqrt{2\sum_{t=1}^k n_t(n-n_t)}}=\frac{1}{\sqrt{2}}H_k. \tag{11}$$

Here $|\phi(x)\rangle=\otimes_{i=1}^n|x_i\rangle$ and $|\phi(y)\rangle=\otimes_{i=1}^n|y_i\rangle$ are fully separable states in which $|x_i\rangle$ and $|y_i\rangle$ are orthogonal. The proof of the two lower bounds above is given in the Appendix.

B. Examples

Example 1. Consider the n-qubit state family given by a mixture of the identity matrix, the W state, and the anti-W state:

$$\rho_n=\frac{1-(a+b)}{2^n}I_{2^n}+a|W_n\rangle\langle W_n|+b|\tilde{W}_n\rangle\langle\tilde{W}_n|, \tag{12}$$

where $|W_n\rangle=\frac{1}{\sqrt{n}}(|00\cdots001\rangle+|00\cdots010\rangle+\cdots+|10\cdots000\rangle)$ and $|\tilde{W}_n\rangle=\frac{1}{\sqrt{n}}(|11\cdots110\rangle+|11\cdots101\rangle+\cdots+|01\cdots111\rangle)$. Let $|\phi(0)\rangle=|0\rangle^{\otimes n}$ and $|\phi(1)\rangle=|1\rangle^{\otimes n}$; then $|\phi_i(0)\rangle=|0\cdots010\cdots0\rangle$ and $|\phi_i(1)\rangle=|1\cdots101\cdots1\rangle$ can be obtained by applying the bit-flip operation σ_x on the ith qubit of $|\phi(0)\rangle$ and $|\phi(1)\rangle$, respectively.

When $n>3$,

$$I_k(\rho_n,\phi(0))=(k-1)a-\frac{n(2n-k-1)(1-a-b)}{2^n}, \tag{13}$$

$$I_k(\rho_n,\phi(1))=(k-1)b-\frac{n(2n-k-1)(1-a-b)}{2^n}. \tag{14}$$

When $n=3$,

$$I_k(\rho_3,\phi(0))=(k-1)a-\frac{3}{4}\sqrt{\frac{(1-a-b)(3-3a+5b)}{3}}-\frac{3(3-k)(1-a-b)}{2^3}, \tag{15}$$

$$I_k(\rho_3,\phi(1))=(k-1)b-\frac{3}{4}\sqrt{\frac{(1-a-b)(3+5a-3b)}{3}}-\frac{3(3-k)(1-a-b)}{2^3}. \tag{16}$$

Our bound 1 inequality (6) is

$$C_{k-\rm ME}\geqslant\begin{cases}\max\{H_kI_k(\rho_n,\phi(0)),H_kI_k(\rho_n,\phi(1))\}, & n>3,\\ \max\{H_kI_k(\rho_3,\phi(0)),H_kI_k(\rho_3,\phi(1))\}, & n=3,\end{cases} \tag{17}$$

where $H_k=\min_{\sum_{t=1}^k n_t=n}\frac{\sqrt{k}}{\sqrt{n^2-\sum_{t=1}^k n_t^2}}$.

Particularly,

$$C_{2-\rm ME}\geqslant\begin{cases}\max\left\{\frac{2}{n}I_2(\rho_n,\phi(0)),\frac{2}{n}I_2(\rho_n,\phi(1))\right\}, & n>3\text{ and }n\text{ is even},\\ \max\left\{\frac{2}{\sqrt{n^2-1}}I_2(\rho_n,\phi(0)),\frac{2}{\sqrt{n^2-1}}I_2(\rho_n,\phi(1))\right\}, & n>3\text{ and }n\text{ is odd},\\ \max\left\{\frac{1}{\sqrt{2}}I_2(\rho_3,\phi(0)),\frac{1}{\sqrt{2}}I_2(\rho_3,\phi(1))\right\}, & n=3.\end{cases} \tag{18}$$

Lower bound 1 in [41] gives

$$C_{\rm GME}\geqslant\begin{cases}\max\left\{\frac{1}{\sqrt{2(n-1)}}I_2(\rho_n,\phi(0)),\frac{1}{\sqrt{2(n-1)}}I_2(\rho_n,\phi(1))\right\}, & n>3,\\ \max\left\{\frac{1}{2\sqrt{2}}I_2(\rho_3,\phi(0)),\frac{1}{2\sqrt{2}}I_2(\rho_3,\phi(1))\right\}, & n=3.\end{cases} \tag{19}$$

Obviously, for the genuine multipartite entanglement measure, our lower bound 1 (inequality (18)) is better than that in [41].

The detection parameter spaces of our bound 1 and bound 1 in [41] of genuine five-partite entanglement are illustrated in Fig. 2 for the family ρ_5 of five-qubit states. The area detected by our bound 1 is larger than bound 1 of [41] when the two lower bounds are equal.

Our bound 2 inequality (10) is as follows:

$$C_{k-\mathrm{ME}} \geqslant \begin{cases} \frac{1}{\sqrt{2}} H_k[I_k(\rho_n,\phi(0)) + I_k(\rho_n,\phi(1))], & n > 3, \\ \frac{1}{\sqrt{2}} H_k[I_k(\rho_3,\phi(0)) + I_k(\rho_3,\phi(1))], & n = 3. \end{cases} \tag{20}$$

Particularly,

$$C_{2-\mathrm{ME}} \geqslant \begin{cases} \frac{\sqrt{2}}{n}[I_2(\rho_n,\phi(0)) + I_2(\rho_n,\phi(1))], & n > 3 \quad \text{and} \quad n \text{ is even}, \\ \frac{\sqrt{2}}{\sqrt{n^2-1}}[I_2(\rho_n,\phi(0)) + I_2(\rho_n,\phi(1))], & n > 3 \quad \text{and} \quad n \text{ is odd}, \\ \frac{1}{2}[I_2(\rho_3,\phi(0)) + I_2(\rho_3,\phi(1))], & n = 3. \end{cases} \tag{21}$$

