Deformed Heisenberg algebra with minimal length and the equivalence principle

V. M. Tkachuk*

Department for Theoretical Physics, Ivan Franko National University of Lviv, 12 Drahomanov St., Lviv, UA-79005, Ukraine

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Studies in string theory and quantum gravity lead to the generalized uncertainty principle (GUP) and suggest the existence of a fundamental minimal length which, as was established, can be obtained within the deformed Heisenberg algebra. The first look on the classical motion of bodies in a space with corresponding deformed Poisson brackets in a uniform gravitational field can give an impression that bodies of different mass fall in different ways and, thus, the equivalence principle is violated. Analyzing the kinetic energy of a composite body, we find that the motion of its center of mass in the deformed space depends on some effective parameter of deformation. It gives a possibility to recover the equivalence principle in the space with deformed Poisson brackets and, thus, GUP is reconciled with the equivalence principle. We also show that the independence of kinetic energy on composition leads to the recovering of the equivalence principle in the space with deformed Poisson brackets.

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I. INTRODUCTION

Recently, much attention has been devoted to studies of different systems in a space with a deformed Heisenberg algebra that takes into account the quantum nature of space on the phenomenological level. These works are motivated by several independent lines of investigations in string theory and quantum gravity (see, e.g., Refs. [1–3]) which lead to the generalized uncertainty principle (GUP),

$$\Delta X \geqslant \frac{\hbar}{2}\left(\frac{1}{\Delta P} + \beta \Delta P\right), \tag{1}$$

and suggest the existence of the fundamental minimal length, $\Delta X_{\min} = \hbar\sqrt{\beta}$, which is of order of Planck's length, $l_p = \sqrt{\hbar G/c^3} \simeq 1.6 \times 10^{-35}$ m.

It was established that minimal length can be obtained in the frame of small quadratic modification (deformation) of the Heisenberg algebra [4,5],

$$[X,P] = i\hbar(1 + \beta P^2). \tag{2}$$

In the classical limit $\hbar \to 0$ the quantum-mechanical commutator for operators is replaced by the Poisson bracket for corresponding classical variables,

$$\frac{1}{i\hbar}[X,P] \to \{X,P\}, \tag{3}$$

which, in the deformed case, reads

$$\{X,P\} = (1 + \beta P^2). \tag{4}$$

We point out that, historically, the first algebra of that kind in the relativistic case was proposed by Snyder in 1947 [6]. But only after investigations in string theory and quantum gravity did considerable interest in the studies of physical properties of classical and quantum systems in spaces with deformed algebras appear. Observation that GUP can be obtained from the deformed Heisenberg algebra opens the possibility to study the influence of minimal length on properties of physical systems on the quantum level as well as on the classical one.

Deformed commutation relations bring new difficulties into quantum mechanics as well as classical mechanics. Only a few problems are known to be solved exactly. They include one-dimensional harmonic oscillator with minimal uncertainty in position [4] and with minimal uncertainty in position and momentum [7,8], D-dimensional isotropic harmonic oscillator [9,10], three-dimensional Dirac oscillator [11], $(1+1)$-dimensional Dirac oscillator within Lorentz-covariant deformed algebra [12], one-dimensional Coulomb problem [13], and the singular inverse-square potential with a minimal length [14,15]. The three-dimensional Coulomb problem with deformed Heisenberg algebra was studied within perturbation theory [16–20]. In Ref. [21] the scattering problem in the deformed space with minimal length was studied. The ultracold neutrons in gravitational field with minimal length were considered in Refs. [22–24]. The influence of minimal length on the Lamb shift, Landau levels, and tunneling current in a scanning tunneling microscope was studied [25,26]. The Casimir effect in a space with minimal length was examined in Ref. [27]. In Ref. [28] the effect of noncommutativity and of the existence of a minimal length on the phase space of cosmological model was investigated. The authors of paper [29] studied various physical consequences which follow from the noncommutative Snyder space-time geometry. The classical mechanics in a space with deformed Poisson brackets was studied in Refs. [30–32]. The composite system (N-particle system) in the deformed space with minimal length was studied in Refs. [33,34].

