

**High-fidelity entanglement purification using chains of atoms and optical cavities**Denis Gonça<sup>1,\*</sup> and Peter van Loock<sup>2,†</sup><sup>1</sup>*Institut of Optics, Information und Photonics, Friedrich-Alexander-University Erlangen-Nuremberg, Staudtstrasse 7, 91058 Erlangen, Germany*<sup>2</sup>*Institut of Physics, Johannes Gutenberg University Mainz, Staudingerweg 7, 55128 Mainz, Germany*

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In our previous paper [D. Gonça and P. van Loock, *Phys. Rev. A* **84**, 042303 (2011)], we proposed an efficient scheme to purify dynamically a bipartite entangled state using short chains of atoms coupled to high-finesse optical cavities. In contrast to conventional entanglement purification protocols, we avoid controlled-NOT gates and thus reduce complicated pulse sequences and superfluous qubit operations. In this paper, we significantly improve the output fidelity of remotely entangled atoms by introducing one additional entanglement protocol in each of the repeater nodes and by optimizing the laser beams required to control the entire scheme. Our improved distillation scheme yields an almost-unit output fidelity that, together with the entanglement distribution and swapping, opens an attractive route towards an efficient and experimentally feasible quantum repeater for long-distance quantum communication.

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**I. INTRODUCTION**

In classical data transmission, repeaters are used to amplify the data signals (bits) when they become weaker during their propagation. In contrast to classical information, the above mechanism is impossible to realize when the transmitted data signals carry bits of quantum information (qubits). In an optical-fiber system, for instance, a qubit is typically encoded by a single photon which cannot be amplified or cloned without destroying quantum coherence associated with this qubit [1,2]. Therefore, the photon has to propagate along the entire length of the fiber, which causes an exponentially decreasing probability to receive this photon at the end of the channel.

To avoid this exponential decay of a photon wave packet and preserve its quantum coherence, the quantum repeater was proposed [3]. This repeater can be divided into three building blocks which have to be applied sequentially. First, a large set of entangled photon pairs distributed over sufficiently short fiber segments is generated. The two subsequent steps, (i) entanglement purification [4,5] and (ii) entanglement swapping [6], are employed to extend the short-distance entangled photon pairs over the entire length of the channel. Using the entanglement purification, high-fidelity entangled pairs are distilled from a larger set of low-fidelity entangled pairs by means of local operations performed in each of the repeater nodes and classical communication between these nodes. The entanglement swapping, finally, combines two entangled pairs of neighboring segments into one entangled pair, gradually increasing the distance of shared entanglement.

Because of the fragile nature of quantum correlations and inevitable photon loss in the transmission channel, in practice, it poses a serious challenge to outperform the direct transmission of photons along the fiber. Up to now, only particular building blocks of a quantum repeater have

been experimentally demonstrated, i.e., bipartite entanglement purification [7,8], entanglement swapping [9,10], and entanglement distribution between two neighboring nodes [11,12]. Motivated by both an impressive experimental progress and theoretical advances, moreover, various revised and improved implementations of repeaters and their building blocks have been recently proposed [13–17].

Practical schemes for implementing a quantum repeater are not straightforward. The two mentioned protocols, entanglement purification and entanglement swapping, in general, require feasible and reliable quantum logic, such as single- and two-qubit gates. Because of the high complexity and demand of physical resources, entanglement purification is the most delicate and cumbersome part of a quantum repeater. The conventional purification protocols [5,18], moreover, involve multiple controlled-NOT (CNOT) gates which pose a serious challenge for most physical realizations of qubits, involving complicated pulse sequences and superfluous qubit operations [8,19–23].

In our previous paper [24], we suggested a more practical scheme to purify a bipartite entangled state by exploiting the natural evolution of spin chains instead of CNOT gates. The realization of this dynamical scheme was proposed in the framework of cavity QED using short chains of atoms and optical cavities. In the present paper, we propose a modified purification scheme, in which we significantly improve the output fidelity of remotely entangled atoms. By introducing one additional entanglement protocol in each repeater node and by optimizing the laser beams required to control the entire scheme, we reach an almost-unit output fidelity after the same number of purification rounds as before. This dramatic improvement, therefore, allows for multiple entanglement swapping operations and opens a route towards an efficient and experimentally feasible quantum repeater for long-distance quantum communication.

The paper is organized as follows. In the next section, we describe in detail the original purification scheme presented in our previous paper. In Sec. III, we present our modified high-fidelity purification scheme. We analyze the atomic evolution

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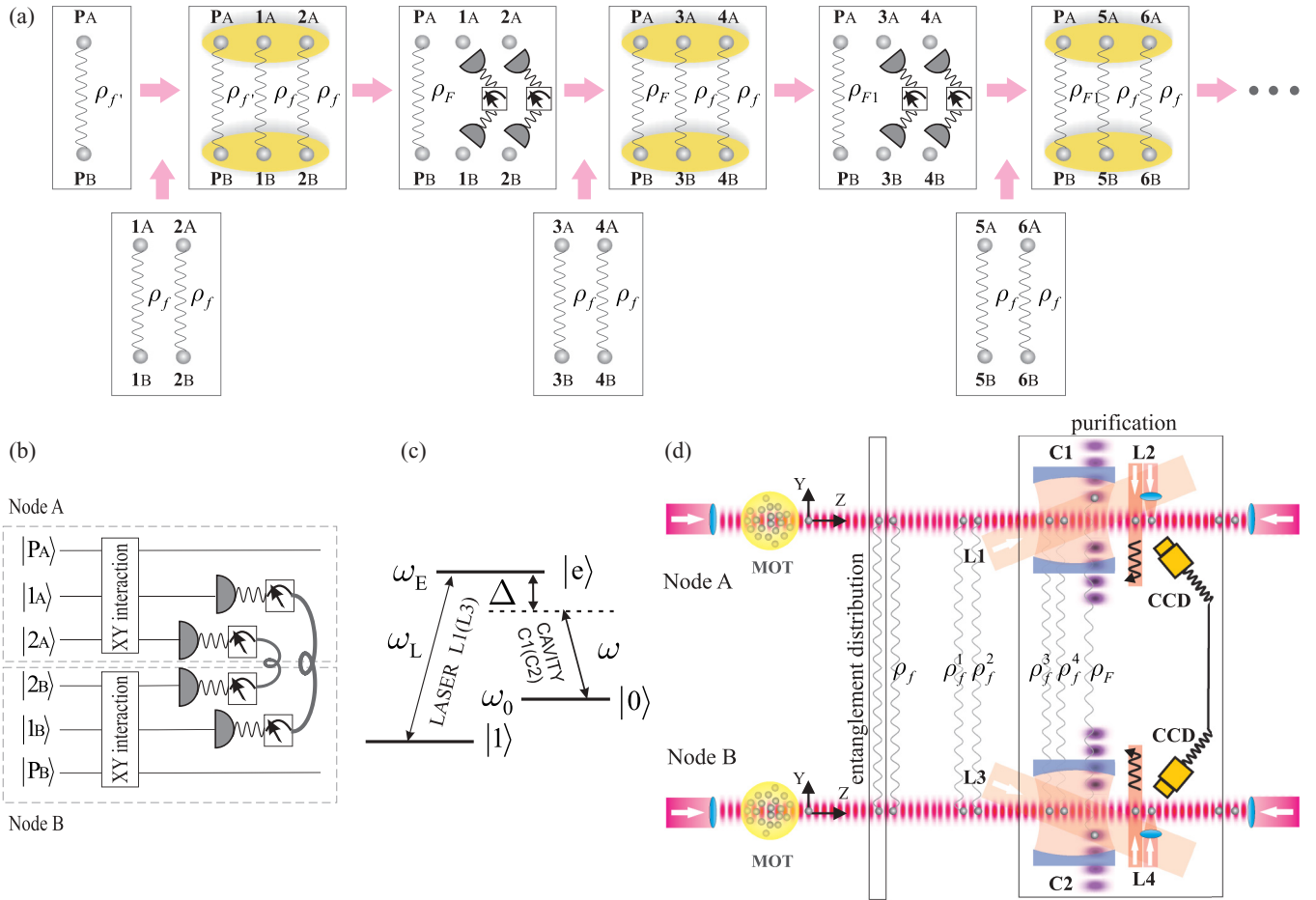


FIG. 1. (Color online) (a) Sequence of steps in the original purification scheme. (b) Quantum circuit corresponding to the interaction indicated above by gray ellipses. (c) Structure of a three-level atom in the  $\Lambda$ -type configuration. (d) Experimental setup that realizes the purification scheme (a) and is incorporated into a quantum repeater segment with two neighboring nodes. See text for description.

mediated by the cavity and laser field, and we determine the main properties which are relevant for our scheme in Sec. III A. In Sec. III B, we discuss a few relevant issues related to the implementation of our purification scheme, while a short summary and outlook are given in Sec. IV.