When $n > 3$, and $|\Phi\rangle = |0\rangle^{\otimes n}|1\rangle^{\otimes n}$ or $(\frac{|0\rangle+|1\rangle}{\sqrt{2}})^{\otimes n}(\frac{|0\rangle-|1\rangle}{\sqrt{2}})^{\otimes n}$, bound of Ref. [38] cannot detect entanglement at all. When $n \geqslant 4$, and $|\psi\rangle = |\phi(0)\rangle, |\phi(1)\rangle$, or $|\phi_i(0)\rangle$, lower bound 2 in [41] cannot detect entanglement at all. Therefore, for the family of n-qubit states that is the mixture of the W state and the anti-W state, dampened with white noise, our lower bounds are better than bounds 1 and 2 of Ref. [41] and the bound of Ref. [38].

Example 2. Let us consider the family of n-qubit states

$$\rho^{(G_n-W_n)} = \alpha|G_n\rangle\langle G_n| + \beta|W_n\rangle\langle W_n| + \frac{1-\alpha-\beta}{2^n}\mathrm{I}, \tag{22}$$

which is the mixture of the Greenberger-Horne-Zeilinger (GHZ) state, the W state, and the white noise. Here $|G_n\rangle = \frac{1}{\sqrt{2}}(|00\cdots 0\rangle + |11\cdots 1\rangle)$ and $|W_n\rangle = \frac{1}{\sqrt{n}}(|00\cdots 001\rangle + |00\cdots 010\rangle + \cdots + |10\cdots 000\rangle)$.

For the selection $|\phi(0)\rangle = \otimes_{i=1}^n |x_i\rangle = |0\rangle^{\otimes n}$ and $|x_i'\rangle = |1\rangle$, our bound 1 gives

$$C_{k-\mathrm{ME}}(\rho^{(G_n-W_n)}) \geqslant H_k\left[(n-1)\beta - n(n-1)\sqrt{\left(\frac{\alpha}{2} + \frac{1-\alpha-\beta}{2^n}\right)\frac{1-\alpha-\beta}{2^n}} - n(n-k)\left(\frac{\beta}{n} + \frac{1-\alpha-\beta}{2^n}\right)\right]. \tag{23}$$

Let $|\phi(x)\rangle = \otimes_{i=1}^n |x_i\rangle = (\frac{|0\rangle-|1\rangle}{\sqrt{2}})^{\otimes n}$ and $|x_i'\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$; our bound 1 gives

$$C_{k-\mathrm{ME}}(\rho^{(G_n-W_n)}) \geqslant \begin{cases} H_k\Big[\frac{(n-1)(n-2)^2\beta}{2^n} - n(n-1)\sqrt{\big(\frac{1+\alpha-\beta}{2^n} + \frac{(n-4)^2\beta}{2^n n}\big)\frac{1+\alpha-\beta+n\beta}{2^n}} - (n-k)\big(\frac{(n-2)^2\beta+n(1-\alpha-\beta)}{2^n}\big)\Big], & n \text{ is even}, \\ H_k\Big[(n-1)\big(\frac{2n\alpha+(n-2)^2\beta}{2^n} - n\sqrt{\big(\frac{1-\alpha-\beta}{2^n} + \frac{(n-4)^2\beta}{2^n n}\big)\frac{1-\alpha-\beta+n\beta}{2^n}}\big) - (n-k)\big(\frac{(n-2)^2\beta+n(1+\alpha-\beta)}{2^n}\big)\Big], & n \text{ is odd}. \end{cases} \tag{24}$$

For the selection $|\Phi\rangle = |0\rangle^{\otimes n}|1\rangle^{\otimes n}$, from (17) in [38],

$$C_{\mathrm{GME}}(\rho^{(G_n-W_n)}) \geqslant \begin{cases} 2\big[\frac{\alpha}{2} - C_n^1\big(\frac{\beta}{n} + \frac{1-\alpha-\beta}{2^n}\big)^{\frac{1}{2}}\big(\frac{1-\alpha-\beta}{2^n}\big)^{\frac{1}{2}} - \big(C_n^2 + \cdots + \frac{1}{2}C_n^{\frac{n}{2}}\big)\big(\frac{1-\alpha-\beta}{2^n}\big)\big], & n \text{ is even}, \\ 2\big[\frac{\alpha}{2} - C_n^1\big(\frac{\beta}{n} + \frac{1-\alpha-\beta}{2^n}\big)^{\frac{1}{2}}\big(\frac{1-\alpha-\beta}{2^n}\big)^{\frac{1}{2}} - \big(C_n^2 + \cdots + C_n^{\lfloor\frac{n}{2}\rfloor}\big)\big(\frac{1-\alpha-\beta}{2^n}\big)\big], & n \text{ is odd}. \end{cases} \tag{25}$$

Here C_n^i is a binomial coefficient, and $\lfloor\frac{n}{2}\rfloor$ is a nonnegative integer no greater than $\frac{n}{2}$. Let $|\Phi\rangle = (\frac{|0\rangle+|1\rangle}{\sqrt{2}})^{\otimes n}(\frac{|0\rangle-|1\rangle}{\sqrt{2}})^{\otimes n}$; from (17) in [38],

$$C_{\mathrm{GME}}(\rho^{(G_5-W_5)}) \geqslant 2\left[\frac{15}{32}\left(1-\alpha+\frac{4\beta}{5}\right)^{\frac{1}{4}}\left(1+\alpha+\frac{4\beta}{5}\right)^{\frac{1}{4}}\right]. \tag{26}$$

The detection quality of our bound 1 and the bound in [38] on the genuine multipartite entanglement is illustrated in Fig. 3 for the family $\rho^{(G_5-W_5)}$.