Note that deformation of Heisenberg algebra brings not only technical difficulties in solving of corresponding equations but also problems of a fundamental nature. One of them is the violation of the equivalence principle in space with minimal length [35]. This is the result of the assumption that the parameter of deformation for macroscopic bodies of different mass is unique. In Ref. [33] we showed that the center of mass of a macroscopic body in deformed space is described by an effective parameter of deformation, which is essentially smaller than the parameters of deformation for particles comprising the body. Using the result of Ref. [33] for the effective parameter of deformation, we show that the

*tkachuk@ktf.franko.lviv.ua, voltkachuk@gmail.com

equivalence principle in the space with minimal length can be recovered. In Sec. III we reproduce the result of Ref. [33] concerning the effective parameter of deformation for the center of mass on the classical level and, in addition, show that the independence of kinetic energy on the composition leads to the recovering of the equivalence principle in the space with a deformed Poisson bracket.

II. FREE FALL OF PARTICLE IN A UNIFORM GRAVITATIONAL FIELD

The Hamiltonian of a particle (a macroscopic body which we consider as a point particle) of mass m in a uniform gravitational field reads

$$H = \frac{P^2}{2m} - mgX, \tag{5}$$

where the gravitational field is characterized by the factor g is directed along the x axis. Note that here the inertial mass (m in the first term) is equal to the gravitational mass (m in the second one). The Hamiltonian equations of motion in space with deformed Poisson brackets are as follows:

$$\dot{X} = \{X,H\} = \frac{P}{m}(1+\beta P^2), \tag{6}$$

$$\dot{P} = \{P,H\} = mg(1+\beta P^2). \tag{7}$$

We impose zero initial conditions for position and momentum, namely $X = 0$ and $P = 0$ at $t = 0$. These equations can be solved easily. From the second equation we find

$$P = \frac{1}{\sqrt{\beta}}\tan(\sqrt{\beta}mgt). \tag{8}$$

From the first equation we obtain for velocity

$$\dot{X} = \frac{1}{m\sqrt{\beta}}\frac{\tan(\sqrt{\beta}mgt)}{\cos^2(\sqrt{\beta}mgt)} \tag{9}$$

and for position

$$X = \frac{1}{2gm^2\beta}\tan^2(\sqrt{\beta}mgt). \tag{10}$$

One can verify that the motion is periodic with period $T = \frac{\pi}{m\sqrt{\beta}g}$. The particle moves from $X = 0$ to $X = \infty$ and then reflects from ∞ and moves in the opposite direction to $X = 0$. But from the physical point of view, this solution is correct only for time $t \ll T$, when the velocity of particle is much smaller than the speed of light. In other cases, the relativistic mechanics must be considered.

It is instructive to write out the results for velocity and coordinate in the first order over β,

$$\dot{X} = gt\left(1 + \frac{4}{3}\beta m^2 g^2 t^2\right), \tag{11}$$

$$X = \frac{gt^2}{2}\left(1 + \frac{2}{3}\beta m^2 g^2 t^2\right). \tag{12}$$

In the limit $\beta \to 0$ we reproduce the well-known results

$$\dot{X} = gt, \quad X = \frac{gt^2}{2}, \tag{13}$$

where kinematic characteristics, such as velocity and position of a free-falling particle depend only on initial position and velocity of the particle and do not depend on the composition and mass of the particle. It is in agreement with the weak equivalence principle, also known as the universality of free fall or the Galilean equivalence principle. Note that in the nondeformed case, when the Newtonian equation of motion in a gravitational field is fulfilled, that the weak equivalence principle abides by the statement of equivalence of inertial and gravitational masses.

