Lifshitz formula by a spectral summation method

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The Lifshitz formula is derived by making use of the spectral summation method, which is a mathematically rigorous simultaneous application of both the mode-by-mode summation technique and scattering formalism. The contributions to the Casimir energy of electromagnetic excitations of different types (surface modes, waveguide modes, and photonic modes) are clearly retraced. A correct transition to imaginary frequencies is accomplished with allowance for all the peculiarities of the frequency equations and pertinent scattering data in the complex ω plane, including, in particular, the cuts connecting the branch points and complex roots of the frequency equations (quasinormal modes). The important feature of our approach is a special choice of appropriate passes in the contour integrals which are used for transition to imaginary frequencies. As a result, the longstanding problem of cuts in the complex ω plane is solved completely. Some subtleties and vague points in previous derivations of the Lifshitz formula are elucidated. For completeness of the presentation, the necessary mathematical facts are also stated, namely, solution of the Maxwell equations for configurations under consideration, scattering formalism for parallel-plane interfaces, determination of the frequency equation roots, and others.

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I. INTRODUCTION

It is not overstatement to say that the Lifshitz formula is the basis for practically all the Casimir calculations dealing with plane boundaries [1]. However, it was recognized in the literature [2–5] that the derivation of this formula in the original paper [6,7] is very complicated. It proved that especially involved is the transition to the imaginary frequencies when obtaining the Lifshitz formula by deforming the contours of integration in the complex frequency plane. As far as we know, nobody has succeeded in repeating this part of Lifshitz' calculations. Furthermore, the Lifshitz paper [7] did not trace in detail the contributions to the vacuum energy generated by different branches of the electromagnetic excitation spectrum (propagating waves and evanescent waves) and, in particular, different presentations of these contributions in terms of real and imaginary frequencies. Now these points have become important in searching for the materials with the properties necessary for obtaining the desired characteristics of the Casimir forces.

Later on, the Lifshitz formula was obtained by making use of different methods, namely: the quantum field theory technique in condensed matter physics [8,9], mode-by-mode summation [1,3,4,10–15], methods of quantized surface modes and quantized Fresnel modes [16–18], scattering formalism [19–21], local Green's functions [22–24], and so on. These approaches are quite different, and at first glance it is difficult to reveal their interrelation, although these methods lead to the same Lifshitz formula, which has never been in doubt.

For the sake of understanding this situation, we propose in the present paper another derivation of the Lifshitz formula, namely, this formula will be obtained by making use of the spectral summation method. This approach is a mathematically rigorous simultaneous application of both the mode-by-mode summation technique and the scattering formalism. At the same time, some subtleties and vague points in previous derivations of the Lifshitz formula will be elucidated.

Our derivation of the Lifshitz formula includes two steps: First, we determine the spectrum of electromagnetic excitations for given boundaries, taking into account the material properties of the media under consideration [25], and afterward we accomplish the summation of the relevant zero-point energies [23,26,27] connected with all branches of this spectrum. For a description of the electromagnetic fields in matter, we shall use the dielectric formalism with allowance for the usual temporal dispersion. In the framework of this approach, one can concentrate on the electromagnetic field dynamics, while the dynamics of induced charges and currents is taken into account by introducing the permittivity and permeability $\varepsilon(\omega)$, $\mu(\omega)$.[1] Some nontrivial points in calculating the electromagnetic energy with allowance for the media dispersion in the Casimir studies are recently clarified in Ref. [29].

When determining the spectrum of the electromagnetic field, all physically relevant solutions to the Maxwell equations should be taken into account. Obviously, such solutions are all squared integrable solutions which belong to the L_2 functional space. In addition, this set of solutions should be complete. In the rigorous spectral theory of differential operators [30,31], the following assertion is proved: the functional space including the solutions that describe the bound states (discrete spectrum branch) and scattering states (continuous branch of the spectrum) is complete with respect to the L_2 norm. It is these solutions that we shall find in the problem under study, and on this basis we shall determine the relevant

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[1]The electromagnetic field is coupled, through the Maxwell equations, with charges and currents; therefore one can take, as a dynamic variable, the local displacement of a continuous charged liquid that describes the free electrons inside the medium [28].

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spectrum, i.e., the admissible values of the electromagnetic oscillation frequencies ω.[2] Upon determining the complete set of solutions to the Maxwell equations, quantization of the electromagnetic field is trivial because we deal here with the sum of two infinite sets (discrete and continuous) of noninteracting oscillators. The discrete set of oscillators describes the evanescent waves including the surface and waveguide modes,[3] and a continuous set of oscillators is responsible for the scattering states. All these modes will be rigorously defined in the course of constructing the complete set of solutions to the Maxwell equations (see below).

The summation of zero-point energies of the discrete modes can be accomplished, obviously, in a straightforward way, while for the summation of such energies appertaining to scattering states, one has to take advantage of the scattering formalism [34]. Preceding derivations of the Lifshitz formula in the framework of the mode-by-mode summation alone or by making use of the scattering formalism only seem somewhat contradictory at first glance. Indeed, each of these methods can be rigorously applied to one branch of the spectrum separately, namely, the mode-by-mode summation is applicable to the discrete part of the spectrum solely and the scattering formalism enables one to take into account the scattering states, i.e., it is applicable to the continuous part of the electromagnetic spectrum only.

However, it turns out that the "naive" presentation, by the contour integral in the complex frequency plane, of the contribution to the vacuum energy generated by one spectrum branch automatically incorporates the contribution of the other branch. In order to show this, we perform rigorous summation of the zero-point energies appertaining to both the spectrum branches (discrete and continuous ones). It is this approach that enables one to overcome known drawbacks in this field, for example, to take into account the cuts in the complex frequency plane [3,4,11–14], to define correctly the contributions to the vacuum energy due to different branches of the electromagnetic spectrum [12,13], and so on.

The important step in the Casimir calculations is the transition to the imaginary frequencies. In our opinion this transition was not rigourously justified at least in the framework of the mode-by-mode summation technique. It is this issue that resulted in some confusion when deriving the Lifshitz formula by the mode summation [3,4,10–14,16–18] (see Conclusion in the present paper). In order to accomplish this transition in a consistent way, the analysis of the analytical properties of the frequency equations and the scattering data in the complex ω plane should be conducted as a preliminary, and the needed cuts connecting the branch points in this plane should be done. In the pertinent literature [1–4,10–22] these issues were not examined. We are going to fill in this gap.

It is worth noting here that the analytic properties of the scattering matrix (or the Jost function) in the Casimir studies are different in comparison with the standard theory of potential scattering. In fact, they are close to those for the Klein-Gordon equation, the role of mass squared being played by k^2, where k is the wave vector along the unbounded dimensions. This implies, in particular, that the analytic properties of the scattering matrix with respect to the complex frequency ω in the Casimir calculations should be revealed by direct analysis of its explicit form without referring to the nonrelativistic potential scattering in the framework of the Schrödinger equation.

The layout of the paper is the following. Section II is devoted to the spectrum of electromagnetic excitations in the problem under consideration. For the Casimir studies the materials permitting the surface waves are of current interest. The permittivity of these media can acquire, in certain frequency bands, negative values. Typical examples of these materials (metals and isotropic dielectrics) are discussed briefly. In Sec. II A, the general solutions to the Maxwell equations for plane parallel interfaces are constructed. Both branches of the spectrum are considered in detail. In Sec. III the Lifshitz formula for the Casimir energy is derived by summing up the zero-point energies of discrete modes and scattering states. In Sec. IV (Conclusion), the obtained results are summed and unsolved problems in this field are briefly outlined.

II. THE SPECTRUM OF ELECTROMAGNETIC EXCITATIONS

The electromagnetic excitations in the media are diverse, and they essentially depend on the dielectric and magnetic properties of the background. For simplicity, only isotropic media are considered in this paper.

In the Casimir studies we are interested in the long-wave excitations with $\lambda \gtrsim d$, where d is the characteristic scale in this field ($d \sim 10$–100 nm) and λ is the considered wavelength. The description of the condensed matter in terms of the dielectric permittivity assumes the long-wave approximation as well.

In applications the dielectric properties of the materials play the leading role; therefore in what follows we put the magnetic permeability equal to one.

Of particular interest for theoretical and experimental Casimir studies are the media with dielectric permittivity taking on negative values over a certain frequency range. Only interfaces of such media support surface electromagnetic waves which can be treated as collective excitations of electron density. The surface waves together with waveguide solutions belong to the discrete branch of the spectrum (see Sec. II B).

Metals are typical media of this sort. The overall picture of the electromagnetic waves (excitations) in metal resembles that in plasma but is not identical with it. In the bulk of the metal the local electron density oscillates with the characteristic plasma frequency

$$\omega_p^2 = \frac{4\pi n e^2}{m}, \tag{1}$$

where n is the mean electron density in the metal and m is the electron mass, the role of the restoring force being played by the electrostatic field. The volume plasma excitations give

[2]In the general case, the spectrum of the differential operator may be more complicated [32]. However, in the Casimir calculations such problems do not occur.