V. EXPERIMENTAL IMPLEMENTATION OF LOWER BOUNDS

The two lower bounds, (6) and (10), are experimentally accessible by means of local observables, without quantum state tomography, which requires exponentially increasing measurements. Since the nonlocal observable is not straightforward to measure in practice, the observables that can easily be measured in any experiment are local observables. In order to be useful in practice, measures for multipartite entanglement need to be experimentally implementable by means of local observables without resorting to a full quantum state tomography. Lower bounds (6) and (10) satisfy these demands because, for fixed $|\phi(x)\rangle$, their computations only require at most n^2+1 and $2n^2+2$ measurements, respectively. Furthermore, they can be implemented locally as explicitly shown in [34]. In total at most $\frac{5(n^2-n)}{2} + n + 1$

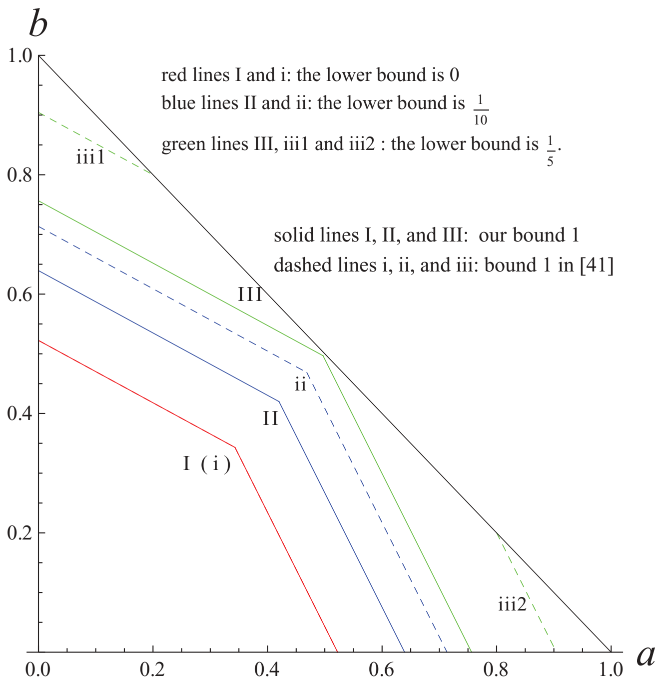

FIG. 2. (Color online) The detection quality of our lower bound 1 and that in [41] on the genuine multipartite entanglement concurrence is shown for the family $\rho_5 = \frac{1-a-b}{32} I_{32} + a|W_5\rangle\langle W_5| + b|\tilde{W}_5\rangle\langle\tilde{W}_5|$ of five-qubit states, where $|W_5\rangle = \frac{1}{\sqrt{5}}(|00001\rangle + |00010\rangle + |00100\rangle + |01000\rangle + |10000\rangle)$ and $|\tilde{W}_5\rangle = \frac{1}{\sqrt{5}}(|11110\rangle + |11101\rangle + |11011\rangle + |10111\rangle + |01111\rangle)$. The region above line I (red) corresponds to the genuine five-partite entanglement detected by our bound 1, our criteria in [32,34], and bound 1 of [41]. The regions above line II (blue) and line III (green) correspond to the genuine five-partite entanglement detected by our bound 1 when it is equal to or greater than $\frac{1}{10}$ and $\frac{1}{5}$, respectively. The states above dashed line ii (blue), dashed line iii1 (green), and dashed line iii2 (green) are detected by bound 1 of Ref. [41] when it is equal to or greater than $\frac{1}{10}$, $\frac{1}{5}$, and $\frac{1}{5}$, respectively. Thus, the area detected by our bound 1 is visibly larger than that of [41] when the two bounds are equal.

and $5n^2 - 3n + 2$ local observables are needed to implement our bound 1 and bound 2, respectively. In an experimental situation, it is now possible to choose the corresponding $|\phi(x)\rangle$ and not only detect the state as being k-nonseparable but also have a reliable statement about the amount of multipartite entanglement the state exhibits.

VI. CONCLUSION

We have presented a measure of multipartite entanglement called k-ME concurrence that unambiguously detects all k-nonseparable states, and we have studied multipartite entanglement of quantum states in arbitrary dimensional systems. This measure satisfies important characteristics of an entanglement measure, such as the entanglement monotone and vanishing on all k-separable states. The three main advantages are that k-ME concurrence is convex, subadditive, and strictly greater than zero for all k-nonseparable states. The GME concurrence [38,41] is a special case of our k-ME concurrence when $k = 2$. Two powerful lower bounds of k-ME concurrence $C_{k-\mathrm{ME}}(\rho)$ for n-partite mixed quantum states through inequality (3) from