## II. DYNAMICAL ENTANGLEMENT PURIFICATION

In our previously proposed purification scheme, two repeater nodes,  $A$  and  $B$ , share one permanent qubit pair  $P_{AB}$  and a finite set of temporary (low-fidelity) entangled pairs grouped into elementary blocks of two qubit pairs as displayed in Fig. 1(a). Each temporary entangled pair is given by the rank 2 mixed state,<sup>1</sup>

$$\rho_f = f \Phi_{A,B}^+ + (1-f)\Phi_{A,B}^-, \quad (1)$$

<sup>1</sup>In our previous paper, we considered the Werner state  $\rho_f = f\Phi^+ + \frac{1-f}{3}(\Phi^- + \Psi^+ + \Psi^-)$  to describe low-fidelity entangled pairs. In this paper, instead, we consider the state, (1), which can be efficiently generated between two remote nodes of a repeater using an optimal, ultimate entanglement distribution and detection protocol [25]. In the last section, we discuss this protocol and provide evidence supporting our choice.

where  $\Phi_{A,B}^\pm \equiv |\phi_{A,B}^\pm\rangle\langle\phi_{A,B}^\pm|$  are the Bell states in the *qubit-storage* basis  $\{|0\rangle, |1\rangle\}$ , and where the fidelity

$$\mathbf{F}(\rho_f) \equiv \text{Tr}[\Phi_{A,B}^+ \rho_f] = f > 0.5 \quad (2)$$

is above the threshold value of  $1/2$ . The qubit-storage states  $|0\rangle$  and  $|1\rangle$  correspond to the two long-living states of a three-level atom in the  $\Lambda$  configuration as displayed in Fig. 1(c). In order to protect this qubit against the decoherence caused by the fast-decaying excited state  $|e\rangle$ , states  $|0\rangle$  and  $|1\rangle$  are chosen as the stable ground and long-living metastable states or as the two hyperfine levels of the ground state.

The permanent pair  $P_{AB}$ , characterized by the density operator  $\rho_f$ , is supplemented by two temporary pairs  $1_{AB}$  and  $2_{AB}$ , characterized by the density operators  $\rho_f^1$  and  $\rho_f^2$ , respectively, as shown in Fig. 1(a). Each of the repeater nodes  $A$  and  $B$ , therefore, contains one triplet of qubits,  $P_A, 1_A, 2_A$  and  $P_B, 1_B, 2_B$ , respectively. Each of these triplets evolves due to the isotropic Heisenberg  $XY$  Hamiltonian [26]

$$H_{xy} = \frac{\hbar J_1}{2} \sum_{i=1}^3 (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y), \quad (3)$$

over the time period ( $n = 0, 1, 2, \dots$ )

$$T = \frac{\pi}{3} \left( n + \frac{1}{2} \right) J_1^{-1}, \quad (4)$$

where  $\sigma_i^x$  and  $\sigma_i^y$  are the respective Pauli operators in the *cavity-active* basis  $\{|0\rangle, |e\rangle\}$ , such that  $\sigma_4^x = \sigma_1^x$  and  $\sigma_4^y = \sigma_1^y$  and where  $J_1$  is the coupling between the qubits. The above Hamiltonian with periodic boundaries is produced deterministically in our scheme by coupling of three (three-level) atoms to the same mode of a high-finesse resonator [see Fig. 1(c)]. In our previous paper, we identified the Hamiltonian, (3), with the Jaynes-Cummings Hamiltonian in the large detuning limit, i.e.,  $\Delta \gg g$ , where  $g$  is the atom-cavity coupling strength,  $\Delta$  is the atom-cavity detuning, and  $J_1 \equiv g^2/\Delta$  is the coupling between the atoms subject to the same cavity mode.

The evolution governed by Hamiltonian (3) over time period (4) is referred to below as the purification gate and is indicated by ellipses in Fig. 1(a) and by rectangles in Fig. 1(b). Once the purification gate is performed, the qubit pairs  $1_{AB}$  and  $2_{AB}$  are pairwise projected in the computational (qubit-storage) basis  $\{|0\rangle, |1\rangle\}$  and the outcome of these projections is exchanged between the two nodes by means of classical communication [see Fig. 1(b) and the third box in Fig. 1(a)]. Entanglement purification is successful if the outcome of projections reads

$$\{0_{1A}, 1_{2A}, 0_{1B}, 1_{2B}\} \text{ or } \{1_{1A}, 0_{2A}, 1_{1B}, 0_{2B}\}. \quad (5)$$

In this case, the (unprojected) permanent qubit pair  $P_{AB}$  is described by the density operator

$$\rho_F = F(f, f') \Phi_{A,B}^+ + [1 - F(f, f')] \Phi_{A,B}^-, \quad (6)$$

where

$$F(f, f') = \frac{f' - 16(f' - 2)f + 32(3f' - 1)f^2}{81 + 32f^2 - 80f' + 16(10f' - 7)f}, \quad (7)$$

such that  $\mathbf{F}(\rho_F) > \mathbf{F}(\rho_{f'})$ . The entanglement purification is unsuccessful, if the mentioned outcome of projections  $1_{AB}$  and  $2_{AB}$  disagrees with (5). In this case, the permanent pair  $P_{AB}$  should be reinitialized and the entire sequence from Fig. 1(a) restarted.

The density operator (6) ensures that the (permanent) qubit pair  $P_{AB}$  preserves its rank 2 form after each successful purification round. Unlike the conventional purification protocol, therefore, the purified state, (6), is completely characterized by the fidelity  $\mathbf{F}(\rho_F) = F(f, f')$ . Expression (7), furthermore, describes quantitatively how the input fidelity  $f'$  of the permanent qubit pair is modified due to one single (and successful) purification round. In Fig. 2(a), we compare the fidelity

$$F(f, f) = \frac{f(11 - 16f + 32f^2)}{27 - 64f + 64f^2} \quad (8)$$

(solid curve) with the respective fidelity given by Eq. (34) in Ref. [24] (dotted curve) that was obtained within the same

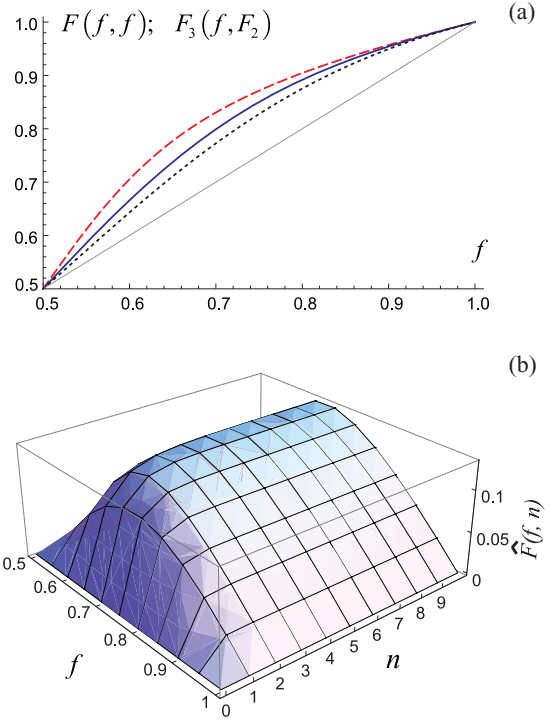


FIG. 2. (Color online) (a) Fidelities  $F(f, f)$  (solid curve) and  $F_3(f, F_2)$  (dashed curve) given by Eqs. (8) and (11), respectively. The dotted curve displays the fidelity given by Eq. (34) in Ref. [24]. (b) Plot of  $\widehat{F}(f, n)$  given by Eq. (10) as a function of input fidelity  $f$  and number  $n$  of purification rounds.

scheme, however, by considering the Werner state instead of the rank 2 mixed state, (1), in this paper. As expected due to the vanishing contribution of  $\Psi^\pm$  in (1), the growth of fidelity in the case of a rank 2 mixed state is larger as for the Werner state.

Assuming that each purification round is successful, the sequence from Fig. 1(a) leads to the gradual growth of entanglement fidelity (of stationary atoms) with regard to the respective fidelity obtained in the previous round

$$f' < F_1(f, f') < F_2(f, F_1) < \dots < F_n(f, F_{n-1}). \quad (9)$$

In order to understand how much the output fidelity increases with each purification round, we analyze quantitatively the following sequence:

$$f < F_1(f, f) < \dots < F_n(f, F_{n-1}) \equiv f + \widehat{F}(f, n). \quad (10)$$

In Fig. 2(b), we show a plot of function  $\widehat{F}(f, n)$  that describes the difference between the final fidelity  $F_n(f, F_{n-1})$  obtained after  $n$  (successful) rounds and the initial fidelity  $f$  ( $n = 0$ ). It is clearly shown that during the first three rounds, this function exhibits a notably fast growth that saturates and, with increasing  $n$ , yields a negligible growth with regard to the *fixed point* fidelity:

$$F_3(f, F_2) = \frac{f(70859 - 377904f + 950112f^2 - 1368064f^3 + 1278976f^4 - 671744f^5 + 294912f^6)}{177147 - 1051072f + 2792896f^2 - 4204544f^3 + 3904512f^4 - 2162688f^5 + 720896f^6}. \quad (11)$$

Regardless of the number of (successful) purification rounds, therefore, the final fidelity is bounded by a fixed point value that determines the optimal number of purification rounds required to reach the best performance in a resource- and time-efficient way. Such behavior is the common feature of the (so-called) *entanglement pumping* purification scheme introduced by W. Dür and coauthors in Ref. [18]. We refer to this property as the saturation of entanglement purification. Corresponding to  $n = 3$  successful purification rounds, for which the final fidelity reaches its saturation level, we display the (fixed point) fidelity  $F_3(f, F_2)$  by the dashed curve in Fig. 2(a). By comparing this curve to the solid curve, we conclude that the growth of fidelity due to three successive rounds is notably larger compared to the case of a single purification round.