[3]In physical problems dealing with the plane boundaries, the surface modes and waveguide modes are called sometimes the guided modes [33].

rise to surface excitations of the electron density (surface plasmons) which propagate along the metal-dielectric or metal-vacuum interface. Obviously, the oscillations of the free charge density cause, according to the Maxwell equations, the oscillations of the electromagnetic field.

For metals the following dielectric function is commonly used (plasma model),

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}\,, \tag{2}$$

where ω_p is the plasma frequency [Eq. (1)]. The dielectric permittivity $\varepsilon(\omega)$ takes negative values in the region $\omega < \omega_p$. The field oscillations brought about by the surface excitations of the electron density are also localized near the surface and are referred to as the surface plasmons (see the corresponding analysis of the solutions to the Maxwell equations in Sec. II A). Inside the metal, the bulk waves propagate with the frequencies $\omega > \omega_p$. The transversal and longitudinal bulk waves form the continuous branch of the spectrum.

It is worth noting here the following. In order to describe the dielectric properties of a metal more precisely, instead of the simple plasma model [Eq. (2)], one has to use the Drude model [1], which takes into account dissipation,

$$\varepsilon_D(\omega) = 1 - \frac{\omega_p^2}{\omega(\omega + i\gamma)}\,, \tag{3}$$

where γ is the relaxation parameter. However, all this immediately results in the necessity to consider the complex-valued eigenfrequencies. So far, the spectral summation method has not been extended to this case (see also Sec. IV Conclusion). Therefore, we shall further describe the dielectric function for metal by the simple plasma model[4] [Eq. (2)].

In the isotropic dielectric the permittivity is described with a good accuracy by an oscillator model [35]. To gain better understanding, one can consider the atoms in a dielectric as weakly damped harmonic oscillators with eigenfrequencies ω_i excited by an external electric field $\mathbf{E} = \mathbf{E_0}e^{-i\omega t}$. The N-oscillator model gives the following dielectric function:

$$\varepsilon(\omega) = \varepsilon(\infty) + \sum_j^N \frac{S_j\,\omega_j^2}{\omega_j^2 - \omega^2 - i\Gamma_j\omega}, \tag{4}$$

where S_j is the jth oscillator strength and Γ_j is its relaxation parameter. In what follows, we neglect again the absorption (undamped oscillators, $\Gamma_j = 0$) and consider real dielectric permittivity. It is negative for ω approaching ω_j from the right. Thus, the condition for the existence of the surface plasmon at the flat dielectric-vacuum interface is fulfilled near the narrow absorption lines of the dielectric media. The number of the plasmons in this case is equal to the number of oscillators in the model. The bulk waves in dielectrics (continuous spectrum) lie in the ranges of frequencies where $\varepsilon(\omega)$ is positive.

[4]At separations below 1 μm, the Drude and plasma models in the Lifshitz formula lead to the values of the Casimir energy and Casimir force differing by less than 2%. One might expect that at such separations thermal corrections are not important [1].

A. General solutions to the Maxwell equations

First, we recall briefly the formulation of the Maxwell theory for compound media with constant permittivity ε and permeability μ in each region. In this case, the harmonic in time electric ($\mathbf{E}$) and magnetic ($\mathbf{H}$) fields are described by the Maxwell equations [36],

$$\nabla \times \mathbf{E} = i\,\frac{\omega}{c}\,\mathbf{B}\,, \qquad \nabla\cdot\mathbf{D} = 0\,, \tag{5}$$

$$\nabla \times \mathbf{H} = -i\,\frac{\omega}{c}\,\mathbf{D}, \qquad \nabla\cdot\mathbf{B} = 0\,, \tag{6}$$

$$\mathbf{D} = \varepsilon\mathbf{E}\,, \qquad \mathbf{B} = \mu\mathbf{H}\,, \quad \mathbf{x} \notin \Sigma\,, \tag{7}$$

which hold outside the interface Σ separating the regions with different ε and μ, and by matching conditions on Σ. These conditions require the continuity of tangential components of the fields $\mathbf{E}$ and $\mathbf{H}$ when crossing Σ :

$$\text{discont}\,(\mathbf{E}_\parallel) = 0, \quad \text{discont}\,(\mathbf{H}_\parallel) = 0\,. \tag{8}$$

The Gauss units are used and it is assumed that external charges and currents are absent (both volume and surface ones). The common time factor $e^{-i\omega t}$ will be dropped.

When allowing for the time dispersion, the permittivity ε and permeability μ in material equations (7) and further should be treated as functions of the frequency ω. General solution to Eqs. (5)–(7) can be represented in terms of two independent Hertz vectors in the following way [36–38]:

$$\mathbf{E} = \nabla\times\nabla\times\mathbf{\Pi}', \qquad \mathbf{H} = -i\varepsilon\,\frac{\omega}{c}\,\nabla\times\mathbf{\Pi}' \quad \text{(TM modes)}; \tag{9}$$

$$\mathbf{E} = i\mu\,\frac{\omega}{c}\,\nabla\times\mathbf{\Pi}'', \qquad \mathbf{H} = \nabla\times\nabla\times\mathbf{\Pi}'' \quad \text{(TE modes)}\,. \tag{10}$$

Here $\mathbf{\Pi}'$ is the electric Hertz vector, $\mathbf{\Pi}''$ is the magnetic Hertz vector, and c is the velocity of light in vacuum.

In each region with given $\varepsilon(\omega)$ and $\mu(\omega)$ the Hertz vectors obey the Helmholtz vector equation

$$(\nabla^2 + k^2)\mathbf{\Pi} = 0\,, \tag{11}$$

where the wave number k is given by

$$k^2 = \varepsilon(\omega)\mu(\omega)\,\frac{\omega^2}{c^2}\,. \tag{12}$$

The boundary conditions [Eq. (8)] involve the complete fields $\mathbf{E}$ and $\mathbf{H}$, i.e., the sums of the TE and TM modes. Fortunately, for some interface geometries these conditions do not couple the TE and TM polarizations. It is true for plane interfaces (see below), for spheres, and in some other cases [1,23,26,27,36].

It is known that in the source-free case the general solution of Maxwell's equations can be derived from two real scalar functions [39–41] which may be chosen in different ways. Thus, the Hertz potentials have in fact only one nonvanishing component. The precise choice of these components is determined by the interface geometry and for reasons of simplicity and convenience.

We consider the flat interface parallel to the (x,y) coordinate plane and cutting the z axis at the point $z = z_0$. Let $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ be the unit base vectors in the chosen coordinate system.

The z components of the Hertz potentials are treated as two independent functions in the general solution to the Maxwell equations:

$$\mathbf{\Pi}' = \mathbf{e}_z e^{i\mathbf{k}\cdot\mathbf{s}}\Phi(z), \quad \mathbf{\Pi}'' = \mathbf{e}_z e^{i\mathbf{k}\cdot\mathbf{s}}\Psi(z). \tag{13}$$

Here $\mathbf{k}$ is a two-component wave vector parallel to the interface, $\mathbf{k} = (k_x, k_y)$, and $\mathbf{s} = (x,y)$. The common time-dependent factor $e^{-i\omega t}$ is dropped as usual.

By substituting the Hertz potentials [Eq. (13)] in Eqs. (9) and (10) we get the fields for the TE modes,

$$\mathbf{E} = \mu \frac{\omega}{c} k_x \mathbf{e}_y e^{i\mathbf{k}\cdot\mathbf{s}}\Psi(z), \tag{14}$$

$$\mathbf{H} = ik_x\mathbf{e}_x e^{i\mathbf{k}\cdot\mathbf{s}}\Psi'(z) + \mathbf{e}_z k^2 e^{i\mathbf{k}\cdot\mathbf{s}}\Psi(z), \tag{15}$$

and for the TM modes,

$$\mathbf{E} = ik_x\mathbf{e}_x e^{i\mathbf{k}\cdot\mathbf{s}}\Phi'(z) + \mathbf{e}_z k^2 e^{i\mathbf{k}\cdot\mathbf{s}}\Phi(z), \tag{16}$$

$$\mathbf{H} = -\varepsilon \frac{\omega}{c} k_x \mathbf{e}_y e^{i\mathbf{k}\cdot\mathbf{s}}\Phi(z). \tag{17}$$

Without loss of generality, we directed the x axis along the vector $\mathbf{k}$: $\mathbf{k} = (k_x = \pm k, 0)$.