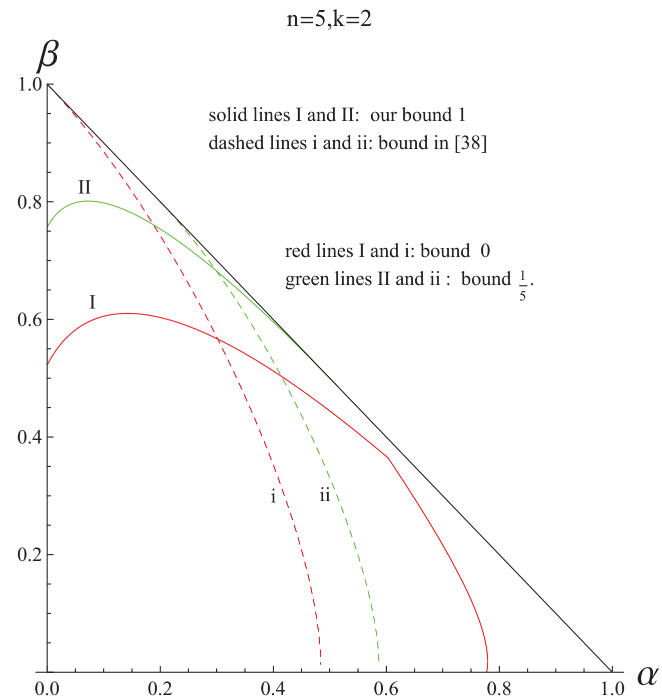

FIG. 3. (Color online) The detection quality of our lower bound 1 and the bound in [38] on the GME-concurrence is shown for the family of five-qubit states $\rho_5 = \alpha|G_5\rangle\langle G_5| + \beta|W_5\rangle\langle W_5| + \frac{1-a-b}{32} I_{32}$ given by the convex combination of a GHZ state, a W state, and the maximally mixed state. The areas above the solid red line labeled I and the dashed red line labeled i are the genuine five-partite entangled states detected by our bound 1 and the bound of [38], respectively. The states in the areas above the solid green line labeled II (dashed green line labeled ii) are the genuine five-partite entangled states detected by our lower bound 1 (the bound of [38]) when the bound is equal to or greater than $\frac{1}{5}$.

Ref. [34] are given. We provide examples in which the lower bounds perform better than other previously known methods.

ACKNOWLEDGMENTS

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APPENDIX: PROOF OF TWO LOWER BOUNDS

Any pure quantum state of an n-particle system can be denoted by vectors in Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$ as follows:

$$|\psi\rangle = \sum_{i_1,i_2,\cdots,i_n} c_{i_1 i_2 \cdots i_n} |i_1 i_2 \cdots i_n\rangle, \tag{A1}$$

which can be rewritten as

$$|\psi\rangle = \sum_{\gamma_{A_t},\gamma_{\bar{A}_t}} c_{\gamma_{A_t}\gamma_{\bar{A}_t}} |\gamma_{A_t}\gamma_{\bar{A}_t}\rangle, \tag{A2}$$

where $\{|i_j\rangle\}$ is the orthonormal basis of $\mathcal{H}_j$ and a basis vector of subsystem A_t is denoted by $|\gamma_{A_t}\rangle = |i_{j_1^t} i_{j_2^t} \cdots i_{j_{n_t}^t}\rangle$. Here $A_1|A_2|\cdots|A_k = j_1^1 j_2^1 \cdots j_{n_1}^1 | j_1^2 j_2^2 \cdots j_{n_2}^2 | \cdots | j_1^k j_2^k \cdots j_{n_k}^k$ is a k-partition of $\{1,2,\ldots,n\}$, and $\bar{A}_t$ is the complement of subsystem A_t in $\{1,2,\ldots,n\}$. Thus,

$$\rho_{A_t} = \mathrm{Tr}_{\bar{A}_t}(|\psi\rangle\langle\psi|) = \sum_{\gamma_{A_t},\eta_{A_t}} \left(\sum_{\gamma_{\bar{A}_t}} c_{\gamma_{A_t}\gamma_{\bar{A}_t}} c^*_{\eta_{A_t}\gamma_{\bar{A}_t}} \right) |\gamma_{A_t}\rangle\langle\eta_{A_t}| \equiv \sum_{\gamma_{A_t},\eta_{A_t}} \rho_{\gamma_{A_t},\eta_{A_t}} |\gamma_{A_t}\rangle\langle\eta_{A_t}| \tag{A3}$$

and

$$\mathrm{Tr}\big(\rho_{A_t}^2\big) = \sum_{\gamma_{A_t},\eta_{A_t}} \big|\rho_{\gamma_{A_t},\eta_{A_t}}\big|^2 = \sum_{\gamma_{A_t}} \big|\rho_{\gamma_{A_t},\gamma_{A_t}}\big|^2 + 2 \sum_{s_{\gamma_{A_t}} < s_{\eta_{A_t}}} \big|\rho_{\gamma_{A_t},\eta_{A_t}}\big|^2, \tag{A4}$$

where $s_{\gamma_{A_t}} = \sum_{l=1}^{n_t} i_{j_l^t} d_{j_l^t+1} d_{j_l^t+2} \cdots d_n d_{n+1}$ and $d_{n+1} = 1$. It follows that

$$\begin{aligned}
1 - \mathrm{Tr}\big(\rho_{A_t}^2\big) &= \sum_{\gamma_{A_t}} \rho_{\gamma_{A_t},\gamma_{A_t}} \big(1 - \rho_{\gamma_{A_t},\gamma_{A_t}}\big) - 2 \sum_{s_{\gamma_{A_t}} < s_{\eta_{A_t}}} \big|\rho_{\gamma_{A_t},\eta_{A_t}}\big|^2 = 2 \sum_{s_{\gamma_{A_t}} < s_{\eta_{A_t}}} \big(\rho_{\gamma_{A_t},\gamma_{A_t}} \rho_{\eta_{A_t},\eta_{A_t}} - \big|\rho_{\gamma_{A_t},\eta_{A_t}}\big|^2\big) \\
&= 2 \sum_{s_{\gamma_{A_t}} < s_{\eta_{A_t}}} \left(\sum_{\gamma_{\bar{A}_t},\eta_{\bar{A}_t}} \big|c_{\gamma_{A_t}\gamma_{\bar{A}_t}} c_{\eta_{A_t}\eta_{\bar{A}_t}}\big|^2 - \sum_{\gamma_{\bar{A}_t},\eta_{\bar{A}_t}} c_{\gamma_{A_t}\gamma_{\bar{A}_t}} c_{\eta_{A_t}\eta_{\bar{A}_t}} c^*_{\eta_{A_t}\gamma_{\bar{A}_t}} c^*_{\gamma_{A_t}\eta_{\bar{A}_t}} \right) \\
&= 2 \sum_{s_{\gamma_{A_t}} < s_{\eta_{A_t}}} \sum_{s_{\gamma_{\bar{A}_t}} < s_{\eta_{\bar{A}_t}}} \big|c_{\gamma_{A_t}\gamma_{\bar{A}_t}} c_{\eta_{A_t}\eta_{\bar{A}_t}} - c_{\eta_{A_t}\gamma_{\bar{A}_t}} c_{\gamma_{A_t}\eta_{\bar{A}_t}}\big|^2.
\end{aligned} \tag{A5}$$