We remark that the described purification scheme is based on the effect of entanglement transfer between the networks of evolving spin chains that was introduced and investigated in Ref. [27]. In the same reference, it was suggested that this effect plays the key role in the entanglement concentration once a part of the spins from two such networks is locally measured. One similar entanglement purification protocol, which is based on the natural spin dynamics, has been proposed independently in Ref. [28]. In our scheme, the role of (spin-chain) networks is played by the atomic triplets located in two repeater nodes, while the cavity-mediated interaction governed by Hamiltonian (3) reproduces the spin-chain dynamics. From a more fundamental point of view, the mentioned effect of entanglement transfer originates the constructive and destructive interference of the quantized spin waves (magnons) in an evolving spin chain (see [28] and references therein).

The main physical resources of the proposed purification scheme are (i) short chains of atoms, (ii) two high-finesse optical cavities, and (iii) detectors for projective measurements of atomic states. In Fig. 1(d) we show the experimental setup of a quantum repeater segment that includes two neighboring nodes ( $A$  and  $B$ ). In this setup, each repeater node consists of one optical cavity  $C_1$  ( $C_2$ ) acting along the  $y$  axis, a laser beam  $L_1$  ( $L_3$ ), a chain of atoms transported by means of an optical lattice along the same axis, one stationary atom trapped inside the cavity with the help of a vertical lattice, a laser beam  $L_2$  ( $L_4$ ) acting along the  $y$  axis, a magneto-optical trap (MOT), and a CCD camera connected to the neighboring node through a classical communication channel.

We associated the permanent qubits with the stationary atoms trapped inside cavities  $C_1$  and  $C_2$  and the temporary qubits with (the chains of) atoms inserted into the horizontal lattices and transported along the  $z$  axis. According to the experimental scheme in Fig. 1(d), this identification implies that atoms pass sequentially through the cavity, such that only two atoms from the chain couple simultaneously to the same cavity mode. These two atoms together with the stationary (trapped) atom form an atomic triplet in each repeater node as assumed by our purification scheme.

Right before an atom from node  $A$  enters the cavity, it becomes entangled with the respective atom from node  $B$  as depicted in Fig. 1(d) by wavy lines. This entanglement is generated nonlocally by means of an entanglement distribution block (indicated by the rectangle), such that each produced entangled pair is described by Eq. (1) in the qubit-storage basis

$\{|0\rangle, |1\rangle\}$ . During the transition of an atomic pair through the cavity, the triplet of atoms has to undergo the cavity-mediated evolution governed by Hamiltonian (3) in each of the repeater nodes over time period (4). Since the Hamiltonian acts solely on the cavity-active states  $\{|0\rangle, |e\rangle\}$ , the atomic population has to be mapped from the qubit-storage basis to the cavity-active basis in order to make possible the interaction of atoms with the cavity mode and, moreover, to protect the qubits against the decoherence caused by the fast-decaying excited state  $|e\rangle$ . This mapping is realized using short resonant light pulses produced by the laser beam  $L_1$  ( $L_3$ ). Each pulse transfers the electronic population from the qubit-storage states to the cavity-active states (or backwards), such that the atoms couple to (or decouple from) the cavity field in a controlled fashion.

According to the sequence in Fig. 1(a), furthermore, the purification sequence is completed once the states of an (conveyed) atomic pair are projectively measured and the outcome of projections is pairwise exchanged between the repeater nodes in order to decide if the purification was successful or not. In our experimental scheme, the latter projections are performed by means of the laser beam  $L_2$  ( $L_4$ ) and a CCD camera in each of the repeater nodes as displayed in Fig. 1(d). While the laser beam  $L_2$  ( $L_4$ ) removes atoms in a given (storage-basis) state from the chain without affecting atoms in the other state (so-called push-out technique [29]), the CCD camera is used to detect the presence of remaining atoms via fluorescence imaging and, therefore, determine the state of each atom that leaves the cavity.

In the successful case, furthermore, the next atomic pair is transported into the cavity and the next purification round takes place with the same stationary atom (permanent qubit). In the unsuccessful case, however, the stationary atoms have to be reinitialized and the entire sequence from Fig. 1(a) should be restarted.

The approach presented in this section requires that short atomic chains are transported at a constant velocity along the experimental setup and coupled to the cavity-laser fields in a well-controllable fashion. For this purpose, we introduced in our setup [see Fig. 3(d)] (i) a MOT that plays the role of an atomic source and (ii) an optical lattice (conveyor belt) that transports atoms into the cavity from the MOT with a position and velocity control over the atomic motion. The proposed setup is compatible with existing experimental setups [30–32], in which the above devices, i and ii, are integrated into the same framework together with a high-finesse optical cavity. The number-locked insertion technique [33], moreover, enables one to extract atoms from the MOT and insert a predefined pattern of them into an optical lattice with a single-site precision. It was already demonstrated that an optical lattice preserves the coherence of transported atoms and can be utilized as a holder of a quantum register. By encoding the qubits by means of hyperfine atomic levels, a qubit storage time of the order of seconds has been demonstrated within this register [29,34].

### III. HIGH-FIDELITY DYNAMICAL ENTANGLEMENT PURIFICATION

As shown in Fig. 2(a), the output fidelity  $F_3(f, F_2)$  (dashed curve) obtained after three successful purification rounds is

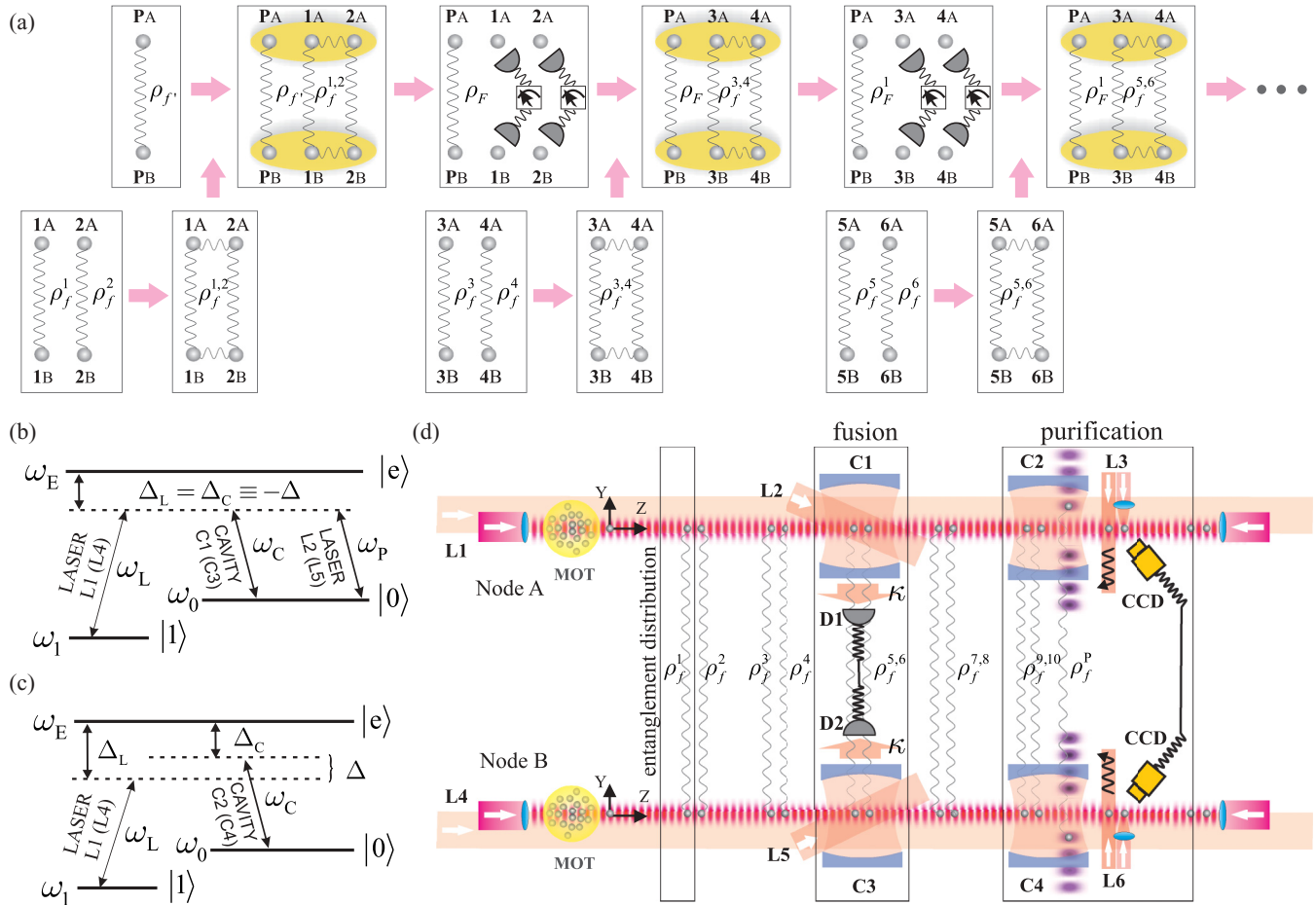


FIG. 3. (Color online) (a) Sequence of steps in the modified purification scheme. (b, c) Structure of a three-level atom in the  $\Lambda$ -type configuration subjected to cavity and laser fields. See text for description. (d) Experimental setup that realizes the purification scheme (a) and is incorporated into a quantum repeater segment with two neighboring nodes.

still far from unit fidelity as required by a realistic quantum repeater. In fact, this output fidelity enables one to perform only a few swapping operations between the purified entangled pairs of neighboring repeater segments until the fidelity of the resulting pair (distributed over a larger distance) drops to the initial fidelity  $f$ . Another bottleneck in our scheme is the necessity to transfer the electronic population from the qubit-storage states to the cavity-active states (and backwards) in order to control the cavity-mediated evolution of atoms inside the cavity and protect our qubits against the decoherence caused by the fast-decaying excited state  $|e\rangle$  [see Fig. 1(c)]. Obviously, these two obstacles make our scheme less attractive to be considered in practice.