For the fields [Eqs. (14)–(17)] to obey the Maxwell equations (5)–(7), the Hertz potentials should meet the Helmholtz equation (11), which now assumes the form

$$-\Phi''(z) = \left(\varepsilon\mu\frac{\omega^2}{c^2} - k^2\right)\Phi(z) \quad (\text{TM modes}), \tag{18}$$

$$-\Psi''(z) = \left(\varepsilon\mu\frac{\omega^2}{c^2} - k^2\right)\Psi(z) \quad (\text{TE modes}),$$
$$-\infty < z < \infty, \quad z \neq z_0, \quad k^2 = k_x^2 + k_y^2. \tag{19}$$

Substitution of the fields [Eqs. (14)–(17)] into the continuity equations (8) results in the matching conditions at $z = z_0$ separately for the functions Φ and Ψ,

$$[\varepsilon(z_0)\Phi(z_0)] = 0, \quad [\Phi'(z_0)] = 0 \quad (\text{TM modes}), \tag{20}$$

$$[\mu(z_0)\Psi(z_0)] = 0, \quad [\Psi'(z_0)] = 0 \quad (\text{TE modes}), \tag{21}$$

where the notation

$$[F(z)] \equiv F(z+0) - F(z-0) \tag{22}$$

is introduced.

For a given value of $k^2 > 0$ the differential equations (18) and (19), with matching conditions (20) and (21), and physical conditions at infinity $z \to \pm\infty$ result in two spectral problems for the TE and TM polarizations. The frequency ω plays part of the spectral parameter. For complicated functions $\varepsilon(\omega)$ and $\mu(\omega)$ the spectral parameter may enter into the initial differential equations nonlinearly.

B. The branches of the electromagnetic spectrum relevant for calculation of vacuum energy

In order to calculate the vacuum energy of the electromagnetic field, one has preliminarily to quantize this field with allowance for given boundary conditions. To this end, the full set of solutions to the Maxwell equations including all physically relevant ones is needed. In the directions parallel to the interfaces the electromagnetic field dynamics is free. Therefore, we are concerned with the behavior of the solutions normal to the boundaries, along the z axis.

As was mentioned in the Introduction, the full set of physically relevant solutions in the problem under consideration comprises discrete natural modes and scattering states. Normal modes are the solutions to the Maxwell equations which are localized near and between the interfaces, their energy being localized too. They are analogous to the bound states in quantum mechanics. These solutions are square-integrable on the whole z axis. The corresponding frequencies (the eigenvalues of the spectral problem) take on discrete real values.

The scattering states (propagating modes) are described by incident waves and outgoing scattered (or reflected and transmitted) waves. Their frequencies are real and positive, forming a continuous spectrum.

1. Discrete part of the spectrum

We start with considering a single flat interface in order to elucidate the properties of dielectric and magnetic functions required for the existence of normal modes in the problem under study.

a. Normal modes for a single-plane interface. Let the plane $z = 0$ separate two uniform half spaces characterized by ε_1, μ_1 and ε_2, μ_2, or symbolically $(\varepsilon_1, \mu_1 | \varepsilon_2, \mu_2)$. The general solutions to the Maxwell equations are defined by two functions $\Phi(z)$ and $\Psi(z)$, obeying (18) and (19) and the matching conditions (20) and (21). The eigenfunctions in the problem should decrease exponentially in both directions from the interface, i.e., they behave like outgoing waves with imaginary wave vectors,

$$\Phi(z) = \begin{cases} A_1 e^{-ik_1 z}, & z < 0, \\ A_2 e^{ik_2 z}, & z > 0, \end{cases} \tag{23}$$

where A_1 and A_2 are the constant amplitudes,

$$k_\alpha^2 = \varepsilon_\alpha\mu_\alpha\frac{\omega^2}{c^2} - k^2 = \frac{\omega^2}{c_\alpha^2} - k^2 < 0,$$
$$c_\alpha^2 = \frac{c^2}{\varepsilon_\alpha\mu_\alpha}, \quad \alpha = 1,2, \tag{24}$$

provided that

$$k_\alpha = +i|k_\alpha|, \quad \alpha = 1,2. \tag{25}$$

On substituting (23) and the analogous representation for the function $\Psi(z)$ into the matching conditions (20) and (21) we arrive at the following equations determining the eigenfrequencies:

$$\varepsilon_1 k_2 + \varepsilon_2 k_1 = 0 \quad (\text{TM modes}), \tag{26}$$

$$\mu_1 k_2 + \mu_2 k_1 = 0 \quad (\text{TE modes}). \tag{27}$$

In view of condition (25) the frequency equations (26) and (27) may have real roots (real eigenfrequencies) only if the permittivities $\varepsilon_1, \varepsilon_2$ and permeabilities μ_1, μ_2 have different signs.

This condition is fulfilled, for instance, at the metal-vacuum interface, provided the dielectric function of the metal $\varepsilon_1(\omega)$ is described by the plasma model [Eq. (2)]. In this case we put $\varepsilon_2 = \mu_1 = \mu_2 = 1$. By substituting all this into Eqs. (24) and (26), one obtains the dispersion equation for the surface plasmon in the TM polarization,

$$\Omega^4 - (1 + 2q^2)\,\Omega^2 + q^2 = 0\,, \tag{28}$$

where the dimensionless variables

$$\Omega = \frac{\omega}{\omega_p}\,, \quad q = \frac{k}{k_p}\,, \quad k_p = \frac{\omega_p}{c} \tag{29}$$

are introduced. The solution to Eq. (28) is given by

$$\Omega(q) = \sqrt{\frac{1}{2} + q^2 - \sqrt{\frac{1}{4} + q^4}}\,. \tag{30}$$

The dispersion curve $\Omega(q)$ is presented in Fig. 1(a). Another solution to Eq. (28) leads to positive k_1^2 that corresponds to propagating waves in medium 1 [dotted curve in the upper-left corner in Fig. 1(a)]. In view of the asymptotic behavior

$$\begin{aligned} \Omega(q) &\to q\left(1 - \frac{q^2}{2} + \cdots\right), \quad q \to 0, \\ \Omega(q) &\to \frac{1}{\sqrt{2}}\left(1 - \frac{1}{8q^2} + \cdots\right), \quad q \to \infty\,, \end{aligned} \tag{31}$$

the dispersion curve $\Omega(q)$ tends in these limits to two straight lines $\Omega = q$ and $\Omega = 1/\sqrt{2}$ from below. Here $\Omega = 1/\sqrt{2}$ is the dimensionless frequency of the surface plasma oscillations [see Fig. 1(a)]. With $\mu_1 = \mu_2 = 1$, obviously, there is no surface modes in the TE polarization.

The surface waves can be sustained by the dielectric-vacuum interface too, as the dielectric permittivity, with account for the dispersion (4), may take on negative values. The corresponding dispersion law can be found, for example, in Ref. [42].

b. Normal modes for two parallel-plane interfaces. Now let us consider two interfaces parallel to the xy plane and cutting the axis z at the points $z = -a$ and $z = a$. For simplicity, we consider from the beginning the symmetric configuration that can be symbolically represented as

$$(\varepsilon_1,\mu_1 \,|\, \varepsilon_2,\mu_2 \,|\, \varepsilon_1,\mu_1)\,.$$

For a given polarization, the normal modes can be subdivided into symmetric or asymmetric solutions. Thus we have for the TM modes

$$\Phi(z) = \begin{cases} A\, e^{-ik_1(z+a)}, & z < a; \\ B \begin{pmatrix}\cos k_2 z \\ \sin k_2 z\end{pmatrix}, & -a < z < a; \\ \pm A\, e^{ik_1(z-a)}, & z > a\,, \end{cases} \tag{32}$$

and the analogous ansatz for the function $\Psi(z)$ in the case of the TE modes. The wave vectors k_1 and k_2 in [Eq. (32)] are defined, as before, by Eq. (24). For $|z| > a$ we again consider the "outgoing waves" with the imaginary wave vector $k_1^2 < 0$, while in the region $|z| < a$ the wave vector k_2 may be