1. Bound 1

From (A5) we have

$$\begin{aligned}
\frac{2\sum_{t=1}^k \big[1 - \mathrm{Tr}\big(\rho_{A_t}^2\big)\big]}{k} &= \frac{4\sum_{t=1}^k \sum_{s_{\gamma_{A_t}} < s_{\eta_{A_t}}} \sum_{s_{\gamma_{\bar{A}_t}} < s_{\eta_{\bar{A}_t}}} \big|c_{\gamma_{A_t}\gamma_{\bar{A}_t}} c_{\eta_{A_t}\eta_{\bar{A}_t}} - c_{\eta_{A_t}\gamma_{\bar{A}_t}} c_{\gamma_{A_t}\eta_{\bar{A}_t}}\big|^2}{k} \\
&\geqslant \frac{4\sum_{t=1}^k \sum_{|\eta_{A_t}|=1,|\eta_{\bar{A}_t}|=1} \big|c_{\eta_{A_t}0_{\bar{A}_t}} c_{0_{A_t}\eta_{\bar{A}_t}} - c_{0_{A_t}0_{\bar{A}_t}} c_{\eta_{A_t}\eta_{\bar{A}_t}}\big|^2}{k},
\end{aligned} \tag{A6}$$

where $0_{A_t} = (i_{j_1^t}, i_{j_2^t}, \ldots, i_{j_{n_t}^t}) = (0,0,\ldots,0)$. Here $|\eta_{A_t}|$ and $|\eta_{\bar{A}_t}|$ represent the number of 1 in η_{A_t} and $\eta_{\bar{A}_t}$, respectively. Next, we deal with (A6). By using the inequality $n\sum_{i=1}^n |a_i|^2 \geqslant (\sum_{i=1}^n |a_i|)^2$ (a_i is a complex number) and the triangle inequality, we obtain

$$\begin{aligned}
\sqrt{\frac{2\sum_{t=1}^k \big[1 - \mathrm{Tr}\big(\rho_{A_t}^2\big)\big]}{k}} &\geqslant \frac{2}{\sqrt{k\sum_{t=1}^k n_t(n-n_t)}} \sum_{t=1}^k \sum_{\substack{|\eta_{A_t}|=1 \\ |\eta_{\bar{A}_t}|=1}} \big(\big|c_{\eta_{A_t}0_{\bar{A}_t}} c_{0_{A_t}\eta_{\bar{A}_t}} - c_{0_{A_t}0_{\bar{A}_t}} c_{\eta_{A_t}\eta_{\bar{A}_t}}\big|\big) \\
&\geqslant \frac{2}{\sqrt{k\sum_{t=1}^k n_t(n-n_t)}} \sum_{t=1}^k \sum_{\substack{|\eta_{A_t}|=1 \\ |\eta_{\bar{A}_t}|=1}} \big(\big|c_{\eta_{A_t}0_{\bar{A}_t}} c_{0_{A_t}\eta_{\bar{A}_t}}\big| - \big|c_{0_{A_t}0_{\bar{A}_t}} c_{\eta_{A_t}\eta_{\bar{A}_t}}\big|\big) \geqslant H_k Q_k,
\end{aligned} \tag{A7}$$

from which it follows

$$C_{k-\mathrm{ME}}(|\psi\rangle) = \min_A \sqrt{\frac{2\sum_{t=1}^k \big[1 - \mathrm{Tr}\big(\rho_{A_t}^2\big)\big]}{k}} \geqslant H_k Q_k, \tag{A8}$$

where

$$H_k = \min_A \frac{\sqrt{k}}{\sqrt{\sum_{t=1}^k n_t(n-n_t)}} \tag{A9}$$

and

$$Q_k = 2 \sum_{\substack{s_{i_1\cdots i_n} < s_{l_1\cdots l_n} \\ |(i_1,\ldots,i_n)|=1 \\ |(l_1,\ldots,l_n)|=1}} \big|c_{i_1\cdots i_n} c_{l_1\cdots l_n}\big| - 2 \sum_{|(i_1,\ldots,i_n)|=2} \big|c_{0\cdots 0} c_{i_1\cdots i_n}\big| - (n-k) \sum_{|(i_1,\cdots,i_n)|=1} \big|c_{i_1\cdots i_n}\big|^2. \tag{A10}$$

Here $|(i_1,\ldots,i_n)|$ denote the number of $i_l = 1$ in $\{i_1,\ldots,i_n\}$.

