

Informational completeness of continuous-variable measurements

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We justify that homodyne tomography turns out to be informationally complete when the number of independent quadrature measurements is equal to the dimension of the density matrix in the Fock representation. Using this as our thread, we examine the completeness of other schemes when continuous-variable observations are truncated to discrete finite-dimensional subspaces.

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I. INTRODUCTION

Measurement lies at the very heart of quantum information. It is an indispensable tool for identifying how well one can prepare or can create a particular quantum state. Such an assessment is achieved by performing suitable measurements on a sequence of identically prepared systems. A set of measurements, whose outcome probabilities are sufficient to determine an arbitrary quantum state, is called informationally complete (IC) [1], whereas, the process of reconstructing the state itself is broadly called quantum tomography [2].

Investigations on IC measurements have been extensively carried out for discrete Hilbert spaces [3]. For continuous variables, the archetypical example is homodyne tomography, first suggested in the seminal paper of Vogel and Risken [4] and implemented by Smithey *et al.* [5]. Still, the reconstruction of the measured state is inexorably accomplished in some finite subspace chosen by an educated guess.

In many experimental situations, one has some additional prior information that can be efficiently used for reducing the number of relevant unknown parameters, making the reconstruction much simpler. For example, the knowledge about the Gaussian character of the state reduces the problem to the evaluation of its 2×2 covariance matrix, which is formally analogous to the estimation of a spin $1/2$ [6]. In the same vein, if the system can be represented (say, in the Fock basis) by a finite density matrix, the number of quadratures needed for an accurate reconstruction in homodyne tomography is precisely equal to the dimension of that matrix [7]. Note, in passing, that this is indeed very closely related to the quantum version of the sampling theorem [8].

With the advent of powerful nonlinear estimation techniques that offer amazing performance and robustness in most applications, matching the signal space used for coding the information in an experimental setup is more relevant than ever. This is, in fact, a dual problem: One is interested not only in what can be reconstructed from a particular scheme, but also in identifying the best option for a given signal. Any measurement is represented by an operator, and what matters for tomography is the number of linearly independent operators required for the reconstruction. Keeping in mind the lesson from homodyne detection, the question of how to select those operators appears far from trivial.

In this paper, we examine the informational completeness of various schemes. We derive a surprisingly simple connection

between the different measurement settings and the number of linearly independent elements induced by them. In addition, the analysis displays the information provided by each acquired piece. Such knowledge is certainly of great interest both for the optimization of feasible tomographic protocols and for the design of new ones.

II. PRELIMINARIES

A d -dimensional quantum system is represented by a positive semidefinite $d \times d$ density matrix that requires $d^2 - 1$ independent real numbers for its specification. A von Neumann measurement fixes at most $d - 1$ real parameters, so $d + 1$ tests have to be performed to reconstruct the state. This means that $d^2 + d$ histograms have to be recorded: The approach is, thus, suboptimal for this number is higher than the number of parameters in the density matrix. The von Neumann strategy can be further optimized regarding this redundancy when the bases in which the measurements are performed are mutually unbiased [9].

Besides, it is known that more general measurements exist. Such generalized measurements appeared as positive operator-valued measures (POVMs) in the quantum theory of detection in the early 1970s. It was soon realized that, for numerous reasons, they provided greater efficiency [10–14].

In short, a POVM is described by a set of linear operators $\{\hat{E}_n\}$ furnishing the correct probabilities in any experiment (we assume, for simplicity, discrete outcomes) through the Born rule,

$$p_n = \text{Tr}(\hat{\rho}\hat{E}_n) \quad (1)$$

for any state described by the density operator $\hat{\rho}$. Here, Tr denotes the trace of a complex matrix. Compatibility with the properties of ordinary probability imposes the requirements of positivity, Hermiticity, and resolution of the identity,

$$\hat{E}_n \geq 0, \quad \hat{E}_n = \hat{E}_n^\dagger, \quad \sum_n \hat{E}_n = \hat{1}. \quad (2)$$

The case of the projective von Neumann test is recovered when all the operators in set $\{\hat{E}_n\}$ commute.