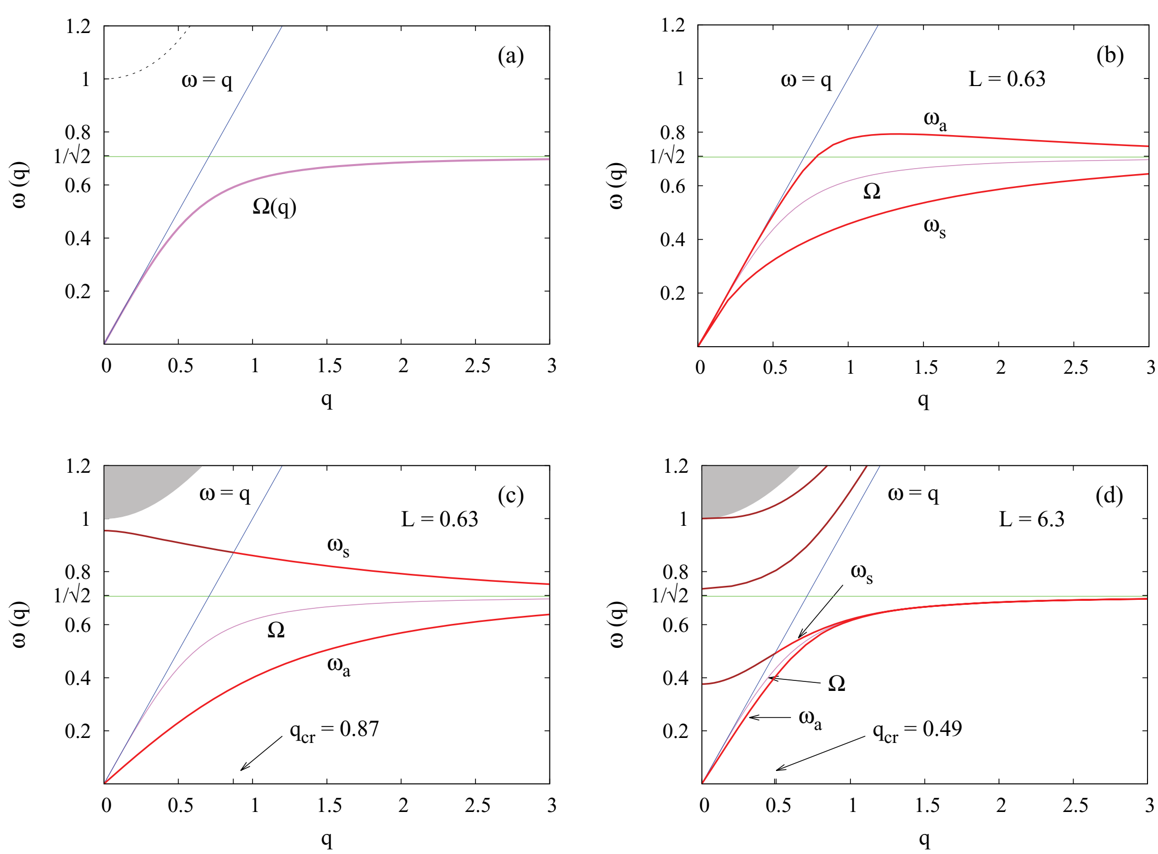

FIG. 1. (Color online) Dispersion curves for the discrete modes in the TM polarization for different configurations: (a) single metal-vacuum interface; (b) metal plate of thickness $L = 2ak_p = 0.63$ in vacuum; (c) vacuum gap of the same width $L = 0.63$ in the metal bulk; and (d) vacuum gap of the width $L = 6.3$ in the metal bulk. Frequencies Ω, ω, and the wave number q are presented in the dimensionless units defined in Eq. (29). The gray filled regions correspond to the continuous spectrum.

both imaginary, $k_2^2 < 0$, or real, $k_2^2 > 0$. In the first case, the waves are just surface ones localized near the interfaces and in the second case, they are the waveguide solutions[5] describing the standing waves between the interfaces and the evanescent waves outside this region. Indeed, both solutions are normal modes because they are squire integrable on the whole axis z.

The matching conditions at points $z = -a$ and $z = a$ result in the respective frequency equations. For the TM modes these equations read

$$L_a \equiv i k_1 \varepsilon_2 + \varepsilon_1 k_2 \tan(a k_2) = 0 \quad \text{(asymmetric modes)}, \tag{33}$$

$$L_s \equiv i k_1 \varepsilon_2 - \varepsilon_1 k_2 \cot(a k_2) = 0 \quad \text{(symmetric modes)}. \tag{34}$$

When defining the spatial symmetry of the electromagnetic modes we follow the papers [44–46], namely, the symmetric TM modes are constructed by making use of the symmetric function $\Phi'(z)$ and, consequently, antisymmetric function $\Phi(z)$ in Eqs. (16) and (17). It implies that the longitudinal component of the electric field $E_x(z)$ is symmetric with respect to the plane $z = 0$ and the transverse components of the fields $E_z(z)$ and $H_y(z)$ are antisymmetric. The antisymmetric TM modes have the contrary symmetry properties, i.e., $E_x(z)$ is antisymmetric with respect to the $z = 0$ plane, and $E_z(z)$ and $H_y(z)$ are symmetric. In the literature [47,48] reverse notation is used too. The eigenfrequencies of the symmetric modes are denoted by $\omega_s(k)$ and those for the antisymmetric modes by $\omega_a(k)$. In the literature different notation for these frequencies is used, for example, sometimes $\omega^+(k)$ is referred to the higher frequency modes and $\omega^-(k)$ applies to the lower frequency collective oscillations [44,45].

By making use of the trigonometric relations

$$\tan 2x = \frac{2\tan x}{\tan^2 x - 1} = \frac{2\cot x}{\cot^2 x - 1}\,,$$

the frequency equations (33) and (34) can be combined into one equation,

$$\tan(2ak_2) = -\frac{2i\eta}{1+\eta^2}, \quad \eta = \frac{\varepsilon_2 k_1}{\varepsilon_1 k_2}\,. \tag{35}$$

The frequency equations (33) and (34) can also be rewritten in the exponential form

$$1 + r_{12} e^{2iak_2} = 0 \quad (L_a = 0), \tag{36}$$

$$1 - r_{12} e^{2iak_2} = 0 \quad (L_s = 0). \tag{37}$$

Multiplying together these two equations we arrive at the exponential form for the total frequency equation (35)

$$D(\omega) \equiv 1 - r_{12}^2 e^{4iak_2} = 0 \quad (L_a L_s = 0), \tag{38}$$

where the notation

$$r_{12} \equiv \frac{\varepsilon_2 k_1 - \varepsilon_1 k_2}{\varepsilon_2 k_1 + \varepsilon_1 k_2} = \frac{\eta - 1}{\eta + 1} \tag{39}$$

is introduced. It will be shown later [see Eq. (51)] that r_{12} is the reflection amplitude for a single interface. The frequency equations for the TE polarization are obtained from the above equations by replacement $\varepsilon_\alpha \to \mu_\alpha, \quad \alpha = 1,2$.

To comprehend the structure of the discrete spectrum in the presence of two interfaces, we address typical examples. The solutions to the frequency equations (33) and (34) are studied in detail in the theory of dielectric waveguides [47] and optical waveguides [49,50], in the theory of surface plasmons [42,44,45,51–53] and so on. However, in these applications the roots of the frequency equations are often sought in the form $k = k(\omega)$ (wave vector as function of frequency). We are interested in the inverse function $\omega = \omega(k)$.

While studying the frequency equations (33) and (34), their plots in terms of the variables k_1 and k_2 instead of the initial variables k and ω are helpful and transparent. To this end, the considered frequency equation is appended with the relation between k_1^2 and k_2^2 following from their definition [Eq. (24)]. At fixed ω one obtains two equations connecting k_1 and k_2. The intersection points of the curves defined by the left-hand sides of these equations give the solutions of the initial frequency equation (see, for example, [47,54]).

Let us briefly describe the structure of the discrete spectrum of electromagnetic excitations for some configurations. Not all these examples can have a direct relation to the experimental studies of the Casimir forces. Nevertheless, all of them turn out to be instructive in revealing the general properties of the electromagnetic spectrum for two plane parallel interfaces.

We specify **the configuration I** in the following way:

$$\varepsilon_1 = \mu_1 = 1, \quad \varepsilon_2 \text{ and } \mu_2 \text{ are some constants} \tag{40}$$

(a material plate in vacuum without dispersion). This configuration is studied in detail in radio engineering as the simplest example of a waveguide [37,47]. When the conditions

$$\varepsilon_2 > 0, \quad \mu_2 > 0, \quad \varepsilon_2\mu_2 > 1 \tag{41}$$

or

$$\varepsilon_2 < 0, \quad \mu_2 < 0, \quad \varepsilon_2\mu_2 > 1 \tag{42}$$

are satisfied, there exist waveguide solutions in the form (32) with $k_1^2 < 0$ and $k_2^2 > 0$ for both the TE and TM polarizations. For a given value of k^2, the number of such solutions is finite and is increasing with ω. The respective real eigenfrequencies lie in the interval $(c_2 k, c_1 k)$ with $c_1 = c$.

As expected, the surface modes ($k_1^2 < 0$, $k_2^2 < 0$) in the present configuration exist only under the conditions (42), in other words, if the plate is made of the left-handed material (both ε_2 and μ_2 are negative [55,56]). In the general case, the number of modes of this kind is equal to 4: there are two surface modes with the TE polarization (symmetric oscillation and antisymmetric oscillation) and two analogous modes with the TM polarization.