As we see from (9) and (10) or (11) and (12), in the deformed space, the trajectory of the point mass in the gravitational field depends on the mass of the particle if we suppose that parameter of deformation is the same for all bodies. So, in this case, the equivalence principle is violated. In Ref. [33] we showed on the quantum level that, in fact, the motion of the center of mass of a composite system in deformed space is governed by an effective parameter (in Ref. [33] it is denoted as $\tilde{\beta}_0$, and here we denote it as β). So, the parameter of deformation for a macroscopic body is

$$\beta = \sum_i \mu_i^3 \beta_i, \tag{14}$$

where $\mu_i = m_i/\sum_i m_i$, m_i and β_i are the masses and parameters of deformation of particles which form composite system (body). Note that in the next section we derive this result considering the kinetic energy of a body consisting of N particles.

First, let us consider a special case, $m_i = m_1$ and $\beta_i = \beta_1$, when a body consists of the same elementary particles. We then find

$$\beta = \frac{\beta_1}{N^2}, \tag{15}$$

where N is the number of particles of a body with mass $m = Nm_1$. Note that expressions (9) and (10) contain combination $\sqrt{\beta}m$. Substituting the effective parameter of deformation β_1/N^2 instead of β we find

$$\sqrt{\beta}m = \sqrt{\beta_1}m/N = \sqrt{\beta_1}m_1. \tag{16}$$

As a result, the trajectory now depends not on the mass of the macroscopic body but on $\sqrt{\beta_1}m_1$, which is the same for bodies of different mass. So the equivalence principle is recovered.

The general case when a body consists of different elementary particles is more complicated. The situation then is possible when different combinations of elementary particles lead to the same mass but with different effective parameters of deformation. The motion of bodies of equal mass but different composition then will differ. This also violates the weak equivalence principle. The equivalence principle can be recovered when we suppose that

$$\sqrt{\beta_1}m_1 = \sqrt{\beta_2}m_2 = \cdots = \sqrt{\beta_N}m_N = \gamma. \tag{17}$$

The effective parameter of deformation for a macroscopic body is

$$\beta = \sum_i \frac{m_i^3}{\left(\sum_i m_i\right)^3}\beta_i = \frac{\gamma^2}{\left(\sum_i m_i\right)^2} = \frac{\gamma^2}{m^2} \tag{18}$$

and, thus,

$$\sqrt{\beta}m = \gamma, \tag{19}$$

which is the same as (17). Note that the trajectory of motion in this case does not depend on mass and depends only on

γ, which takes the same value for all bodies. It means that bodies of different mass and different composition move in a gravitational field in the same way and, thus, the weak equivalence principle is not violated when (17) is satisfied. Equation (17) brings one new fundamental constant, γ. Note that parameter $1/\gamma$ has the dimension of velocity. The parameters of deformation β_i of particles or macroscopic bodies of mass m_i are determined by fundamental constant γ as follows:

$$\beta_i = \frac{\gamma^2}{m_i^2}, \tag{20}$$

So the parameter of deformation is completely determined by the mass of a particle. In the next section we derive formula (14) on the classical level and give some arguments concerning the relation (17).

III. KINETIC ENERGY OF A COMPOSITE SYSTEM IN DEFORMED SPACE AND PARAMETER OF DEFORMATION

In this section we use the following statement: *The kinetic energy has the additivity property and does not depend on the composition of a body but only on its mass.*

First, we consider *the additivity property of the kinetic energy*. Let us consider N particles with masses m_i and deformation parameters β_i. It is equivalent to the situation when the macroscopic body is divided into N parts which can be treated as point particles with corresponding masses and parameters of deformation. We consider the case when each particle of the system moves with the same velocity as the whole system.