A quite general way of implementing a POVM is to properly combine the system to be tested with an auxiliary system whose initial state is known and then to perform a von Neumann test on the combined system [15]. When the coupling with the auxiliary system and the von Neumann measurement

are judiciously chosen, one can get IC POVMs, which, in addition, are optimal in the sense that minimize the number of independent detections that must be obtained during the tomographic process [16]. Below, we will exploit a similar idea to perform IC measurements from incomplete ones by cutting the dimensionality of the signal space [17].

III. QUADRATURE MEASUREMENT

To get the backbone of our proposal, we start with states that can be written as a finite sum in the Fock basis. We wish to characterize a d -dimensional subspace $\hat{1}_d = \sum_{n=0}^{d-1} |n\rangle\langle n|$ from projections onto the quadrature eigenstates $|x_\theta\rangle$, which can be expressed as

$$\langle n|x_\theta\rangle \propto H_n(x)e^{-x^2/2}e^{in\theta}. \quad (3)$$

Our task is to find out how many different phase settings θ (also referred to as cuts) are needed to make the scheme complete. It is clear that the complex argument of $\langle n|x_\theta\rangle$ is independent of x , so one quadrature can never generate a complete POVM. For example, the three states $(|0\rangle\langle 0| + |1\rangle\langle 1|)/2$ and $(|0\rangle \pm i|1\rangle)/\sqrt{2}$ have the same position distribution, and hence, the position measurement ($\theta = 0$) cannot discriminate among them. Obviously, since $\langle n|x_{\theta=0}\rangle$ is real for all x , such projections fail to generate the σ_y component. This difficulty persists in higher dimensions.

Fortunately enough, this topic can be fully analyzed in a closed form. Indeed, the probability distribution of the homodyne detection can be written, up to a normalization factor, as

$$p(x, \theta) \propto \sum_{k, \ell=0}^{d-1} \rho_{k\ell} H_k(x) H_\ell(x) e^{-x^2} e^{i(k-\ell)\theta}, \quad (4)$$

where x is the measured amplitude. Let us take all matrix elements $\rho_{k\ell}$ to be nonzero and calculate the number N of linearly independent combinations of the density matrix (4) that can be generated from different choices of x and θ . For every polynomial term with a maximum power x^n , we have the following different combinations of Hermite polynomials [omitting, for brevity, the associated factor $e^{i(k-\ell)\theta}$]:

$$\begin{aligned} x^0: & H_0 H_0, \\ x^1: & H_0 H_1, H_1 H_0, \\ x^2: & H_0 H_2, H_2 H_0, H_1 H_1, \\ x^3: & H_0 H_3, H_3 H_0, H_1 H_2, H_2 H_1, \\ & \dots \\ x^{2d-4}: & H_{d-2} H_{d-2}, H_{d-1} H_{d-3}, H_{d-3} H_{d-1}, \\ x^{2d-3}: & H_{d-2} H_{d-1}, H_{d-1} H_{d-2}, \\ x^{2d-2}: & H_{d-1} H_{d-1}. \end{aligned} \quad (5)$$

For a single quadrature θ_1 , the number N_1 is determined by the term $H_{d-1} H_{d-1}$ with the highest polynomial power and is equal to $N_1 = 2d - 1$ (the total number of first elements $H_k H_\ell$ in all rows of the above array). Since we have *all* the polynomial powers from 0 up to $2d - 2$, there are no more linearly independent elements. The second value θ_2 additionally gives us $N_2 = 2d - 3$ linearly independent combinations of $\rho_{k\ell}$ (the total number of second elements $H_k H_\ell$ in all rows), the third value θ_3 gives $N_3 = 2d - 5$, and so on.