For symmetric surface modes it is enough to fulfill the condition (42). For antisymmetric surface modes, one has to impose, in addition to (42), the following conditions:

$$0 < \frac{a\omega}{c}(\varepsilon_2\mu_2 - 1) < \begin{cases} |\varepsilon_2|^{-1} & \text{TM polarization;} \\ |\mu_2|^{-1} & \text{TE polarization.} \end{cases} \tag{43}$$

The real frequencies of the surface modes lie in the interval $(0, c_2 k)$.

As the next example we consider **configuration II**, which is complimentary to the previous one, namely, ε_1 and μ_1 are

[5] In Ref. [43], such solutions are referred to as the cavity modes.

some positive or negative constants such that $\varepsilon_1\mu_1 > 1$ and $\varepsilon_2 = \mu_2 = 1$ (two semispaces of the same material without dispersion separated by the vacuum gap).

This configuration is not investigated in radio engineering and waveguide optics, but it is closely related to the Casimir calculations. In the case of left-handed material obeying the conditions

$$\varepsilon_1 < 0, \quad \mu_1 < 0, \quad \varepsilon_1\mu_1 > 1, \tag{44}$$

there exist surface symmetric and antisymmetric waves in both polarizations [56,57]. Indeed, one of the branches of the hyperbola defined by the equation

$$a^2\kappa_2^2 - a^2\kappa_1^2 = \frac{a^2\omega^2}{c^2}(\varepsilon_1\mu_1 - 1) > 0\,, \quad k_\alpha = i\kappa_\alpha, \quad \alpha = 1,2 \tag{45}$$

at negative ε_1 and μ_1 always crosses in the first quadrant the curves defined by the equations

$$a\kappa_1 = -\frac{\varepsilon_1}{\varepsilon_2}(a\kappa_2)\tanh(a\kappa_2)\,, \quad a\kappa_1 = -\frac{\varepsilon_1}{\varepsilon_2}(a\kappa_2)\coth(a\kappa_2),$$

and two more obtained by replacing ε_α by μ_α, $\alpha = 1,2$. Therefore, the discrete spectrum for the present configuration exists only in the case of left-handed materials and consists of four surface modes. The frequencies of these modes belong to the interval $0 < \omega < c_1 k$.

Keeping in mind the possibility of total reflection at the interface of the media 1 and 2, one can try to find the solutions for this configuration with $k_1^2 > 0$ and $k_2^2 < 0$. However, it is easy to see that there are no such solutions with real frequencies as the equations are complex. Even if such solutions existed, they are not eigenfunctions of the discrete spectrum because they are not squared integrable on the infinite axis z.

For the configuration under consideration there are no waveguide solutions to the Maxwell equations. Indeed, for the existence of such solutions, it is required that $k_1^2 < 0$, and $k_2^2 > 0$. From these conditions we arrive at the conflicting inequalities $0 < \omega < c_1 k$ and $\omega > ck$, with $c_1 = c/\sqrt{\varepsilon_1\mu_1} < c$.

Now let us consider the examples of the discrete electromagnetic spectrum explicitly allowing for the temporal dispersion in the media. As **configuration III** we take a material (metal) slab in vacuum, the dielectric properties of the slab being described by the simple plasma model [Eq. (2)] [see Fig. 1(b)]:

$$\varepsilon_1 = \mu_1 = 1, \quad \varepsilon_2(\omega) = 1 - \frac{\omega_p^2}{\omega^2}, \quad \mu_2 = 1\,. \tag{46}$$

This configuration supports two surface waves in the TM polarization. These modes are well studied theoretically [46,56] and experimentally, for instance, in measurements of the energy losses by the electrons passing through the metal films [44,45]. There is vast literature on this subject [42,51,53]. Our citing is minimal.

The physical picture of these collective excitations is the following. At both boundaries of the metal plate ($z = \pm a$) with $a \to \infty$ there exist plasma oscillations of the electron density with the same frequency $\omega_p/\sqrt{2}$ and dispersion law [Eq. (30)]. If the thickness of the plate or film is finite, these oscillations interact reciprocally and the energy level [Eq. (30)] splits, as usual, in two levels (asymmetric and symmetric ones) with frequencies $\omega_a(k)$ and $\omega_s(k)$ defined by Eqs. (33) and (34), respectively [see Fig. 1(b)].

It is important that both curves $\omega_s(k)$ and $\omega_a(k)$ are placed to the right of the line $\omega = ck$ and below the horizontal line $\omega = \omega_p$. Thus the eigenfrequencies of the surface modes lie in the interval $0 \leqslant \omega_s(k), \omega_a(k) < ck$ for $0 < k < \infty$.

The curve $\omega_a(k)$ at all k lies above the curve $\omega_s(k)$. Near the origin both the curves touch the line $\omega = ck$ from the right. With $k \to \infty$ the curves $\omega_s(k)$ and $\omega_a(k)$ tend to the horizontal line $\omega = \omega_p/\sqrt{2}$, respectively, from below and from above. For $2a \to \infty$ the frequencies $\omega_s(k)$ and $\omega_a(k)$ tend to the universal curve [Eq. (30)] from below and from above.

The characteristic value of the longitudinal wave vector k in the problem under consideration is $k_p = \omega_p/c$. It is within the vicinity of $k \sim k_p$ where the "bend" of the curves $\omega_s(k)$ and $\omega_a(k)$ is observed.

It is evident that there are no waveguide solutions ($k_1^2 < 0$, $k_2^2 > 0$) in the present case. Indeed, from the definition of the wave vectors k_1 and k_2 [see Eqs. (2) and (24)] we obtain

$$-k_1^2 + k_2^2 = -\frac{\omega_p^2}{c^2}\,.$$

For the waveguide solutions the left-hand side of this equation should be positive. Hence, this equality cannot be satisfied.

Now we consider **configuration IV**, complementary to the previous one, namely, a vacuum slot, $2a$ in width, in the bulk metal described by the plasma model [see Figs. 1(c) and 1(d)]:

$$\varepsilon_1(\omega) = 1 - \frac{\omega_p^2}{\omega^2}, \quad \mu_1 = 1, \quad \varepsilon_2 = \mu_2 = 1\,. \tag{47}$$

First, we address the surface waves in the present setup, $k_1^2 < 0$, $k_2^2 < 0$, which exist in the TM polarization [46]. Again, we have two such collective modes, symmetric and antisymmetric ones, their frequencies being placed in the interval $(0,ck)$. The layout of the dispersion curves is interchanged in comparison with the previous case, $\omega_s(q) > \omega_a(q)$. The curve $\omega_a(q)$ lies below the horizontal line $\omega = 1/\sqrt{2}$ and to the right from the line $\omega = q$. At the point $q = q_{\text{cr}} = 1/\sqrt{1 + L/2}$ the dispersion curve $\omega_s(k)$ crosses the light line $\omega = q$ and becomes the lowest waveguide mode (see Figs. 1(c) and 1(d) and [46]).[6] Thus, the symmetric surface mode $\omega_s(q)$ becomes in this configuration a "hybrid" one. It is this fact that leads to the known result, namely, the contribution of the surface modes to the Casimir energy for a given configuration proves to be considerable only at small distances. Indeed, because of the hybrid nature of the symmetric mode, the dispersion curves $\omega_s(q)$ and $\omega_a(q)$ move apart in the region $q < q_{\text{cr}}$ [Figs. 1(c) and 1(d)], this divergence being maximum at small L [compare Figs. 1(c) and 1(d)]. Obviously, this implies in turn that the mutual interaction of these collective excitations is maximum at small L.

[6]Probably, this implies that the symmetric surface TM mode in this configuration may directly be excited by the light beam contrary to the plate configuration [Fig. 1(b)] [42,44,45,51].

With increasing the gap width L, in this configuration there appear waveguide solutions ($k_1^2 < 0$, $k_2^2 > 0$) satisfying the conditions

$$c^2k^2 < \omega_{wg}^2(k) < \omega_p^2 + c^2k^2 \,. \tag{48}$$

Thus, they lie to the left from the line $\omega = ck$ and below the curve $\omega = \sqrt{\omega_p^2 + c^2k^2}$. Approximately, the frequency of these modes and their total numbers, for each polarization, are given by Ref. [58]

$$\begin{aligned} &\omega^2(k) \approx k^2c^2 + n^2\pi^2c^2/4a^2, \\ &n = 0,1,2,\ldots,n < 2a\omega_p/\pi c \,. \end{aligned} \tag{49}$$

These modes exist only at large gaps [compare Figs. 1(c) and 1(d)]. Indeed, for gold $\omega_p = 1.38 \times 10^{16}\ \mathrm{s}^{-1}$. Therefore, the dimensionless width of the gap $L = 0.63$ implies $2a = 14$ nm and $L = 6.3$ corresponds to $2a = 140$ nm. In the first case, there are no pure waveguide modes [Fig. 1(c)] and, at the same time, the surface modes interact strongly because their dispersion curves are distinguished considerably. In the second case [Fig. 1(d)], only two pure waveguide modes appear while the surface modes at $q > 1$ practically coincide with each other and with the single interface dispersion curve $\Omega(q)$. The latter means that in this region the surface modes do not couple. For copper, we have close figures: $\omega_p = 1.46 \times 10^{16}\ \mathrm{s}^{-1}$, $2a = 12.6$ nm ($L = 0.63$), and $2a = 126$ nm ($L = 6.3$).