In this section, we propose a modified purification scheme, in which we significantly improve the output fidelity of remotely entangled atoms and get rid of the superfluous laser pulses required to transfer the electronic population of atoms. By introducing one additional entanglement protocol in each repeater node and by optimizing the laser beams required to control the entire scheme, we achieve an almost-unit output fidelity after the same number of successful purification rounds. This dramatic improvement, therefore, allows for multiple entanglement swapping operations on the purified pairs.

Similar to the original scheme that we presented in the previous section, the modified scheme includes two repeater nodes, A and B, sharing one permanent qubit pair  $P_{AB}$ , characterized by the density operator  $\rho_f$ , and a finite set of temporary entangled pairs as displayed in Fig. 3(a). Each temporary entangled pair is given by the rank 2 mixed state, (1), in the basis  $\{|0\rangle, |1\rangle\}$ , such that the fidelity, (2), of each pair is above the threshold value of  $1/2$ . In contrast to the original scheme, however, right before the permanent pair is supplemented by the temporary pairs  $1_{AB}$  and  $2_{AB}$ , these two (separate) entangled pairs are merged into the four-qubit entangled state:

$$\begin{aligned} \rho_f^{1,2} = & \frac{1}{2} (|\phi_{1A,2A}^-, \phi_{1B,2B}^- \rangle \langle \phi_{1A,2A}^-, \phi_{1B,2B}^- | \\ & + |\psi_{1A,2A}^-, \psi_{1B,2B}^- \rangle \langle \psi_{1A,2A}^-, \psi_{1B,2B}^- |) \\ & + \frac{2f-1}{2(1-2f+2f^2)} (|\phi_{1A,2A}^-, \phi_{1B,2B}^- \rangle \langle \psi_{1A,2A}^-, \psi_{1B,2B}^- | \\ & + |\psi_{1A,2A}^-, \psi_{1B,2B}^- \rangle \langle \phi_{1A,2A}^-, \phi_{1B,2B}^- |). \end{aligned} \quad (12)$$

This entangled state is generated using an additional entanglement protocol that occurs prior to the purification gate in our scheme. By this protocol, the pairs  $1_{A,2A}$  and  $1_{B,2B}$  interact locally within the repeater nodes A and B,

respectively, such that state (12) is generated. In Fig. 3(d) we display the experimental setup of our modified scheme. In contrast to the setup displayed in Fig. 1(d), we added (i) high-finesse cavities  $C_1$  and  $C_3$ , (ii) photon detectors  $D_1$  and  $D_2$ , and (iii) laser beams  $L_2$  and  $L_5$  to each of the repeater nodes  $A$  and  $B$ , respectively. These ingredients are compatible with the resources utilized in the original purification scheme and together they form the *fusion* block that is framed by the rectangle in Fig. 3(d). Finally, the laser beams  $L_1$  and  $L_4$  act continuously along the  $z$  axis and, together with the cavity field of  $C_1$  ( $C_2$ ) and  $C_3$  ( $C_4$ ), respectively, produce the two-photon (Raman) transition between state  $|0\rangle$  and state  $|1\rangle$  of the coupled atoms, such that the fast-decaying excited state  $|e\rangle$  remains almost unpopulated (see below).

Being transported from the entanglement distribution block into the cavity  $C_1$  ( $C_3$ ), the atoms  $1_A, 2_A$  ( $1_B, 2_B$ ) couple simultaneously to the same cavity mode and both laser beams  $L_2$  ( $L_5$ ) and  $L_1$  ( $L_4$ ) as displayed in Fig. 3(b). Assuming the nonzero cavity relaxation rate  $\kappa$  associated with  $C_1$  ( $C_3$ ), the evolution of the coupled atom-cavity-laser system is governed by the master equation [35]

$$\dot{\rho} = -\frac{i}{\hbar}[H_S, \rho] + \frac{\kappa}{2}(2a\rho a^\dagger - a a^\dagger \rho - \rho a^\dagger a) \equiv \mathcal{L}\rho, \quad (13)$$

$$H_S = \frac{\hbar J_2}{2}(a + a^\dagger)(\sigma_1^X + \sigma_2^X), \quad (14)$$

where  $\rho$  is the density operator describing the state of the two atoms together with the cavity mode,  $\mathcal{L}$  is the Lindbladian superoperator that acts on the density operator,  $\sigma_i^X$  is the respective Pauli operator in the basis  $\{|0\rangle, |1\rangle\}$ , and  $J_2$  is the coupling between the atoms inserted into the same cavity mode and subjected to the two laser beams. We show in Appendix A that the above Hamiltonian is produced deterministically in our setup assuming both (i) the strong driving regime of atoms and (ii) the large detuning limit for laser and cavity fields.

The evolution of atoms  $1_A, 2_A$  ( $1_B, 2_B$ ) coupled to the field of cavity  $C_1$  ( $C_3$ ) due to Eq. (13) is completely determined by the exponent  $e^{\mathcal{L}t}$  and the initial state of the cavity and the atoms. Since the pairs of atoms  $1_{AB}$  and  $2_{AB}$  are initially entangled, we have to consider the composite density operator

$$\tilde{\rho}_{AB} = e^{(\mathcal{L}_A + \mathcal{L}_B)t} (\rho_f^1 \otimes \rho_f^2 \otimes |\bar{0}_A, \bar{0}_B\rangle\langle\bar{0}_A, \bar{0}_B|), \quad (15)$$

describing the state of two atomic pairs and two initially empty cavities at a given time  $t$ . In this expression,  $\rho_f^1$  and  $\rho_f^2$  are the density operators of the entangled pairs  $1_{AB}$  and  $2_{AB}$ , respectively, while  $|\bar{0}_A\rangle$  and  $|\bar{0}_B\rangle$  denote the vacuum states of cavities  $C_1$  and  $C_3$ , respectively. In Appendix B we show, moreover, that, conditioned upon the no-photon measurement of the leaked cavity field in both repeater nodes, state (15) reduces to state (12) in the steady-state regime ( $\kappa t \gg 1$ ), that is,

$$\rho_f^{1,2} = \frac{\langle \bar{0}_A, \bar{0}_B | \tilde{\rho}_{AB}^{ss} | \bar{0}_A, \bar{0}_B \rangle}{\text{Tr}[\langle \bar{0}_A, \bar{0}_B | \tilde{\rho}_{AB}^{ss} | \bar{0}_A, \bar{0}_B \rangle]}, \quad (16)$$

where  $\tilde{\rho}_{AB}^{ss}$  is operator (15) in the steady-state regime. The measurement of the leaked cavity field is performed using the photon detector  $D_1$  ( $D_2$ ), which is connected to the neighboring repeater node through a classical communication channel. We stress that since the detection of the leaked cavity field in our scheme discriminates between a vacuum state (no

clicks) and a strong coherent state (many clicks), the efficiency of the detectors  $D_1$  and  $D_2$  can take rather moderate values.

Assuming that the four-qubit entangled state, (12), has been successfully generated, the permanent pair  $P_{AB}$  is supplemented by two temporary pairs,  $1_{AB}$  and  $2_{AB}$ , as displayed in Fig. 3(a). Similarly to the original scheme, each repeater node now contains one triplet of qubits and each of these triplets evolves due to the isotropic Heisenberg  $XY$  Hamiltonian

$$H_{XY} = \frac{\hbar J_3}{2} \sum_{i=1}^3 (\sigma_i^X \sigma_{i+1}^X + \sigma_i^Y \sigma_{i+1}^Y) \quad (17)$$

over the time period ( $n = 0, 1, 2, \dots$ )

$$T = \frac{\pi}{3} \left( n + \frac{1}{2} \right) J_3^{-1}, \quad (18)$$

such that  $\sigma_4^X = \sigma_1^X$  and  $\sigma_4^Y = \sigma_1^Y$  and where  $J_3$  is the coupling between the qubits. In Appendix C, we show that Hamiltonian (17) is produced deterministically in our scheme by coupling simultaneously three atoms to the same cavity mode  $C_2$  ( $C_4$ ) and the laser beam  $L_1$  ( $L_4$ ) in the large detuning limit [see Fig. 3(c)].