Now suppose that $\rho = \sum_m p_m \rho^m = \sum_m p_m |\psi_m\rangle\langle\psi_m|$ is an n-partite mixed state where $|\psi_m\rangle = \sum_{i_1,\ldots,i_n} c^m_{i_1\cdots i_n}|i_1\cdots i_n\rangle$. Using (4) and (A8), we see

$$C_{k-\mathrm{ME}}(\rho) = \inf_{\{p_m,|\psi_m\rangle\}} \sum_m p_m C_{k-\mathrm{ME}}(|\psi_m\rangle) \geqslant H_k \inf_{\{p_m,|\psi_m\rangle\}} \sum_m p_m Q^m_k. \tag{A11}$$

Let $|\phi(0)\rangle = |00\cdots 0\rangle$ and $0' = 1$; we have

$$I_k(\rho,\phi(0)) = 2\sum_{i<j}\left|\rho_{\prod_{l=i+1}^{n+1} d_l,\,\prod_{l=j+1}^{n+1} d_l}\right| - 2\sum_{i<j}\sqrt{\rho_{0,0}\rho_{\prod_{l=i+1}^{n+1} d_l+\prod_{l=j+1}^{n+1} d_l,\,\prod_{l=i+1}^{n+1} d_l+\prod_{l=j+1}^{n+1} d_l}} - (n-k)\sum_i \rho_{\prod_{l=i+1}^{n+1} d_l,\,\prod_{l=i+1}^{n+1} d_l}. \tag{A12}$$

Here $d_{n+1} = 1$. Considering the three terms of (A12), we get

$$2\sum_{i<j}\left|\rho_{\prod_{l=i+1}^{n+1} d_l,\,\prod_{l=j+1}^{n+1} d_l}\right| \leqslant 2\sum_m p_m \sum_{i<j}\left|\rho^m_{\prod_{l=i+1}^{n+1} d_l,\,\prod_{l=j+1}^{n+1} d_l}\right| = \sum_m p_m\left(2\sum_{\substack{s_{i_1\cdots i_n} < s_{l_1\cdots l_n}\\ |(i_1,\ldots,i_n)|=1\\ |(l_1,\ldots,l_n)|=1}}\left|c^m_{i_1\cdots i_n}c^m_{l_1\cdots l_n}\right|\right), \tag{A13}$$

$$\begin{aligned}2\sum_{i<j}\sqrt{\rho_{0,0}\rho_{\prod_{l=i+1}^{n+1} d_l+\prod_{l=j+1}^{n+1} d_l,\,\prod_{l=i+1}^{n+1} d_l+\prod_{l=j+1}^{n+1} d_l}} &= 2\sum_{i<j}\sqrt{\left(\sum_m p_m\rho^m_{0,0}\right)\left(\sum_m p_m\rho^m_{\prod_{l=i+1}^{n+1} d_l+\prod_{l=j+1}^{n+1} d_l,\,\prod_{l=i+1}^{n+1} d_l+\prod_{l=j+1}^{n+1} d_l}\right)}\\ &\geqslant \sum_m p_m\left(2\sum_{|(i_1,\ldots,i_n)|=2}\left|c^m_{0\cdots 0}c^m_{i_1\cdots i_n}\right|\right),\end{aligned} \tag{A14}$$

$$(n-k)\sum_i \rho_{\prod_{l=i+1}^{n+1} d_l,\,\prod_{l=i+1}^{n+1} d_l} = \sum_m p_m(n-k)\sum_{|(i_1,\ldots,i_n)|=1}\left|c^m_{i_1\cdots i_n}\right|^2. \tag{A15}$$

Combining (A13)–(A15), we obtain

$$I_k(\rho,\phi(0)) \leqslant \sum_m p_m\left(2\sum_{\substack{s_{i_1\cdots i_n} < s_{l_1\cdots l_n}\\ |(i_1,\ldots,i_n)|=1\\ |(l_1,\ldots,l_n)|=1}}\left|c^m_{i_1\cdots i_n}c^m_{l_1\cdots l_n}\right| - 2\sum_{|(i_1,\ldots,i_n)|=2}\left|c^m_{0\cdots 0}c^m_{i_1\cdots i_n}\right| - (n-k)\sum_{|(i_1,\ldots,i_n)|=1}\left|c^m_{i_1\cdots i_n}\right|^2\right) = \sum_m p_m Q^m_k, \tag{A16}$$

which implies that

$$I_k(\rho,\phi(0)) \leqslant \inf_{\{p_m,|\psi_m\rangle\}} \sum_m p_m Q^m_k. \tag{A17}$$

Therefore, from (A11),

$$C_{k-\mathrm{ME}}(\rho) \geqslant H_k I_k(\rho,\phi(0)). \tag{A18}$$

Since, for any fully separable state $|\phi(x)\rangle = \otimes_{i=1}^n |x_i\rangle = |x_1x_2\cdots x_n\rangle$, there exists a local unitary transformation $U = U_1\otimes U_2\otimes\cdots\otimes U_n$ such that $U|\phi(0)\rangle = |\phi(x)\rangle$, $H_k I_k(\rho,\phi(x))$ is also a lower bound because of the invariance of $C_{k-\mathrm{ME}}(\rho)$ under local unitary transformations. Therefore we have

$$C_{k-\mathrm{ME}}(\rho) \geqslant \max_{\{|\phi(x)\rangle\}} H_k I_k(\rho,\phi(x)) \geqslant H_k I_k(\rho,\phi(x)), \tag{A19}$$

as desired.

Particularly, when $k = 2$,

$$C_{2-\mathrm{ME}}(\rho) \geqslant \begin{cases}\frac{2}{n} I_2(\rho,\phi(x)), & n \text{ is even},\\ \frac{2}{\sqrt{n^2-1}} I_2(\rho,\phi(x)), & n \text{ is odd}.\end{cases} \tag{A20}$$

Since H_2 is greater than $\frac{1}{\sqrt{2}(n-1)}$, our lower bound 1 is stronger than that in [41].