The fact that waveguide modes exist only for large gaps suggests that they contribute appreciably to the dispersion force at large distances [58], where contributions of the surface modes practically vanish.

The examples studied above reveal general properties of the discrete spectrum in the problem at hand and permit us to establish the lower boundary of the continuous spectrum. Let us summarize these properties:

(i) The discrete spectrum may comprise the solutions of two types: the surface modes and the waveguide modes;

(ii) At fixed k the frequencies of the surface modes (if they exist) lie in the interval $[0,\omega_-(k)]$, while the frequencies of the waveguide solutions (if they exist) belong to the interval $[\omega_-(k),\omega_+(k)]$, where the boundary values $\omega_-(k)$ and $\omega_+(k)$ $[\omega_-(k) < \omega_+(k)]$ are defined by the explicit form of the dielectric function $\varepsilon(\omega)$;

(iii) The lower boundary of the continuous spectrum is defined by the condition $k_1^2(\omega) > 0$.

Here we list the values of $\omega_+(k)$ and $\omega_-(k)$ for the configurations I–IV.

$$\text{I:}\quad \omega_-(k) = c_2k \equiv \frac{ck}{\sqrt{\varepsilon_2\mu_2}}\,, \quad \omega_+(k) = ck, \quad c_2 < c\,.$$

In this case, the discrete spectrum may include the surface modes and the waveguide modes with the frequencies from the above-specified intervals; the continuous spectrum ω is defined by $\omega > \omega_+(k)$.

$$\text{II:}\quad \omega_-(k) = c_1k \equiv \frac{ck}{\sqrt{\varepsilon_1\mu_1}}\,, \quad \omega_+(k) = ck, \quad c_1 < c\,.$$

The discrete spectrum may consist of the surface modes only with the frequencies from the above specified interval; the continuous spectrum ω exists for $\omega > \omega_-(k) = c_1k$.

$$\text{III:}\quad \omega_-(k) = ck\,, \quad \omega_+(k) = \sqrt{c^2k^2 + \omega_p^2}\,.$$

The discrete spectrum is formed by the surface modes with frequencies from the interval specified above; the continuous spectrum is defined by $\omega > \omega_-(k) = ck$.

$$\text{IV:}\quad \omega_-(k) = ck\,, \quad \omega_+(k) = \sqrt{c^2k^2 + \omega_p^2}\,.$$

The discrete spectrum comprises the surface modes and the waveguide modes with frequencies in the corresponding intervals; the continuous spectrum exists for $\omega > \omega_+(k) = \sqrt{c^2k^2 + \omega_p^2}$.

In what follows it is significant that the points $\omega_-(k)$ and $\omega_+(k)$ are the branch points for the root functions $k_1(\omega)$ and $k_2(\omega)$ (see Sec. III A). The configurations considered above are unbounded, i.e., open. As a result, in addition to the real roots the frequency equations have also the discrete complex roots $\omega(k) = \omega'(k) + i\omega''(k)$ which correspond to the quasinormal modes [34]. These modes are not squared integrable and therefore they are not normalizable. That is why the quasinormal modes are not included in the completeness relation. However, their existence should be taken into account when choosing the contours for integration in the complex ω plane (see Sec. III A).

It is worth noting here that the quasinormal modes are explored in fact in the experiments dealing with the radiating plasmons [45]. Usually, in this field the complex wave numbers are considered $k(\omega) = k'(\omega) + ik''(\omega)$ instead of the complex eigenfrequencies.

2. *Treatment of the continuous part of the spectrum*

To study the contribution of the continuous spectrum to the vacuum energy, the scattering matrix is required, i.e., the S matrix for the wave equations (18) and (19) with the matching conditions (20) and (21), respectively. Here we deal with the one-dimensional scattering on the infinite z axis, $-\infty < z < \infty$. This problem has some peculiarities [59–62]. Specifically, one has to consider the incidence of the initial waves from the left and from the right separately. Thus, we must add such initial waves to the outgoing and standing waves in Eqs. (23) and (32) considered in the pertinent eigenvalue problems. The respective reflection amplitudes, r_l, r_r, and transition amplitudes, t_l, t_r, make up a (2×2) matrix that plays the role of the scattering matrix in the problem under consideration:

$$S(\omega) = \begin{pmatrix} t_l & r_l \\ r_r & t_r \end{pmatrix}. \tag{50}$$

a. Single-plane interface. In the case of the one-plane interface $(\varepsilon_1,\mu_1|\varepsilon_2,\mu_2)$ we have for the TM polarization,

$$r_l \equiv r_{12} = \frac{\varepsilon_2k_1 - \varepsilon_1k_2}{\varepsilon_1k_2 + \varepsilon_2k_1}, \quad r_r \equiv r_{21} = -r_{12} = -r_l\,, \tag{51}$$

$$t_l \equiv t_{12} = \frac{2\varepsilon_1k_1}{\varepsilon_1k_2 + \varepsilon_2k_1}, \quad t_r \equiv t_{21} = \frac{2\varepsilon_2k_2}{\varepsilon_1k_2 + \varepsilon_2k_1}, \quad t_{12} \neq t_{21}. \tag{52}$$

The analogous amplitudes for the TE polarization are obtained from Eqs. (51) and (52) by the substitution $\varepsilon_\alpha \to \mu_\alpha$, $\alpha = 1,2$.

The amplitudes t and r for the TM and TE polarizations have the poles at the points where the variable ω coincides with the eigenfrequencies defined by Eqs. (26) and (27). This property is preserved for several interfaces too (see below). The single-plane interface is a nonsymmetric configuration; therefore, the S matrix [Eq. (50)] in this case does not possess appealing physical properties.

b. Electromagnetic scattering by two parallel-plane interfaces. Now we address the scattering matrix for two plane interfaces which are parallel to the xy plane and cut the z axis at the points $z = -a$ and $z = a$, in other words, we envisage the configuration $(\varepsilon_1,\mu_1|\varepsilon_2,\mu_2|\varepsilon_3,\mu_3)$. The explicit form of t and r is again derived from the matching conditions (20) and (21) [35,36] or by making use of the multiple reflection technique [24,35]. By taking careful account of the phase factors at the points $z = -a$ and $z = a$ one obtains

$$r_l \equiv r_{123} = \frac{r_{12} + r_{23}e^{4ik_2a}}{1 + r_{12}r_{23}e^{4ik_2a}}, \quad t_l \equiv t_{123} = \frac{t_{12}t_{23}e^{2ik_2a}}{1 + r_{12}r_{23}e^{4ik_2a}}, \tag{53}$$

$$r_r \equiv r_{321} = \frac{r_{32} + r_{21}e^{4ik_2a}}{1 + r_{32}r_{21}e^{4ik_2a}}, \quad t_r \equiv t_{321} = \frac{t_{32}t_{21}e^{2ik_2a}}{1 + r_{32}r_{21}e^{4ik_2a}}, \tag{54}$$

where r_{ij} and t_{ik} are defined in Eqs. (51) and (52).

Furthermore, we consider the special case of Eqs. (53) and (54), the symmetric one, $(\varepsilon_1,\mu_1|\varepsilon_2,\mu_2|\varepsilon_1,\mu_1)$. In the symmetric setup one has

$$\begin{aligned} r = r_l = r_r \equiv r_{121} &= \frac{r_{12} - r_{12}e^{4ik_2a}}{1 - r_{12}^2e^{4ik_2a}}, \\ t = t_l = t_r \equiv t_{121} &= \frac{t_{12}t_{21}e^{2ik_2a}}{1 - r_{12}^2e^{4ik_2a}}. \end{aligned} \tag{55}$$

The amplitudes r and t in Eq. (55) have the poles at the points of the discrete spectrum [see Eq. (38)], and they obey the relations

$$|t|^2 + |r|^2 = 1, \tag{56}$$

$$r\bar{t} + \bar{r}t = 0, \tag{57}$$

where the bar means complex conjugation. Now the scattering matrix [Eq. (50)] is given by

$$S(\omega) = \begin{pmatrix} t & r \\ r & t \end{pmatrix}. \tag{58}$$

Due to Eqs. (56) and (57) the S matrix [Eq. (58)] is a symmetric unitary matrix

$$SS^\dagger = S^\dagger S = 1. \tag{59}$$

In the case of potential scattering on an infinite line with symmetric potential the amplitudes t and r are parametrized as follows [61]:

$$t = \cos\theta\, e^{i\delta}, \quad r = i\sin\theta\, e^{i\delta}, \tag{60}$$

where θ and δ are the real functions of ω. Further, we have

$$\det S(\omega) = t^2(\omega) - r^2(\omega) = e^{2i\delta(\omega)} \tag{61a}$$

$$= \frac{t(\omega)}{\bar{t}(\omega)}. \tag{61b}$$

For the real ω the function $\det S(\omega)$ is defined by formula (61a), while (61b) gives its continuation to the whole complex plane ω.