Similar to the original scheme, this evolution is followed by the projective measurement of qubit pairs  $1_{AB}$  and  $2_{AB}$  in the basis  $\{|0\rangle, |1\rangle\}$  and the exchange of the projection outcomes between the two repeater nodes by means of classical communication. The entanglement purification is successful if the outcome of projections agrees with (5). In this case, the (unprojected) permanent qubit pair is described again by the density operator (6), where

$$F(f, f') = \frac{(25 - 50f + 194f^2)f'}{169 + 194f^2 - 144f' + (288f' - 338)f}, \quad (19)$$

such that  $\mathbf{F}(\rho_F) = F(f, f') > \mathbf{F}(\rho_{f'})$ . In Fig. 4(a), we compare the fidelity

$$F(f, f) = \frac{(25 - 50f + 194f^2)f}{169 - 482f + 482f^2} \quad (20)$$

(solid curve) with the respective fidelity given by Eq. (8) (dotted curve). We see that the growth of fidelity in the modified scheme is almost twice as large as in the original purification scheme. This nice result, however, relies merely on the input state, (12), which is entangled strongly compared to the separable state  $\rho_f^1 \otimes \rho_f^2$  used in the original scheme. We recall that our scheme relies on the effect of entanglement transfer between the networks of evolving spin chains introduced in Refs. [27] and [28] and realized in our scheme using the cavity QED framework. The stronger the entangled state that we provide as the input for the purification block in our scheme, the more entanglement is transferred to the permanent qubit pair.

Assuming that each purification round is successful, the sequence in Fig. 3(a) leads to a gradual growth of entanglement fidelity (of stationary atoms) with regard to the respective fidelity obtained in the previous round. Similar to the original scheme, we analyze quantitatively sequence (10) in order to understand how much the output fidelity increases with each purification round. In Fig. 4(b), we show the plot of  $\hat{F}(f, n)$  that

describes the difference between the final fidelity  $F_n(f, F_{n-1})$  obtained after  $n$  (successful) rounds and the initial fidelity  $f$  ( $n = 0$ ). For  $f > 0.75$ , this function exhibits a dramatic

growth during the first three rounds that saturates and, with increasing  $n$ , yields a negligible growth with regard to the fixed point fidelity

$$F_3(f, F_2) = \frac{f(25 - 50f + 194f^2)^3}{4826809 - 33772038f + 103411314f^2 - 179097440f^3 + 189095940f^4 - 119456664f^5 + 39818888f^6}, \quad (21)$$

displayed in Fig. 4(a) by the dashed curve. We readily see that we obtain an almost-unit output fidelity for  $f > 0.75$  after the same number of purification rounds as in the original scheme. For  $f < 0.75$ , however, the function  $\widehat{F}(f, n)$  continues to grow with each purification round. In this case, the optimal number

of purification rounds and the respective fixed point fidelity have to be determined for each particular value of  $f$  separately.

#### A. Evolution governed by Hamiltonian (17) and the purification gate

In this section, we already explained that density operator (6) with function (19) characterize completely the permanent qubit pair obtained in our (modified) scheme after a single purification round. In this subsection, we analyze briefly the evolution governed by Hamiltonian (17) and connect it with the main results utilized in this paper.

The atomic evolution governed by Hamiltonian (17),

$$e^{-\frac{i}{\hbar} H_{XY} t} = \sum_{k=1}^8 e^{-\frac{i}{\hbar} E_k t} |k\rangle\langle k|, \quad (22)$$

is completely determined by the energies  $E_k$  and vectors  $|k\rangle$ , which satisfy the eigenvalue equality  $H_{XY} |k\rangle = E_k |k\rangle$  with orthogonality and completeness relations  $\langle k|k'\rangle = \delta_{kk'}$  and  $\sum |k\rangle\langle k| = I$ , respectively. With the help of Jordan-Wigner transformation [36], this eigenvalue problem can be solved exactly (see, e.g., Ref. [37]). Since the evolution operator, (22), acts on the states of one atomic triplet that is entangled with another atomic triplet in the neighboring node, we have to consider the composite evolution operator

$$U(t) = \sum_{k, k'=1}^8 e^{-\frac{i}{\hbar} (E_k + E_{k'}) t} |k_A \otimes k'_B\rangle\langle k_A \otimes k'_B|. \quad (23)$$

Earlier we explained that right before each atomic pair from node  $A$  enters the cavity, it becomes entangled with another atomic pair from node  $B$ , such that the four-qubit state (12) is generated. We denote the density operator of the stationary atoms  $\rho_f^3$ . According to evolution operator (23) and this notation, the state of both atomic triplets in nodes  $A$  and  $B$  is described by the six-qubit density operator

$$\rho^{1,2,3}(t, f, f') = U(t)(\rho_f^{1,2} \otimes \rho_{f'}^3)U^\dagger(t), \quad (24)$$

which evolves over time period  $T$  given by Eq. (18). After this evolution, the state of both atomic triplets in nodes  $A$  and  $B$  is described by the density operator

$$\rho^{1,2,3}(T, f, f') = \sum_{i, j=1}^{64} \rho_{ij}^{1,2,3}(T, f, f') |\mathbf{v}_i\rangle\langle \mathbf{v}_j|, \quad (25)$$

where  $2^6$  composite vectors  $|\mathbf{v}_i\rangle$ , satisfying the orthogonality and completeness relations  $\langle \mathbf{v}_i|\mathbf{v}_j\rangle = \delta_{ij}$  and  $\sum |\mathbf{v}_i\rangle\langle \mathbf{v}_i| = I$ , respectively, have been introduced.

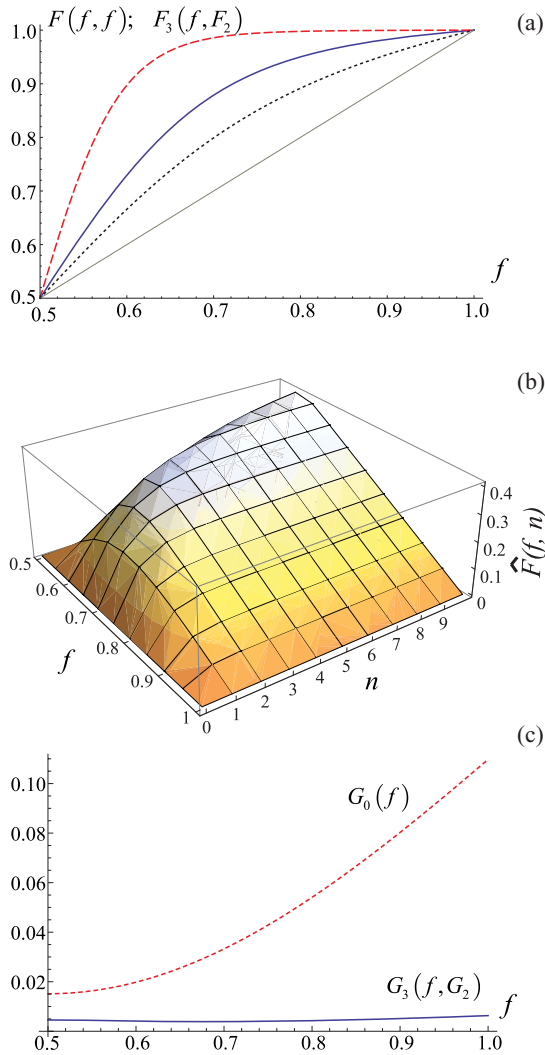


FIG. 4. (Color online) (a) Fidelities  $F(f, f)$  (solid curve) and  $F_3(f, F_2)$  (dashed curve) given by Eqs. (20) and (21), respectively. The dotted curve displays the fidelity given by Eq. (8) obtained in the original scheme. (b) Plot of  $\widehat{F}(f, n)$  in the modified scheme as a function of input fidelity  $f$  and number  $n$  of purification rounds. (c) Off-diagonal contributions  $G_0(f)$  (dashed curve) and  $G_3(f, G_2)$  (solid curve) given by Eqs. (29) and (33), respectively.

In order to finalize one purification round, the conveyed atomic pairs are projectively measured, such that the projected density operator

$$\rho(T, f, f') = \sum_{\alpha, \beta=1}^4 \frac{\rho_{\alpha\beta}^{1,2,3}(T, f, f')}{P_{\text{succ}}(T, f, f')} |\tilde{\mathbf{v}}_{\alpha}\rangle \langle \tilde{\mathbf{v}}_{\beta}|, \quad (26)$$

with

$$P_{\text{succ}}(T, f, f') = \text{Tr} \left[ \sum_{\alpha, \beta=1}^4 \rho_{\alpha\beta}^{1,2,3}(T, f, f') |\tilde{\mathbf{v}}_{\alpha}\rangle \langle \tilde{\mathbf{v}}_{\beta}| \right],$$

describes the state of the stationary atoms. In the above expressions, the Greek indices run over the four different values given by

$$|\tilde{\mathbf{v}}_{\alpha}\rangle \equiv |0_{1A}, 1_{2A}, 0_{1B}, 1_{2B}| \mathbf{v}_{\alpha}\rangle \neq 0 \quad (27a)$$

$$\text{or } |\tilde{\mathbf{v}}_{\alpha}\rangle \equiv |1_{1A}, 0_{2A}, 1_{1B}, 0_{2B}| \mathbf{v}_{\alpha}\rangle \neq 0, \quad (27b)$$

which correspond to the outcomes of the projections, (5).

Using the six-qubit density operator, (24), we have routinely computed the matrix elements  $\rho_{ij}^{1,2,3}(T, f, f')$ , which, however, are rather bulky to be displayed here. With the help of these matrix elements, we confirmed that the density operator (26) coincides with the rank 2 mixed state (6), where the function  $F(f, f')$  is given by Eq. (19).