2. Bound 2

From (A5), we get

$$\frac{2\sum_{t=1}^{k}\left[1-\mathrm{Tr}\left(\rho_{A_t}^2\right)\right]}{k} = \frac{4\sum_{t=1}^{k}\sum_{s_{\gamma_{A_t}}<s_{\eta_{A_t}}}\sum_{s_{\gamma_{\bar{A}_t}}<s_{\eta_{\bar{A}_t}}}\left|c_{\gamma_{A_t}\gamma_{\bar{A}_t}}c_{\eta_{A_t}\eta_{\bar{A}_t}}-c_{\eta_{A_t}\gamma_{\bar{A}_t}}c_{\gamma_{A_t}\eta_{\bar{A}_t}}\right|^2}{k}$$
$$\geqslant \frac{4\sum_{t=1}^{k}\left(\sum_{\substack{|\eta_{A_t}|=1\\|\eta_{\bar{A}_t}|=1}}\left|c_{\eta_{A_t}0_{\bar{A}_t}}c_{0_{A_t}\eta_{\bar{A}_t}}-c_{0_{A_t}0_{\bar{A}_t}}c_{\eta_{A_t}\eta_{\bar{A}_t}}\right|^2+\sum_{\substack{|\eta_{A_t}|=n_t-1\\|\eta_{\bar{A}_t}|=n-n_t-1}}\left|c_{\eta_{A_t}1_{\bar{A}_t}}c_{1_{A_t}\eta_{\bar{A}_t}}-c_{1_{A_t}1_{\bar{A}_t}}c_{\eta_{A_t}\eta_{\bar{A}_t}}\right|^2\right)}{k}. \tag{A21}$$

Similar to the proof of bound 1,

$$\sqrt{\frac{2\sum_{t=1}^{k}\left[1-\mathrm{Tr}\left(\rho_{A_t}^2\right)\right]}{k}}$$
$$\geqslant \frac{2}{\sqrt{2k\sum_{t=1}^{k}n_t(n-n_t)}}\sum_{t=1}^{k}\left(\sum_{\substack{|\eta_{A_t}|=1\\|\eta_{\bar{A}_t}|=1}}\left|c_{\eta_{A_t}0_{\bar{A}_t}}c_{0_{A_t}\eta_{\bar{A}_t}}-c_{0_{A_t}0_{\bar{A}_t}}c_{\eta_{A_t}\eta_{\bar{A}_t}}\right|+\sum_{\substack{|\eta_{A_t}|=n_t-1\\|\eta_{\bar{A}_t}|=n-n_t-1}}\left|c_{\eta_{A_t}1_{\bar{A}_t}}c_{1_{A_t}\eta_{\bar{A}_t}}-c_{1_{A_t}1_{\bar{A}_t}}c_{\eta_{A_t}\eta_{\bar{A}_t}}\right|\right)$$
$$\geqslant \frac{2}{\sqrt{2k\sum_{t=1}^{k}n_t(n-n_t)}}\sum_{t=1}^{k}\left[\sum_{\substack{|\eta_{A_t}|=1\\|\eta_{\bar{A}_t}|=1}}\left(\left|c_{\eta_{A_t}0_{\bar{A}_t}}c_{0_{A_t}\eta_{\bar{A}_t}}\right|-\left|c_{0_{A_t}0_{\bar{A}_t}}c_{\eta_{A_t}\eta_{\bar{A}_t}}\right|\right)+\sum_{\substack{|\eta_{A_t}|=n_t-1\\|\eta_{\bar{A}_t}|=n-n_t-1}}\left(\left|c_{\eta_{A_t}1_{\bar{A}_t}}c_{1_{A_t}\eta_{\bar{A}_t}}\right|-\left|c_{1_{A_t}1_{\bar{A}_t}}c_{\eta_{A_t}\eta_{\bar{A}_t}}\right|\right)\right]$$
$$\geqslant \frac{\sqrt{k}}{\sqrt{2\sum_{t=1}^{k}n_t(n-n_t)}}(Q_k+\bar{Q}_k). \tag{A22}$$

So we get

$$C_{k-\mathrm{ME}}(|\psi\rangle)\geqslant \bar{H}_k(Q_k+\bar{Q}_k), \tag{A23}$$

where

$$\bar{H}_k=\min_A\frac{\sqrt{k}}{\sqrt{2\sum_{t=1}^{k}n_t(n-n_t)}}=\frac{H_k}{\sqrt{2}}, \tag{A24}$$

$$Q_k=2\sum_{\substack{s_{i_1\cdots i_n}<s_{l_1\cdots l_n}\\|(i_1,\ldots,i_n)|=1\\|(l_1,\ldots,l_n)|=1}}\left|c_{i_1\cdots i_n}c_{l_1\cdots l_n}\right|-2\sum_{|(i_1,\ldots,i_n)|=2}\left|c_{0\cdots0}c_{i_1\cdots i_n}\right|-(n-k)\sum_{|(i_1,\ldots,i_n)|=1}\left|c_{i_1\cdots i_n}\right|^2. \tag{A25}$$

$$\bar{Q}_k=2\sum_{\substack{s_{i_1\cdots i_n}<s_{l_1\cdots l_n}\\|(i_1,\ldots,i_n)|=n-1\\|(l_1,\ldots,l_n)|=n-1}}\left|c_{i_1\cdots i_n}c_{l_1\cdots l_n}\right|-2\sum_{|(i_1,\ldots,i_n)|=n-2}\left|c_{1\cdots1}c_{i_1\cdots i_n}\right|-(n-k)\sum_{|(i_1,\ldots,i_n)|=n-1}\left|c_{i_1\cdots i_n}\right|^2. \tag{A26}$$