TABLE I. The number of linearly independent POVM elements induced by m different quadrature measurements in a d -dimensional Fock subspace. The bold font indicates IC POVMs.

d	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$
2	3	4	4	4	4	4
3	5	8	9	9	9	9
4	7	12	15	16	16	16
5	9	16	21	24	25	25
6	11	20	27	32	35	36
7	13	24	33	40	45	48
8	15	28	39	48	55	60

An alternative way to get the same result is by noticing that, after measuring the first quadrature, $2d - 1$ linear combinations of $\rho_{k\ell}$ are fixed, and there remain $d^2 - 1 - (2d - 1) = (d - 1)^2 - 1$ free parameters entering the probability (4). This effectively lowers the dimension of the problem by one. Thus, $N_2 = 2(d - 1) - 1 = 2d - 3$, $N_3 = 2(d - 2) - 1 = 2d - 5$, etc. The procedure can be repeated until the number of cuts m equals d when N reaches its maximum value of d^2 . One can draw an important conclusion from this discussion: By measuring each new quadrature, the prior information is updated in such a way that the dimension of the unknown part of the system becomes one unity less.

In consequence, m different phase settings induce $N = \sum_{k=1}^m N_k$ linearly independent POVM elements; that is,

$$N = \begin{cases} m(2d - m), & m < d, \\ d^2, & m \geq d, \end{cases} \quad (6)$$

which can be conveniently summarized as in Table I. This relation is the main achievement of this paper as it provides a full description of the IC for the problem at hand.

As we can see, to fully characterize a d -dimensional Fock subspace, one needs d different quadratures. For a fixed m , the number N scales quadratically with d until $d = m$ due to the completeness of the generated POVMs. For $d > m$, the size of the POVM grows only linearly, hence, in the asymptotic case of sufficiently large dimensionality, the measurement becomes more and more incomplete.

In this respect, we wish to highlight that our results do not rely on a particular reconstruction scheme as in Ref. [7], but rather, they emerge from the properties of the measurement itself and pervade all the continuous-variable detection schemes.

IV. PHYSICAL DISCUSSION

The analysis presented so far has significant consequences; some of which are explored here. Notice that the IC quadrature operators with d phases approximate the structure of mutually unbiased measurements. Without the Hilbert-space truncation, the projectors with the same phase setting are orthogonal, whereas, for different phases, they overlap. These complementary properties become modified in a finite-dimensional subspace, although they approach their continuous-variable counterpart as long as the cutoff d is sufficiently large.

Loosely speaking, the orthogonal measurements always bring a completely new bit of information, whereas, the

overlapping ones verify the previously acquired information and add a new piece thereof. This is the essence of the tomography where afterwards, all the space is covered by an “operator” mesh. Since the homodyne detection on finite-dimensional subspaces is feasible, this gives an excellent opportunity to investigate the information gained in the process of the measurement.

We emphasize that even a single quadrature actually generates IC POVMs, provided that the signal space is properly matched to such a detection. For example, Table I shows that, in a five-dimensional Fock subspace, a single quadrature generates nine independent POVM elements. These are IC in a suitably chosen three-dimensional subspace. Put differently, a single quadrature provides characterization of a qutrit in a five-dimensional space or two entangled qubits in a nine-dimensional space. Similarly, all Pauli matrices are induced by just two quadratures (say, x and p). Such kinds of prior knowledge about the signal space can significantly reduce the number of measurements required for the complete state reconstruction. A particularly natural situation is the case of symmetric and antisymmetric subspaces [18]: A complete characterization of an arbitrary two-qutrit state requires nine different quadratures, whereas, an antisymmetric two-qutrit state can be inferred with just one quadrature measurement.

The previous results can be simplified in some cases where prior information is available. For instance, some elements of the density matrix can be known to vanish, and we can reach the complete POVM with less settings. Let us consider an analog of a squeezed vacuum of the form

$$|\Psi\rangle = c_0|0\rangle + c_1|4\rangle + c_2|8\rangle + \cdots + c_k|4k\rangle + \cdots \quad (7)$$

The coefficients c_k gradually decrease with k , which makes setting them to zero at some threshold value $c_k = 0$ for $k \geq d$ possible. The resulting state lives in a d -dimensional subspace, so one may think that $m = d$ phase settings would be needed to characterize that subspace. Surprisingly, the prior information about photons being generated in quadruples modifies the structure of the induced POVM, and the number of linearly independent measurements is found to be different from that calculated before. Let us take a look at the case $d = 3$. The coefficients in Eq. (4) contain polynomial powers up to $4(d - 1) \times 4(d - 1) = 16$,

$$\begin{aligned} x^0: & H_0H_0, \\ x^4: & H_0H_4, H_4H_0, \\ x^8: & H_4H_4, H_0H_8, H_8H_0 \\ x^{12}: & H_4H_8, H_8H_4, \\ x^{16}: & H_8H_8. \end{aligned} \quad (8)$$