### B. Remarks on the entanglement distribution between stationary atomic qubits

Throughout the paper, we have assumed that the stationary atoms are initially entangled, such that the fidelity is above the threshold value of  $1/2$ . In our previous paper, we suggested that there is no need to introduce an additional entanglement distribution protocol in our setup in order to entangle the stationary atoms prior to the purification. Instead, it was suggested to prepare initially both permanent atoms in the ground state and run our purification scheme. We showed that one successful purification round *induces* the entanglement of the stationary atoms and ensures that the fidelity of resulting density operator (almost) coincides with the fidelity  $f$  of the temporary pairs  $\rho_f^1$  and  $\rho_f^2$ . In other words, one purification round entangles two (initially separable) stationary atoms, such that the fidelity of the temporary pairs is mapped to the fidelity of the stationary density operator.

In our modified scheme, we utilize the same procedure as in the original scheme. We generate two (separate) entangled pairs and let them be conveyed through cavity  $C_1$  ( $C_3$ ), such that the (probabilistic) entanglement protocol that produces the four-qubit state, (12), is switched off. It can be shown that a successful purification round with these two entangled pairs transforms the state of the permanent atoms (prepared initially in the ground state) into an entangled state described by

$$\begin{aligned} \rho_f = & F_0(f) \Phi_{A,B}^+ + (1 - F_0(f)) \Phi_{A,B}^- \\ & + G_0(f) (|\phi_{A,B}^+\rangle \langle \phi_{A,B}^-| + |\phi_{A,B}^-\rangle \langle \phi_{A,B}^+|), \end{aligned} \quad (28)$$

with

$$F_0(f) = \frac{1 + 48f + 32f^2}{\sum_i (-1)^i c_i f^i}, \quad G_0(f) = \frac{9 - 32f + 32f^2}{\sum_i (-1)^i c_i f^i}, \quad (29)$$

where  $c_0 = 82$ ,  $c_1 = c_2 = 64$  are the only nonzero coefficients.

Obviously, the above state is no longer a rank 2 mixed state as in Eq. (1) because of the off-diagonal contribution  $G_0(f)$  displayed in Fig. 4(c) by the dashed curve. The fidelity, (2), associated with (28), however, is slightly larger than the fidelity  $f$  associated with the temporary pairs  $\rho_f^1$  and  $\rho_f^2$ . The role of this *initialization* round is solely to entangle the stationary atoms, and therefore, it has to be followed by a number of purification rounds leading to the gradual growth of fidelity,

$$F_0(f) < F_1(f, F_0) < \dots < F_n(f, F_{n-1}), \quad (30)$$

and the gradual reduction of (off-diagonal) contributions,

$$G_0(f) > G_1(f, G_0) > \dots > G_n(f, G_{n-1}). \quad (31)$$

Using the above sequences, we calculated the fidelity

$$F_3(f, F_2) = \frac{(1 + 48f + 32f^2)(25 - 50f + 194f^2)^3}{2 \sum_i (-1)^i d_i f^i} \quad (32)$$

and the respective off-diagonal contribution

$$G_3(f, G_2) = \frac{274625(9 - 32f + 32f^2)(1 - 2f + 2f^2)^3}{2 \sum_i (-1)^i d_i f^i}, \quad (33)$$

which are motivated by the optimal number of purification rounds obtained previously [see (11) and (21)] and where

$$\begin{aligned} d_0 = 195493577, \quad d_1 = 1442887766, \quad d_2 = 4716352898, \\ d_3 = 8883640864, \quad d_4 = 10517241220, \quad d_5 = 7944708952, \\ d_6 = 3738576328, \quad d_7 = 934577152, \quad d_8 = 233644288 \end{aligned}$$

are the only nonzero coefficients.

In contrast to the dashed curve describing  $G_0(f)$  in Fig. 4(c), the solid curve describing (33) deviates slightly around the constant value of 0.004. To a good approximation, therefore, the off-diagonal contribution  $G_3(f, G_2)$  can be neglected and the resulting density operator takes the form of the rank 2 mixed state, (1). We have verified, moreover, that the output fidelity, (32), (almost) coincides with the output fidelity, (21), displayed in Fig. 4(a) by the dashed curve. The price we pay for one extra (successful) purification round prior to the main sequence of rounds, therefore, is clearly compensated by the more moderate demand of physical resources in our purification scheme.

## IV. SUMMARY AND DISCUSSION

In this paper, an efficient, high-fidelity scheme was proposed to purify the low-fidelity entangled atoms trapped in two remote optical cavities. This scheme is a modification of the purification scheme proposed in our previous paper [24], which exploits the natural evolution of spin chains instead of CNOT gates. Similarly to the original scheme, the modified scheme uses a cavity-QED framework, namely, (i) short chains of atoms, (ii) high-finesse optical cavities, and (iii) detectors for the projective measurement of atomic states. In contrast to the original scheme, however, one additional entanglement protocol was introduced in each repeater node, and the laser beams which are used to control the entire scheme were optimized. With the help of these modifications, an almost-unit output fidelity was achieved after the same number



of successful purification rounds as in the original scheme. Similarly to the original paper, furthermore, the modified scheme was supplied with a detailed experimental setup, and a complete description of all necessary steps and manipulations has been given. A comprehensive analysis of fidelities obtained after multiple purification rounds was performed and the optimal number of rounds was determined. We also discussed in detail the initial distribution of entanglement between the stationary qubits trapped in two remote cavities.

Throughout the paper, we have assumed that each purification round is finalized successfully, leading to a gradual growth, (9) or (30), of entanglement fidelity. In the case of an unsuccessful purification event, i.e., when the outcomes of the projective measurement disagree with (5), the stationary atoms should be reinitialized and the entire scheme restarted. Since the probability to get the two (of  $2^4$ ) combinations of projective measurements for a successful purification is rather low, the occurrence of multiple unsuccessful events can require a large amount of atomic pairs in the chain and unreasonable operational times. We stress, therefore, that although the proposed purification scheme is experimentally feasible, a practical mechanism that reduces unsuccessful purification events has to be considered. This problem and possible solutions shall be addressed in our future work.

The high-fidelity purification scheme proposed in this paper enables one to perform multiple entanglement swapping operations and thus opens a route towards an efficient and experimentally feasible quantum repeater for long-distance quantum communication. More specifically, in our experimental setup, each atom in node A has to be entangled with another atom from node B right before they enter cavities  $C_1$  and  $C_3$  for further processing. The (low-fidelity) entanglement between these atoms is distributed nonlocally using the entanglement distribution block indicated in Fig. 3(d) by the rectangle. In order to entangle two (three-level) atoms located at distant repeater nodes A and B, we find the entanglement distribution scheme proposed in Ref. [38] the most appropriate. This scheme is also realizable in the framework of cavity-QED and, therefore, it utilizes the same physical resources as our purification scheme.

By this scheme, a coherent-state light pulse interacts with the coupled atom-cavity system in node A, such that the optical field accumulates a phase conditioned upon the atomic state in this node. Afterwards, the light pulse propagates to node B, where it interacts with the second coupled atom-cavity system and accumulates another phase conditioned upon the atomic state in this node. The resulting density operator [25],

$$f |\tilde{\phi}_{A,B}^+\rangle\langle\tilde{\phi}_{A,B}^+| + (1-f) |\tilde{\phi}_{A,B}^-\rangle\langle\tilde{\phi}_{A,B}^-|, \quad (34)$$

with

$$|\tilde{\phi}_{A,B}^\pm\rangle \equiv \frac{1}{\sqrt{2}} |C_0\rangle |\phi_{A,B}^\pm\rangle \pm \frac{1}{2} e^{-i\eta\epsilon} |C_1\rangle |1_A, 0_B\rangle + \frac{1}{2} e^{i\eta\epsilon} |C_2\rangle |0_A, 1_B\rangle,$$

describes the state of both atoms and the coherent light pulse, where  $C_0 \equiv \sqrt{\eta}\alpha$ ,  $C_1 \equiv \sqrt{\eta}\alpha e^{i\theta}$ ,  $C_2 \equiv \sqrt{\eta}\alpha e^{-i\theta}$  denote the phase-rotated and channel-damped coherent state  $\alpha$ ,  $\epsilon \equiv \alpha^2 \sin\theta$ , and  $f \equiv (1 + e^{-(1-\eta)\alpha^2(1-\cos\theta)})/2$  plays the role of the entanglement fidelity.

The resulting (phase-rotated) coherent pulse becomes disentangled from the atoms with the help of homodyne detection followed by postselection [38] or, alternatively, using unambiguous state discrimination [25]. This projects state (34) onto an entangled state of two atoms that coincides with the rank 2 mixed state, (1). We remark that the conditioned phase rotation exploited in the entanglement distribution scheme is naturally realized in a cavity-QED framework using the single atom-cavity evolution in the dispersive interaction regime.