Now suppose that $\rho=\sum_m p_m\rho^m=\sum_m p_m|\psi_m\rangle\langle\psi_m|$ is an n-partite mixed state where $|\psi_m\rangle=\sum_{i_1,\ldots,i_n}c^m_{i_1\cdots i_n}|i_1\cdots i_n\rangle$. Using (4) and (A23), we see that

$$C_{k-\mathrm{ME}}(\rho)=\inf_{\{p_m,|\psi_m\rangle\}}\sum_m p_mC_{k-\mathrm{ME}}(|\psi_m\rangle)\geqslant \bar{H}_k\inf_{\{p_m,|\psi_m\rangle\}}\sum_m p_m\left(Q_k^m+\bar{Q}_k^m\right). \tag{A27}$$

Let $|\phi(1)\rangle=|11\cdots1\rangle$ and $1'=0$; then

$$I_k(\rho,\phi(1))=2\sum_{i<j}\left|\rho_{\sum_{l\neq i}d_{l+1}d_{l+2}\cdots d_{n+1},\,\sum_{l\neq j}d_{l+1}d_{l+2}\cdots d_{n+1}}\right|-2\sum_{i<j}\sqrt{\rho_{\sum_l d_{l+1}d_{l+2}\cdots d_{n+1},\,\sum_l d_{l+1}d_{l+2}\cdots d_{n+1}}\,\rho_{\sum_{l\neq i,j}d_{l+1}d_{l+2}\cdots d_{n+1},\,\sum_{l\neq i,j}d_{l+1}d_{l+2}\cdots d_{n+1}}}$$
$$-(n-k)\sum_i\rho_{\sum_{l\neq i}d_{l+1}d_{l+2}\cdots d_{n+1},\,\sum_{l\neq i}d_{l+1}d_{l+2}\cdots d_{n+1}}, \tag{A28}$$

where $d_{n+1} = 1$. For the first term of (A28),

$$2\sum_{i<j}\left|\rho_{\sum_{l\neq i} d_{l+1}d_{l+2}\cdots d_{n+1},\, \sum_{l\neq j} d_{l+1}d_{l+2}\cdots d_{n+1}}\right| \leqslant \sum_m p_m \left(2 \sum_{\substack{s_{i_1\cdots i_n} < s_{l_1\cdots l_n}\\ |(i_1,\cdots,i_n)| = n-1\\ |(l_1,\cdots,l_n)| = n-1}} \left|c^m_{i_1\cdots i_n} c^m_{l_1\cdots l_n}\right|\right). \tag{A29}$$

For the second term,

$$\begin{aligned} &2\sum_{i<j}\sqrt{\rho_{\sum_l d_{l+1}d_{l+2}\cdots d_{n+1},\, \sum_l d_{l+1}d_{l+2}\cdots d_{n+1}}\,\rho_{\sum_{l\neq i,j} d_{l+1}d_{l+2}\cdots d_{n+1},\, \sum_{l\neq i,j} d_{l+1}d_{l+2}\cdots d_{n+1}}}\\ &= 2\sum_{i<j}\sqrt{\left(\sum_m p_m \rho^m_{\sum_l d_{l+1}d_{l+2}\cdots d_{n+1},\, \sum_l d_{l+1}d_{l+2}\cdots d_{n+1}}\right)\left(\sum_m p_m \rho^m_{\sum_{l\neq i,j} d_{l+1}d_{l+2}\cdots d_{n+1},\, \sum_{l\neq i,j} d_{l+1}d_{l+2}\cdots d_{n+1}}\right)}\\ &\geqslant \sum_m p_m\left(2\sum_{|(i_1,\cdots,i_n)|=n-2}\left|c^m_{1\cdots 1}c^m_{i_1\cdots i_n}\right|\right). \end{aligned} \tag{A30}$$

For the third term,

$$(n-k)\sum_i \rho_{\sum_{l\neq i} d_{l+1}d_{l+2}\cdots d_{n+1},\, \sum_{l\neq i} d_{l+1}d_{l+2}\cdots d_{n+1}} = \sum_m p_m\left[(n-k)\sum_{|(i_1,\cdots,i_n)|=n-1}|c_{i_1\cdots i_n}|^2\right]. \tag{A31}$$

Combining (A29)–(A31) gives

$$I_k(\rho,\phi(1)) \leqslant \sum_m p_m \bar{Q}^m_k. \tag{A32}$$

From (A16), (A27), and (A32), we obtain

$$C_{k-\mathrm{ME}}(\rho) \geqslant \bar{H}_k[I_k(\rho,\phi(0)) + I_k(\rho,\phi(1))]. \tag{A33}$$

Note that for any fully separable state $|\phi(x)\rangle = \otimes_{i=1}^n |x_i\rangle$, there is a local unitary transformation $V = V_1 \otimes V_2 \otimes \cdots \otimes V_n$ satisfying $V|\phi(0)\rangle = |\phi(x)\rangle$ and $V|\phi(1)\rangle = |\phi(y)\rangle$. Thus $\bar{H}_k[I_k(\rho,\phi(x)) + I_k(\rho,\phi(y))]$ is also a lower bound because of the invariance of $C_{k-\mathrm{ME}}(\rho)$ under local unitary transformations, so we have

$$C_{k-\mathrm{ME}}(\rho) \geqslant \max_{\{\phi(x),\phi(y)\}} \bar{H}_k[I_k(\rho,\phi(x)) + I_k(\rho,\phi(y))] \geqslant \bar{H}_k[I_k(\rho,\phi(x)) + I_k(\rho,\phi(y))]. \tag{A34}$$

Here $|\phi(x)\rangle = \otimes_{i=1}^n |x_i\rangle$ and $|\phi(y)\rangle = \otimes_{i=1}^n |y_i\rangle$ are fully separable states in which $|x_i\rangle$ and $|y_i\rangle$ are orthogonal. The proof is complete.

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