In order to consider in this approach the initial asymmetric configuration (1|2|3), one has to proceed from the symmetric multilayered configuration (1|2|3|2|1), calculate the respective total reflection and transition amplitudes, and find $\det S(\omega)$. Transition to the initial asymmetric configuration should be done only at the final stage of the vacuum energy calculation (see Sec. III A) by moving the interface 3|2 to infinity.

III. CASIMIR ENERGY

We define the Casimir energy as the renormalized zero-point energy of the electromagnetic oscillations for the configuration at hand

$$E = \frac{\hbar}{2}\sum_{\{q\}}\omega_q. \tag{62}$$

Here the sum sign implies the summation over the discrete part of the spectrum and the integration over its continuous branch.

The summing up of the discrete eigenfrequencies will be performed by making use of the frequency equations. The integration over the continuous part of the spectrum cannot be carried out by applying the "naive" formula

$$\int \omega\, d\omega, \tag{63}$$

because it does not "feel" the physical nature of the problem under consideration, namely, this integral does not depend on the specific form of the differential operator and the corresponding boundary conditions [34]. Therefore, the summation over the continuous branch of the spectrum should be accomplished, as usual, by making use of the spectral density shift. The rigorous mathematical theory of the scattering problem gives the following expression for this function [63]:

$$\Delta\rho(\omega) \equiv \rho(\omega) - \rho_0(\omega) = \frac{1}{\pi}\frac{d}{d\omega}\delta(\omega) \tag{64a}$$

$$= \frac{1}{2\pi i}\frac{d}{d\omega}\ln\det S(\omega) = \frac{1}{2\pi i}\frac{d}{d\omega}\ln\frac{t(\omega)}{\bar{t}(\omega)}, \tag{64b}$$

where $S(\omega)$ is the S matrix in the spectral problem at hand. In Eq. (64) $\rho(\omega)$ is the density of states for a given potential (or for given matching conditions in the case of the compound media) and $\rho_0(\omega)$ is the spectral density in the respective free spectral problem (for the vanishing potential or for the homogeneous unbounded space). In the case of one-dimensional scattering the calculation of the spectral density was considered in Refs. [61,64]. It is worth noting here that already at the level of the spectral density [Eq. (64)] the removing of infinity is carried out, namely, the contribution to the vacuum energy generated by the unbounded Minkowski space is subtracted here.

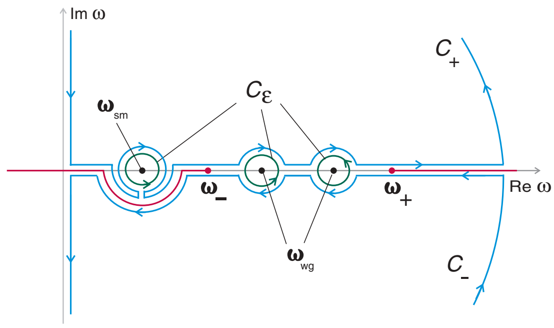

FIG. 2. (Color online) The contours C_+ and C_- in the complex ω plane which are used when going on to the integration over the imaginary frequencies. For simplicity, the spectrum possessing one surface mode ω_{sm}, two waveguide modes ω_{wg}, and the continuous part $\omega > \omega_+$ is considered. The cuts starting at the points ω_- and ω_+ are shown by heavy lines.

Taking into account all this, we can work out Eq. (62) in detail as follows:

$$E(2a) = \frac{\hbar}{2}\sum_\sigma \int_{-\infty}^{\infty} \frac{d^2\mathbf{k}}{(2\pi)^2}\left[\sum_n \omega_n(\sigma,k,a) + \int_{\omega_0}^{\infty} \omega\, \Delta\rho(\sigma,\omega,k,a)\, d\omega\right] - (a\to\infty). \quad (65)$$

Here $\mathbf{k}$ is the wave vector along unbounded dimensions $\mathbf{k}=(k_x,k_y)$; $\omega_n(\sigma,k,a)$ are the frequencies appertaining to the discrete spectrum; the function of the spectral density shift $\Delta\rho(\omega)$ is given by Eq. (64); and ω_0 is the lower bound of the continuous spectrum. Summation over $\sigma = \mathrm{TE},\mathrm{TM}$ in Eq. (65) takes into account two polarizations of the electromagnetic field.

The further treatment of Eq. (65) is aimed at deriving a unique integral representation for the contributions of both the discrete part and the continuous part of the spectrum, the integration being carried out over the imaginary frequencies $\omega = i\zeta$.

To be definite, we shall consider the "richest" electromagnetic spectrum possessing the surface modes, waveguide modes, and the continuous part (see, for example, configurations I and IV in Sec. II B1). As was shown there, the frequencies of the surface modes lie in the interval $0 < \omega_{sm} < \omega_-(k)$, the frequencies of the waveguide modes belong to the band $\omega_-(k) < \omega_{sm} < \omega_+(k)$, and for the frequencies ω of the continuous part of the spectrum we have $\omega_+(k) < \omega$ (see Fig. 2).

A. Transition to imaginary frequencies

In what follows, we have to go out in the complex frequency plane ω. For this the one-valued branches of the integrand in Eq. (65) should be chosen. The square root dependence

$$k_\alpha(\omega) = \sqrt{\varepsilon_\alpha(\omega)\frac{\omega^2}{c^2} - k^2}, \quad \alpha = 1,2 \quad (66)$$

results in branch points. For the usually accepted functions $\varepsilon(\omega)$ there appear four such points on the real axes: $\pm\omega_-(k)$ and $\pm\omega_+(k)$ $[0 \leqslant \omega_-(k) < \omega_+(k)]$ (see Sec. II B1). These are the same points that separate the different parts of the spectrum.

We draw two cuts on the real axes ω: the first cut connects the points $-\omega_-$ and ω_-, and the second one starts at the point ω_+ and goes to infinity (see Fig. 2). The first cut passes round the eigenfrequency of the surface mode ω_{sm} from below. It is not crucial and this cut can go round ω_{sm} from above as well. Further, we assume that $k_1(\omega)$ and $k_2(\omega)$ in Eq. (66) acquires real positive values at the upper edges of the cuts.

The eigenfrequencies $\omega_n(\sigma,k,a)$ are unknown in an explicit form; we have only the respective frequency equations (33) and (34) or in another form Eqs. (36)–(38). This dictates, without choice, making use of the argument principle theorem from the complex analysis when summing over n in Eq. (65). When choosing the contour for integration in the complex ω plane, one should bear in mind that the left-hand sides of the frequency equations $D(\sigma,\omega)$, considered as the functions of the complex variable ω, have a "good" behavior in the upper half plane ω only:

$$D(\sigma,\omega) = 1 - r_\sigma^2 e^{4iak_2}, \quad ck_2 = \sqrt{\omega^2 - \omega_+^2}\,. \quad (67)$$

This forbids from the very beginning the deformation of the contour in such a way that it includes the whole imaginary axes $\mathrm{Im}\,\omega$. An admissible contour is obviously the following one: the integration should be performed along the circles C_ε of the radius ε with $\varepsilon \to 0$, with each circle surrounding a single root of the frequency equation (see Fig. 2).