By explicitly writing the power expansion of the coefficients $H_0H_8 = 256x^8 - 3584x^6 + 13440x^4 - 13440x^2 + 1680$ and $H_4H_4 = (16x^4 - 48x^2 + 12)^2$, we see that they are linearly independent. Indeed, H_0H_8 cannot be written as a linear combination of H_0H_0 , H_0H_4 , and H_4H_4 due to the lack of *all* needed polynomial powers in contrast to the general case. Hence, the number of linearly independent POVM elements is much higher now: Instead of $N_1 = 2d - 1$, $N_2 = 2d - 3$, $N_3 = 2d - 5$, etc., we have the modified expressions $N_1^* = N_1 + N_3 + N_5 + \cdots$, $N_2^* = N_2$, $N_3^* = N_4$, $N_4^* = N_6$, The number of different phase settings for the scheme to be complete is $m^* = [d/2] + 1$,

where $[]$ denotes the integer part. Hence, the family of states (7) can be characterized with roughly half as many quadratures that would be required for a truncated Fock subspace of the same dimension.

As our last contention, we insist that proper matching conditions are essential [19]. Particularly, without a deeper understanding, too dense binning of quadratures may be useless, whereas, the quadrature phase may easily become undersampled. In other words, the number of quadrature bins multiplied by the number of phase settings does not, in general, equal the number of free parameters in the homodyne detection.

V. PHOTON-NUMBER RESOLVING DETECTION

To obtain our main result (6), we have essentially relied on the polynomial representation of the quadrature (3), namely, that the probability distribution (4) can be written with the help of polynomials up to a certain power, defined by dimensionality of the quantum state. This property also holds for many other situations; indeed, almost any physical measurement satisfies this property [12]. For example, an approximation of coherent state $|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \alpha^n / \sqrt{n!} |n\rangle$ by a finite sum, neglecting tiny contributions of sufficiently high n , is essentially the same situation as above: Linearly independent POVM elements can be obtained from projections of a finite-dimensional state onto a properly matched basis.

In the simplest scenario, the strategy can be photon-number-resolving detection. In fact, instead of (4), we now have

$$p(\alpha, n) = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}. \quad (9)$$

An obvious drawback of this choice is phase insensitivity since coherent states with the same amplitude and different phases give identical statistics. To avoid this degeneracy, the photon-number detection can be preceded by a suitable displacement [20] or even more complicated “steering” operation. A detailed examination of this point is out of the scope of the present paper.

VI. CONCLUDING REMARKS

We have addressed the fundamental question of how many independent measurements can be generated by a given experimental setup. The analysis clarifies the issue of necessary and sufficient complexity for tomographical schemes. It turns out that the effective size of quantum tomography for any finite-dimensional system can be related to the number of different quadratures observed, which illustrates an exciting link between discrete and continuous-variable systems. We stress that such a reconstruction cannot be accomplished without the appropriate delimitation of the reconstruction space since the space touched by the observation then exceeds the amount of available data.

Our central result indicates that for a d -dimensional system a single quadrature can be discretized up to $2d - 1$ bins. This appears to be very similar to the classical Nyquist frequency in the Kotelnikov-Shannon coding theorem. By finding a simple analytical rule, we have provided a full characterization of informational completeness for a wide class

of continuous-variable measurements and made a step towards a better understanding of quantum resources in quantum optics and quantum information processing.

The analysis is applicable for arbitrary dimensionality and, thus, serves as a good alternative to other traditional approaches based, e.g., on mutually unbiased bases and symmetric IC POVMs.

Finally, note that only data-acquisition issues have been addressed here: The subsequent reconstruction must be capable of dealing with generic nonequivalent detections. The mathematical method best suited for this purpose is the maximum likelihood estimation; however, even in this case, the continuous data are discretized due to the very nature

of the measurement and the finite amount of memory and computational time available. We believe that our theory can be helpful in future optimizations of continuous-variable measurements.