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## APPENDIX A: DERIVATION OF HAMILTONIAN (14)

In this Appendix, we show that Hamiltonian (14) is produced deterministically in our setup. Specifically, two (three-level) atoms are subjected to the field of the (initially empty) cavity  $C_1$  ( $C_3$ ) and the fields of laser beams  $L_1$  ( $L_4$ ) and  $L_2$  ( $L_5$ ) simultaneously as displayed in Fig. 3(b). The evolution of this coupled atom-cavity-laser system is governed by the Hamiltonian ( $k = 1, 2$ )

$$H_1 = \hbar \omega_C a^\dagger a - i\hbar \sum_k \left[ \frac{g}{2} a |e\rangle_k \langle 0| + \frac{\Omega}{2} (e^{-i\omega_L t} |e\rangle_k \langle 1| + e^{-i\omega_P t} |e\rangle_k \langle 0|) - \text{H.c.} \right] + \hbar \sum_k [\omega_1 |1\rangle_k \langle 1| + \omega_E |e\rangle_k \langle e| + \omega_0 |0\rangle_k \langle 0|], \quad (A1)$$

where  $g$  denotes the coupling strength of an atom to the cavity mode, while  $\Omega$  denotes the coupling strengths of an atom to both laser fields.

We assume that  $\omega_C = \omega_P$  and switch to the interaction picture using the unitary transformation

$$U_1 = e^{-it[\sum(\omega_1|1\rangle_k\langle 1| + \omega_E|e\rangle_k\langle e| + (\omega_L + \omega_1 - \omega_P)|0\rangle_k\langle 0|) + \omega_P a^\dagger a]}.$$

We assume, moreover, that  $\Delta_L = \Delta_C \equiv -\Delta$ , where the notation  $\Delta_L \equiv (\omega_E - \omega_1) - \omega_L$  and  $\Delta_C \equiv (\omega_E - \omega_0) - \omega_C$  has been introduced. In the above interaction picture, therefore, Hamiltonian (A1) takes the following form:

$$H_2 = -i\hbar \sum_k \left[ \frac{g}{2} e^{-i\Delta t} a |e\rangle_k \langle 0| + \frac{\Omega}{2} e^{-i\Delta t} (|e\rangle_k \langle 1| + |e\rangle_k \langle 0|) - \text{H.c.} \right]. \quad (A2)$$

We require that  $\Delta$  is sufficiently far detuned, such that no atomic  $|e\rangle \leftrightarrow |0\rangle$  and  $|e\rangle \leftrightarrow |1\rangle$  transitions can occur. We expand the evolution governed by Hamiltonian (A2) in series and keep the terms up to the second order,

$$U_2 \cong \mathbb{1} - \frac{i}{\hbar} \int_0^t H_2 dt' - \frac{1}{\hbar^2} \int_0^t \left( H_2 \int_0^{t'} H_2 dt'' \right) dt'.$$

By performing integration and retaining only linear-in-time contributions, we express this evolution in the form

$$U_2 \cong \mathbb{I} - \frac{i}{\hbar} H_3 t \cong \exp \left[ -\frac{i}{\hbar} H_3 t \right], \quad (\text{A3})$$

where the effective Hamiltonian is given by

$$H_3 = \frac{\hbar \Omega}{4\Delta} \sum_k [\Omega |1\rangle_k \langle 0| + g |1\rangle_k \langle 0| a + \text{H.c.}], \quad (\text{A4})$$

after removing the constant contributions.

At this stage, we switch from the atomic basis  $\{|0\rangle, |1\rangle\}$  to the basis  $\{|+\rangle, |-\rangle\}$ , where

$$|+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle), \quad |-\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle). \quad (\text{A5})$$

In this basis, Hamiltonian (A4) takes the form

$$H_4 = \frac{\hbar \Omega}{8\Delta} \sum_k [2\Omega S_k^Z + g(S_k^Z(a + a^\dagger) + (S_k^\dagger - S_k)(a - a^\dagger))], \quad (\text{A6})$$

where  $S_k \equiv |-\rangle_k \langle +|$  and  $S_k^Z \equiv |+\rangle_k \langle +| - |-\rangle_k \langle -|$ , and where we have removed any constant contributions. We switch once more to the interaction picture with respect to the first term in (A6). In this interaction picture, we obtain

$$H_5 = \hbar \frac{g \Omega}{8\Delta} \sum_k [S_k^Z(a + a^\dagger) + (S_k^\dagger e^{i\frac{g\Omega}{2\Delta}t} - S_k e^{-i\frac{g\Omega}{2\Delta}t})(a - a^\dagger)]. \quad (\text{A7})$$

In the strong-driving regime, i.e., for  $\Omega \gg \{g, \Delta\}$ , we eliminate the last (fast-oscillating) term using the same arguments as for the rotating-wave approximation. Hamiltonian (A7), therefore, reduces to

$$H_6 = \hbar \frac{g \Omega}{8\Delta} (a + a^\dagger) (\sigma_1^X + \sigma_2^X), \quad (\text{A8})$$

where we have used the identity  $S_k^Z = \sigma_k^X$ . The resulting Hamiltonian, (A8), coincides with Hamiltonian (14) under the notation  $J_2 \equiv g \Omega / (4\Delta)$ .

## APPENDIX B: STEADY-STATE SOLUTION OF EQ. (13)

In this Appendix, we show that mixed state (12) is conditionally generated by means of evolution (13), which takes place simultaneously in both repeater nodes ( $A$  and  $B$ ) with the initial state

$$\tilde{\rho}_{AB}^0 = \rho_f^1 \otimes \rho_f^2 \otimes |\bar{0}_A, \bar{0}_B\rangle \langle \bar{0}_A, \bar{0}_B| \quad (\text{B1})$$

in the steady-state regime. In Sec. III, we considered expression (15) based on the exponent  $e^{\mathcal{L}t}$ , which is difficult to evaluate. Here we apply sequentially the steady-state solution of Eq. (13) to the coupled (atom-atom-cavity) system in node  $A$  and, afterwards, to the coupled system in node  $B$ .

In order to proceed, we consider first the solution of (13) for an initially empty cavity  $C_1$  and a general mixed state of two atoms in node  $A$  ( $i, j = 1, \dots, 4$ ),

$$\rho_A^0 = \sum_{i,j} (\rho_A^0)_{ij} |\mathbf{u}_i^A\rangle \langle \mathbf{u}_j^A| \otimes |\bar{0}_A\rangle \langle \bar{0}_A|, \quad (\text{B2})$$

where we have switched to the atomic basis, (A5), such that  $|\mathbf{u}_1^A\rangle = |_{+1A}, +2A\rangle$ ,  $|\mathbf{u}_2^A\rangle = |_{-1A}, -2A\rangle$ ,  $|\mathbf{u}_3^A\rangle =$

$|_{+1A}, -2A\rangle$ , and  $|\mathbf{u}_4^A\rangle = |_{-1A}, +2A\rangle$  together form an orthogonal basis. To our best knowledge, the master equation, (13), was solved in Refs. [39] and [40] only for an initial pure state of atoms, which is not appropriate for use in our case. We therefore (re)solved this master equation for an initial mixed state of two atoms, (B2). Assuming the strong atom-cavity coupling  $J_2 \gg \kappa$ , the solution we found in the steady-state regime  $\kappa t \gg 1$  can be expressed as

$$\rho_A^{ss} = \sum_{i,j} (\rho_A^0)_{ij} \lambda_{ij} |\mathbf{u}_i^A\rangle \langle \mathbf{u}_j^A| \otimes |_{-u_i} \alpha_A^{ss}\rangle \langle_{-u_j} \alpha_A^{ss}|, \quad (\text{B3})$$

where  $\alpha_A^{ss} = 2i J_2 / \kappa$  is the amplitude of the coherent state,  $\lambda_{ij} = \delta_{(u_i - u_j), 0}$ , with  $u_1 = 1$ ,  $u_2 = -1$ , and  $u_3 = u_4 = 0$ .

Recall that the evolution, (13), takes place simultaneously in both repeater nodes,  $A$  and  $B$ , with initial state (B1), which we cast in the form

$$\begin{aligned} \tilde{\rho}_{AB}^0 &= \sum_{i,j,k,l} (\tilde{\rho}_{AB}^0)_{ijkl} |\mathbf{u}_i^A, \mathbf{u}_k^B\rangle \langle \mathbf{u}_j^A, \mathbf{u}_l^B| \otimes |\bar{0}_A, \bar{0}_B\rangle \langle \bar{0}_A, \bar{0}_B| \\ &\equiv \sum_{i,j} (\rho_A^0)_{ij} |\mathbf{u}_i^A\rangle \langle \mathbf{u}_j^A| \otimes |\bar{0}_A\rangle \langle \bar{0}_A|, \end{aligned} \quad (\text{B4})$$

where we have introduced the matrices

$$(\tilde{\rho}_{AB}^0)_{ijkl} = \langle \mathbf{u}_i^A, \mathbf{u}_k^B | \rho_f^1 \otimes \rho_f^2 | \mathbf{u}_j^A, \mathbf{u}_l^B \rangle, \langle \mathbf{u}_i^A, \mathbf{u}_k^B |, \quad (\text{B5})$$

$$(\rho_A^0)_{ij} = \sum_{k,l} (\tilde{\rho}_{AB}^0)_{ijkl} |\mathbf{u}_k^B\rangle \langle \mathbf{u}_l^B| \otimes |\bar{0}_B\rangle \langle \bar{0}_B|. \quad (\text{B6})$$