When treating the contribution of the continuous part of the spectrum to the vacuum energy one should keep in mind the following. Upon substituting (64b) in $\Delta\rho(\omega)$ the transition amplitude from Eq. (55),

$$t(\omega) = \frac{t_{12}t_{21}e^{2ik_2a}}{D(\omega)}, \quad (68)$$

in vacuum energy [Eq. (65)] there appear terms which are proportional to the distance between the slabs $2a$. These contributions are generated by the factor e^{2ik_2a} and they belong to the internal energy of medium 2 in the configuration under consideration. These terms are removed from the Casimir energy by the subtraction indicated in Eq. (65). Hence, in calculation of the vacuum electromagnetic energy the denominator $D(\omega)$ in Eq. (68) "works" alone. Now the vacuum energy [Eq. (65)] acquires the form

$$E(2a) = \frac{\hbar}{2}\sum_\sigma \int_0^\infty \frac{kdk}{2\pi}\frac{1}{2\pi i}\left[\int_{C_\varepsilon} d\omega\,\omega\frac{d}{d\omega}\ln D(\sigma,\omega) + \int_{\omega_+}^{\infty} d\omega\,\omega\frac{d}{d\omega}\ln\frac{\bar D(\sigma,\omega)}{D(\sigma,\omega)}\right]. \quad (69)$$

Furthermore, we represent the contribution of the continuous spectrum to the vacuum energy [the second term between the square brackets in Eq. (69)] in the form of the contour integrals as well. According to the Cauchy integral theorem, we can write

$$\int_{C_+} d\omega\,\omega\frac{d}{d\omega}\ln D(\sigma,\omega) = 0, \quad (70a)$$

$$\int_{C_-} d\omega\,\omega\frac{d}{d\omega}\ln \bar D(\sigma,\omega) = 0, \quad (70b)$$

where the contours C_+ and C_- are shown in Fig. 2. Here we have taken into account that the complex roots of the equation

$D(\sigma,\omega) = 0$ lie only in the lower half plane ω and, respectively, the complex roots of the equation $\bar{D}(\sigma,\omega) = 0$ are placed only in the upper half plane ω.[7] Such a layout of the complex eigenfrequencies is in a direct relation with the choice of the time dependence in the solutions to the Maxwell equations in the form $e^{-i\omega t}$ (see Sec. II A). In view of relations (70), we can express the contribution of the continuous part of the spectrum in Eq. (69) as the integrals along the contours C_+ and C_- which have no section (ω_+,∞). One can easily see from Fig. 2 that along the contours C_ε the contributions of the discrete spectrum and continuous spectrum are mutually canceled.[8] Besides, at the length $(0,\omega_+)$ the contributions of the continuous part of the spectrum generated by $D(\sigma,\omega)$ and $\bar{D}(\sigma,\omega)$ are mutually canceled too.

As a result, we arrive at a very compact and clear formula for the Casimir energy,

$$E(2a) = \frac{\hbar}{2\pi}\sum_\sigma \int_0^\infty \frac{k\,dk}{2\pi}\int_0^\infty d\zeta\,\ln D(\sigma,i\zeta,k),\quad \sigma = \text{TE, TM}, \tag{71}$$

where

$$\begin{aligned} D(\sigma,i\zeta,k) &= 1 - r_\sigma^2(i\zeta,k)\,e^{4a\kappa_2(\zeta,k)},\quad \sigma = \text{TE, TM},\\ r_{\text{TE}}(i\zeta,k) &= \frac{\kappa_1(\zeta,k) - \kappa_2(\zeta,k)}{\kappa_1(\zeta,k) + \kappa_2(\zeta,k)},\\ r_{\text{TM}}(i\zeta,k) &= \frac{\varepsilon_2(i\zeta)\kappa_1(\zeta,k) - \varepsilon_1(i\zeta)\kappa_2(\zeta,k)}{\varepsilon_2(i\zeta)\kappa_1(\zeta,k) + \varepsilon_1(i\zeta)\kappa_2(\zeta,k)},\\ \kappa_\alpha(\zeta,k) &\equiv ik_\alpha(i\zeta,k) = \sqrt{\varepsilon_\alpha(i\zeta)\frac{\zeta^2}{c^2} + k^2},\quad \alpha = 1,2\,. \end{aligned} \tag{72}$$

In obtaining formula (71) the integration by parts was carried out with respect to the variable ζ: $\omega = i\zeta$. Obviously, this formula holds for an arbitrary spectrum of electromagnetic excitations sustained by the configuration under consideration. Let us remember that the symmetric configuration is considered with nonmagnetic materials $[\varepsilon_1(\omega),\mu_1 = 1|\varepsilon_2(\omega),\mu_2 = 1|\varepsilon_1(\omega),\mu_1 = 1]$.

Differentiation of the vacuum energy [Eq. (71)] with respect to the gap width $2a$ immediately gives the Lifshitz formula for the Casimir force at zero temperature [1]:

$$F(2a) = -\frac{\partial E(2a)}{\partial(2a)} = -\frac{\hbar}{2\pi^2}\int_0^\infty k\,dk\int_0^\infty d\zeta\,\kappa_2(\zeta,k) \times \sum_\sigma \left[r_\sigma^{-2}(i\zeta,k)e^{4a\kappa_2(\zeta,k)} - 1\right]^{-1}. \tag{73}$$

Here $\sigma =$ TE, TM as before and the same notation [Eq. (72)] is used.

In order to reproduce exactly the Lifshitz formula (9) in Ref. [7] for a vacuum gap ($\varepsilon_2 = \mu_2 = 1$) between the identical slabs, a new variable $p = p(k)$ should be introduced:

$$\frac{p\zeta}{c} \equiv \kappa_2 = \left(\frac{\zeta^2}{c^2} + k^2\right)^{1/2}. \tag{74}$$

Taking into account that $p(k = 0) = 1$ and

$$\kappa_2 k\,dk = \frac{p^2\zeta^3}{c^3}dp,$$

we easily convert Eq. (73) to the form

$$F(2a) = -\frac{\hbar^2}{2\pi^2c^3}\int_0^\infty \zeta^3 d\zeta\int_1^\infty p^2\,dp \times \sum_\sigma \left[r_\sigma^{-2}(i\zeta,p)e^{4ap\zeta/c} - 1\right]^{-1}, \tag{75}$$

where

$$r_{\text{TE}}^{-2}(i\zeta,p) = \left(\frac{\sqrt{\varepsilon_1(i\zeta) - 1 + p^2} + p}{\sqrt{\varepsilon_1(i\zeta) - 1 + p^2} - p}\right)^2,$$

$$r_{\text{TM}}^{-2}(i\zeta,p) = \left(\frac{\sqrt{\varepsilon_1(i\zeta) - 1 + p^2} + \varepsilon_1(i\zeta)p}{\sqrt{\varepsilon_1(i\zeta) - 1 + p^2} - \varepsilon_1(i\zeta)p}\right)^2.$$

IV. CONCLUSION

Our derivation of the Lifshitz formula (75) is accomplished within the framework of the rigorous spectral summation method. We believe that in this approach the answers are given to many questions in this field which have needed solutions for a long time. First of all, it is a simultaneous incorporation, on the same footing, of the discrete and continuous parts of the electromagnetic spectrum, with consistent treatment of the branch cuts in the ω plane, with consideration of the complex eigenfrequency location and so on. Our approach is based on a special choice of appropriate passes in the contour integrals which are used for transition to imaginary frequencies. Contours of this type were employed in our preceding paper [67] dealing with a related problem.[9] The main idea of our approach is easy to grasp. With close consideration of Fig. 2, the mechanism of mutual cancellations between contributions of the discrete and continuous parts of the spectrum becomes evident as well as the final result, the Lifshitz formula (71). Without doubt, it is an important advantage of our consideration.

By making use of the contours C_+ and C_- shown in Fig. 2 one can easily “explain” in what way the correct Lifshitz formulas (71) and (72) may be “derived” by summing up only the discrete eigenfrequencies [1,3,4,10–18]. Let us ignore the branch cuts in the ω plane, the complex eigenfrequencies (quasinormal modes), and the bad behavior of the left-hand side of the frequency equation $D(\sigma,\omega)$ in the lower half plane ω. After that one may “glue” the sections of the contours C_+ and C_- which are along the positive Re ω axis and drop them. As a result, we obtain a unique contour involving the whole Im ω axis and a semicircle of large radius in the right half plane.

[7] A simple example of the complex frequencies in solutions to the Maxwell equations for an open system (a perfectly conducting sphere in the vacuum) was discussed in Ref. [34].

[8] In earlier papers dealing with other configurations the analogous cancellation was brought out by making use of the complex k_2 plane (see, for example, [65,66]). However, the Lifshitz formula is presented in terms of the variable $i\zeta = \omega$.

[9] By the way, it is these contours that had to be used in our preceding paper [68] considering dielectric cylinder without dispersion.

By using this contour in the first integral between the square brackets in Eq. (69) we arrive immediately at the final equation (71). Proceeding in an analogous way one may "derive" the Lifshitz formula in the scattering formalism alone [19–21, 69]. It should be noted here that the comparative analysis of the contributions to the vacuum energy generated by surface, waveguide, and photonic modes has recently been carried out by Bordag [70].

In the Lifshitz formula (71), written in terms of the integral over imaginary frequencies, there are no traces of the electromagnetic spectrum details in the problem at hand; in other words, this formula has the same universal form for all physically admissible material media. Presumably, it implies that the problem of finding the vacuum energy with allowance for the material characteristics of media is, on its nature, a *statistical problem*. Therefore, an appropriate mathematical method here would be the dissipation-fluctuation theorem formulated, from the very beginning, in terms of imaginary time. One can anticipate that this may simplify the transition to imaginary frequencies in the Lifshitz approach.

Many important topics are left beyond our consideration. One of them is the allowance for the dissipation when calculating the vacuum energy. There are many approaches to this problem (see, for example, [71] and more recent papers [72]), but until now it has been unclear as to how to come to an agreement regarding the rigorous spectral summation method which is based on the real eigenfrequencies in the pertinent spectral problem.

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