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- [1] E. Prugovečki, *Int. J. Theor. Phys.* **16**, 321 (1977); P. Busch and P. J. Lahti, *Found. Phys.* **19**, 633 (1989).
- [2] *Quantum State Estimation*, edited by M. G. A. Paris and J. Řeháček, Lecture Notes in Physics Vol. 649 (Springer, Berlin, 2004).
- [3] G. M. D'Ariano, P. Perinotti, and M. F. Sacchi, *J. Opt. B* **6**, S487 (2004); S. T. Flammia, A. Silberfarb, and C. M. Caves, *Found. Phys.* **35**, 1985 (2005); S. Weigert, *Int. J. Mod. Phys. B* **20**, 1942 (2006); T. Durt, *Open Syst. Inf. Dyn.* **13**, 403 (2006).
- [4] K. Vogel and H. Risken, *Phys. Rev. A* **40**, 2847 (1989).
- [5] D. T. Smithey, M. Beck, M. G. Raymer, and A. Faridani, *Phys. Rev. Lett.* **70**, 1244 (1993).
- [6] J. Řeháček, S. Olivares, D. Mogilevtsev, Z. Hradil, M. G. A. Paris, S. Fornaro, V. D'Auria, A. Porzio, and S. Solimeno, *Phys. Rev. A* **79**, 032111 (2009).
- [7] U. Leonhardt and M. Munroe, *Phys. Rev. A* **54**, 3682 (1996); U. Leonhardt, *J. Mod. Opt.* **44**, 2271 (1997).
- [8] A. V. Oppenheim, A. S. Willsky, and S. Hamid, *Signals and Systems*, 2nd ed. (Prentice Hall, Upper Saddle River, NJ, 2010).
- [9] J. Schwinger, *Proc. Natl. Acad. Sci. USA* **46**, 570 (1960); W. K. Wootters and B. D. Fields, *Ann. Phys.* **191**, 363 (1989); A. B. Klimov, J. L. Romero, G. Björk, and L. L. Sánchez-Soto, *J. Phys. A: Math. Theor.* **40**, 3987 (2007); A. B. Klimov, D. Sych, L. L. Sánchez-Soto, and G. Leuchs, *Phys. Rev. A* **79**, 052101 (2009); T. Durt, B.-G. Englert, I. Bengtsson, and K. Życzkowski, *Int. J. Quantum Inf.* **8**, 533 (2010).
- [10] R. L. Stratonovich, *J. Stochastics* **1**, 87 (1973).
- [11] B. A. Grishanin, *Tekhn. Kibernetika* **11**, 127 (1973) [arXiv: [quant-ph/0301159v1](https://arxiv.org/abs/quant-ph/0301159v1)].
- [12] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, New York, 1976).
- [13] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory*, 2nd ed. (North-Holland, Amsterdam, 2003).
- [14] V. B. Braginski and F. Y. Khalili, *Quantum Measurement* (Oxford University Press, Oxford, 1992).
- [15] C. W. Helstrom, J. W. S. Liu, and J. P. Gordon, *Proc. IEEE* **58**, 1578 (1970).
- [16] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, *J. Math. Phys.* **45**, 2171 (2004); D. M. Appleby, S. T. Flammia, and C. A. Fuchs, *ibid.* **52**, 022202 (2011); R. Salazar, D. Goyeneche, A. Delgado, and C. Saavedra, *Phys. Lett. A* **376**, 325 (2012).
- [17] J. Řeháček, Z. Hradil, Z. Bouchal, R. Čelechovský, I. Rigas, and L. L. Sánchez-Soto, *Phys. Rev. Lett.* **103**, 250402 (2009).
- [18] D. Sych and G. Leuchs, *New J. Phys.* **11**, 013006 (2009); **11**, 113040 (2009).
- [19] A. Ourjoumtsev, R. Tualle-Brouri, J. Laurat, and P. Grangier, *Science* **312**, 83 (2006).
- [20] C. Wittmann, U. L. Andersen, M. Takeoka, D. Sych, and G. Leuchs, *Phys. Rev. Lett.* **104**, 100505 (2010); *Phys. Rev. A* **81**, 062338 (2010).