By identifying expressions (B2) and (B4), we conclude that the density operator, (B3), gives the steady-state solution of (13) obtained in node  $A$  for initial state (B1). Conditioned upon the no-photon measurement of the leaked field from cavity  $C_1$ , this steady-state solution reduces to

$$\begin{aligned} \langle \bar{0}_A | \tilde{\rho}_{AB}^{ss} | \bar{0}_A \rangle &= \sum_{i,j} (\rho_A^0)_{ij} \lambda_{ij} e^{-|\alpha_A^{ss}|^2 u_i u_j} |\mathbf{u}_i^A\rangle \langle \mathbf{u}_j^A| \\ &\equiv \sum_{k,l} (\rho_B^0)_{kl} |\mathbf{u}_k^B\rangle \langle \mathbf{u}_l^B| \otimes |\bar{0}_B\rangle \langle \bar{0}_B|, \end{aligned} \quad (\text{B7})$$

where

$$(\rho_B^0)_{kl} \equiv \sum_{i,j} (\tilde{\rho}_{AB}^0)_{ijkl} \lambda_{ij} e^{-|\alpha_A^{ss}|^2 u_i u_j} |\mathbf{u}_i^A\rangle \langle \mathbf{u}_j^A|. \quad (\text{B8})$$

Similarly to node  $A$ , the density operator

$$\rho_B^{ss} = \sum_{i,j} (\rho_B^0)_{ij} \lambda_{ij} |\mathbf{u}_i^B\rangle \langle \mathbf{u}_j^B| \otimes |_{-u_i} \alpha_B^{ss}\rangle \langle_{-u_j} \alpha_B^{ss}| \quad (\text{B9})$$

gives the steady-state solution of (13) for node  $B$  with initial state (B8). Conditioned upon the no-photon measurement of the leaked field from cavity  $C_3$ , this solution reduces to

$$\begin{aligned} \langle \bar{0}_A, \bar{0}_B | \tilde{\rho}_{AB}^{ss} | \bar{0}_A, \bar{0}_B \rangle &= \sum_{k,l} \lambda_{kl} (\rho_B^0)_{kl} e^{-|\alpha_B^{ss}|^2 u_k u_l} |\mathbf{u}_k^B\rangle \langle \mathbf{u}_l^B| \\ &= \sum_{i,j,k,l} (\tilde{\rho}^0 \lambda)_{ijkl} e^{-|\alpha^{ss}|^2 \theta_{ijkl}} |\mathbf{u}_i^A, \mathbf{u}_k^B\rangle \langle \mathbf{u}_j^A, \mathbf{u}_l^B|, \end{aligned} \quad (\text{B10})$$

where the notation  $\theta_{ijkl} = u_i u_j + u_k u_l$ ,  $(\tilde{\rho}^0 \lambda)_{ijkl} = (\tilde{\rho}_{AB}^0)_{ijkl} \lambda_{ij} \lambda_{kl}$ , and  $\alpha_A^{ss} = \alpha_B^{ss} \equiv \alpha^{ss}$  was introduced. The atom-cavity strong coupling ensures that  $|\alpha^{ss}|^2 \gg 1$ . To a good approximation, this observation implies that the

exponent in (B10) vanishes for  $\theta_{ijkl} \neq 0$ :

$$\langle \bar{0}_A, \bar{0}_B | \bar{\rho}_{AB}^{ss} | \bar{0}_A, \bar{0}_B \rangle = \sum_{i,j,k,l}^{\theta_{ijkl}=0} (\bar{\rho}^0 \lambda)_{ijkl} | \mathbf{u}_i^A, \mathbf{u}_k^B \rangle \langle \mathbf{u}_j^A, \mathbf{u}_l^B |. \quad (\text{B11})$$

$$\begin{aligned} \rho_f^{1,2} = & \frac{f^2}{2-4f+4f^2} (|\phi_{1A,1B}^+, \phi_{2A,2B}^+\rangle \langle \phi_{1A,1B}^+, \phi_{2A,2B}^+| + |\psi_{1A,1B}^+, \psi_{2A,2B}^+\rangle \langle \psi_{1A,1B}^+, \psi_{2A,2B}^+| \\ & - |\psi_{1A,1B}^+, \psi_{2A,2B}^+\rangle \langle \phi_{1A,1B}^+, \phi_{2A,2B}^+| - |\phi_{1A,1B}^+, \phi_{2A,2B}^+\rangle \langle \psi_{1A,1B}^+, \psi_{2A,2B}^+|) \\ & + \frac{(f-1)^2}{2-4f+4f^2} (|\phi_{1A,1B}^-, \phi_{2A,2B}^-\rangle \langle \phi_{1A,1B}^-, \phi_{2A,2B}^-| + |\psi_{1A,1B}^-, \psi_{2A,2B}^-\rangle \langle \psi_{1A,1B}^-, \psi_{2A,2B}^-| \\ & - |\psi_{1A,1B}^-, \psi_{2A,2B}^-\rangle \langle \phi_{1A,1B}^-, \phi_{2A,2B}^-| - |\phi_{1A,1B}^-, \phi_{2A,2B}^-\rangle \langle \psi_{1A,1B}^-, \psi_{2A,2B}^-|), \end{aligned} \quad (\text{B12})$$

which describes a four-qubit entangled state and coincides with state (12).

### APPENDIX C: DERIVATION OF HAMILTONIAN (17)

In this Appendix, we show that Hamiltonian (17) is produced deterministically in our setup. Specifically, three (three-level) atoms are subject to the field of (initially empty) cavity  $C_2$  ( $C_4$ ) and the field of laser beam  $L_1$  ( $L_4$ ) simultaneously as displayed in Fig. 3(c). The evolution of this coupled atom-cavity-laser system is governed by the Hamiltonian ( $k = 1, 2, 3$ )

$$\begin{aligned} H_1 = & \hbar \omega_C a^\dagger a \\ & - i \hbar \sum_k \left[ \frac{g}{2} a |e\rangle_k \langle 0| + \frac{\Omega}{2} e^{-i\omega_L t} |e\rangle_k \langle 1| - \text{H.c.} \right] \\ & + \hbar \sum_k [\omega_1 |1\rangle_k \langle 1| + \omega_E |e\rangle_k \langle e| + \omega_0 |0\rangle_k \langle 0|], \end{aligned} \quad (\text{C1})$$

where  $g$  denotes the coupling strength of an atom to the cavity mode, while  $\Omega$  denotes the coupling strength of an atom to the laser field.

We switch to the interaction picture using the unitary transformation

$$U_1 = e^{-it[\sum(\omega_1|1\rangle_k \langle 1| + \omega_E |e\rangle_k \langle e| + \omega_0 |0\rangle_k \langle 0|) + (\omega_1 + \omega_L - \omega_0) a^\dagger a]}.$$

In this picture, Hamiltonian (C1) takes the form

$$\begin{aligned} H_2 = & \hbar (\Delta_L - \Delta_C) a^\dagger a - i \hbar \sum_k \left[ \frac{g}{2} a e^{i\Delta_L t} |e\rangle_k \langle 0| \right. \\ & \left. + \frac{\Omega}{2} e^{i\Delta_L t} |e\rangle_k \langle 1| - \text{H.c.} \right], \end{aligned} \quad (\text{C2})$$

Owing to the explicit form of (B1), we have routinely computed the matrix elements  $(\bar{\rho}_{AB}^0)_{ijkl}$ , which, however, are rather bulky to display here. With the help of these matrix elements, we obtained the normalized density operator, (B11), in the form

where the notation  $\Delta_L \equiv (\omega_E - \omega_1) - \omega_L$  and  $\Delta_C \equiv (\omega_E - \omega_0) - \omega_C$  has been introduced.

We require that  $\Delta_L$  and  $\Delta_C$  are sufficiently far detuned, such that no atomic  $|e\rangle \leftrightarrow |0\rangle$  and  $|e\rangle \leftrightarrow |1\rangle$  transitions can occur. We expand the evolution governed by Hamiltonian (C2) in series up to the second order. By performing integration and retaining only linear-in-time contributions, we express this evolution in the form (A3), where the effective Hamiltonian is given by (we assume that the cavity field is initially in the vacuum state)

$$H_3 = \hbar \Delta a^\dagger a + \hbar \frac{g\Omega}{4\Delta_L} \sum_k [a |1\rangle_k \langle 0| + \text{H.c.}], \quad (\text{C3})$$

where  $\Delta \equiv \Delta_L - \Delta_C$ . We switch one more time to the interaction picture with respect to the first term in (C3). In this interaction picture, the Hamiltonian takes the form

$$H_4 = \hbar \frac{g\Omega}{4\Delta_L} \sum_k [a e^{-i\Delta t} |1\rangle_k \langle 0| + \text{H.c.}]. \quad (\text{C4})$$

We require, finally, that  $\Delta$  is sufficiently far detuned. As above, we expand again the evolution governed by Hamiltonian (C4) in series up to the second order and retain only linear-in-time contributions after the integration. This leads to the effective Hamiltonian ( $i, j = 1, 2, 3$ )

$$H_5 = \frac{\hbar g^2 \Omega^2}{16 \Delta_L^2 \Delta} \left[ \sum_{i,j}^{i \neq j} |0_i, 1_j\rangle \langle 1_i, 0_j| + \sum_k |1\rangle_k \langle 1| \right]. \quad (\text{C5})$$

Since the second term in this Hamiltonian commutes with the first term, we eliminate the second term by means of an appropriate interaction picture. The resulting Hamiltonian, i.e., the first term in (C5), coincides with Hamiltonian (17) under the notation  $J_3 \equiv g^2 \Omega^2 / (16 \Delta_L^2 \Delta)$ .

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