Let us rewrite the kinetic energy as a function of velocity. From the relation between velocity and momentum (6) in the first approximation over β we find

$$P = m\dot{X}(1 - \beta m^2 \dot{X}^2). \tag{21}$$

The kinetic energy as a function of velocity in the first-order approximation over β then reads

$$T = \frac{m\dot{X}^2}{2} - \beta m^3 \dot{X}^4. \tag{22}$$

The kinetic energy of the whole system is given by (22), where $m = \sum_i m_i$. On the other hand, the kinetic energy of the whole system is the sum of kinetic energies of particles which constitute the system,

$$T = \sum_i T_i = \frac{m\dot{X}^2}{2} - \sum_i \beta_i m_i^3 \dot{X}^4, \tag{23}$$

where we take into account that velocities of all particles are the same as the velocity of the whole system, $\dot{X}_i = \dot{X}$, $i = 1, \dots, N$. Comparing (22) and (23) we obtain (14).

Now let us consider *the independence of kinetic energy on the composition of a body*. It is enough to consider a body of a fixed mass consisting of two parts (particles) with masses $m_1 = m\mu$ and $m_2 = m(1-\mu)$, where $0 \leqslant \mu \leqslant 1$. Parameters of deformation for the first and second particles are $\beta_1 = \beta_\mu$ and $\beta_2 = \beta_{1-\mu}$; here we write explicitly that parameters of deformations are some function of mass ($\mu = m_1/m$ is dimensionless mass). The particles with different masses constitute the body with the same mass, $m = m_1 + m_2$. So, in this situation, we have a body of the same mass but with a different composition.

The kinetic energy of the whole body is given by (22) with the parameter of deformation

$$\beta = \beta_\mu \mu^3 + \beta_{1-\mu}(1-\mu)^3. \tag{24}$$

Since the kinetic energy does not depend on the composition, the parameter of deformation for the whole body must be fixed, $\beta = \text{const}$, for different μ. Thus (24) is the equation for β_μ as a function of μ at fixed β. One can verify that the solution reads

$$\beta_\mu = \frac{\beta}{\mu^2}. \tag{25}$$

Taking into account that $\mu = m_1/m$ we find

$$\beta_1 m_1^2 = \beta m^2, \tag{26}$$

which corresponds to (17). So the independence of the kinetic energy from the composition leads to the one fundamental constant, $\gamma^2 = \beta m^2$. The parameters of deformation β_i of particles or composite bodies of different masses m_i then are $\beta_i = \gamma^2/m_i^2$, which is in agreement with relation (20).

IV. CONCLUSIONS

One of the main results of the paper is the expression for the parameter of deformation for particles or bodies of different mass (20) which recovers the equivalence principle and, thus, the equivalence principle is reconciled with the generalized uncertainty principle. It is necessary to stress that expression (20) was derived also in Sec. III from the condition of the independence of kinetic energy on composition.

Note that (20) contains the same constant γ for different particles and parameter of deformation is inverse to the squared mass. The constant γ has dimension inverse to velocity. Therefore, it is convenient to introduce a dimensionless constant γc, where c is the speed of light. In order to make some speculations concerning the possible value of γc, we suppose that for the electron the parameter of deformation β_e is related to Planck's length, namely

$$\hbar\sqrt{\beta_e} = l_p = \sqrt{\hbar G/c^3}. \tag{27}$$

We then obtain

$$\gamma c = c\sqrt{\beta} m_e = \sqrt{\alpha \frac{G m_e^2}{e^2}} \simeq 4.2 \times 10^{-23}, \tag{28}$$

where $\alpha = e^2/\hbar c$ is the fine structure constant.

Fixing the parameter of deformation for electron we can calculate the parameter of deformation for particles or bodies of different mass. It is more instructive to write the minimal length for space where the composite body of mass m lives,

$$\hbar\sqrt{\beta} = \frac{m_e}{m}\hbar\sqrt{\beta_e} = \frac{m_e}{m} l_p. \tag{29}$$

As an example, let us consider nucleons (protons or neutrons). The parameter of deformation for nucleons β_{nuc} or the minimal length for nucleons reads $\hbar\sqrt{\beta_{\text{nuc}}} \simeq l_p/1840$. So the effective minimal length for nucleons is three orders smaller than that for electrons.

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