

Multipartite entanglement evolution under separable operations

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We study how multipartite entanglement evolves under the paradigm of separable operations, which include the local operations and classical communication as a special case. We prove that the average “decay” of entanglement induced by a separable operation is measure independent (among SL-invariant ones) and state independent: The ratio between the average output entanglement and the initial entanglement is solely a function of the separable operation, regardless of the input state and of the SL-invariant entanglement measure being used. We discuss the “disentangling power” of a quantum channel and show that it exhibits a similar state invariance as the average entanglement decay ratio. Our Rapid Communication significantly extends the bipartite results of Horodecki and Horodecki, *Quantum Inf. Comput.* **10**, 901 (2010), Konrad *et al.*, *Nat. Phys.* **4**, 99 (2008), and Tiersch *et al.*, *Phys. Rev. Lett.* **101**, 170502 (2008) as well as the multipartite one of Gour, *Phys. Rev. Lett.* **105**, 190504 (2010), all of the previous work being restricted to one-sided or particular noise models.

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Introduction. Entanglement is a key ingredient in many quantum information protocols, such as factoring [1], teleportation [2], quantum dense coding [3], or measurement-based quantum computation [4]. In all these schemes entanglement plays the role of a resource that is “consumed” during the actual implementation of the protocol. For example, in measurement-based quantum computation, a highly multipartite entangled state (the cluster state) is adaptively measured locally until the desired unitary evolution is achieved; the cluster state model is equivalent to the well-known circuit model [5], but, in contrast to the latter, the role of entanglement as a computational resource is now clearly visible.

Decoherence [6] constitutes a major impediment in the physical implementation of quantum information protocols. In general, entangled states are not resilient against noise, and this may first look like an impassible barrier for the construction of a working quantum computer. Quantum error correcting codes can make quantum computation possible provided the level of noise is below a certain threshold, but the current technology is far from achieving it. It is therefore of crucial importance to understand how entanglement behaves under decoherence. Previous work [7–10] addressed this important problem of entanglement evolution (decay), however, the noise model was restricted to one-sided channels; i.e., noise that acts nontrivially only on one subsystem, without affecting the other remaining parts of it.

In this Rapid Communication we investigate multipartite entanglement evolution under a far more general setup: the locally correlated noisy channels—technically separable operations—which includes the local operations and classical communication (LOCC) as a strict subset [11]. Separable operations (and automatically LOCC) constitute a much richer class and encompasses the previous investigated one-sided channels as well as the usual independent noise model assumed in the theory of quantum error correction. The study of

entanglement evolution under such a noise model is therefore deepening our understanding of the fascinating and not-at-all well understood subject of multipartite entanglement [12].

Our most surprising result is that entanglement evolution is *independent* of the initial state and the entanglement measure used to quantify its evolution, and depends solely on the intrinsic properties of the noise. We provide a closed-form formula for how entanglement decays which is straightforward to calculate given the Kraus operator-sum representation of the process.

Recent attempts to characterize entanglement evolution under locally correlated noise, not just one sided, were made in Ref. [13], however, the authors restricted their study to tripartite qubit entanglement and a particular form of LOCC protocols. Our result completely characterizes the average entanglement evolution under the very general class of separable operations and may be of interest from an experimental point of view.

Entanglement evolution. We consider a multipartite system of n qudits, described by a Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$, and let d_i be the dimension of the i th local Hilbert space. Denote by $\mathcal{B}(\mathcal{H})$ the set of all bounded operators (e.g., density matrices) acting on \mathcal{H} . We further define the special linear group $G \equiv \text{SL}(d_1, \mathbb{C}) \otimes \text{SL}(d_2, \mathbb{C}) \otimes \cdots \otimes \text{SL}(d_n, \mathbb{C})$, where $\text{SL}(d, \mathbb{C})$ is the group of $d \times d$ complex matrices of determinant 1.

Definition 1. An SL-invariant multipartite entanglement measure [10], $E_{\text{inv}}(\cdot)$, is a nonzero function initially defined from pure states in \mathcal{H} to non-negative real numbers then extended to mixed states satisfying the following:

- (i) it is SL invariant, i.e., $E_{\text{inv}}(g|\psi\rangle\langle\psi|g^\dagger) = E_{\text{inv}}(|\psi\rangle\langle\psi|)$, for all $g \in G$ and all $|\psi\rangle \in \mathcal{H}$;
- (ii) it is homogeneous of degree 1, i.e., $E_{\text{inv}}(r|\psi\rangle\langle\psi|) = r E_{\text{inv}}(|\psi\rangle\langle\psi|)$, for all non-negative r and all $|\psi\rangle \in \mathcal{H}$; and
- (iii) its convex roof extension to mixed states is given by

$$E_{\text{inv}}(\rho) := \min \sum_i p_i E_{\text{inv}}(|\psi_i\rangle\langle\psi_i|), \quad (1)$$

where the minimum is taken over all possible pure states ensembles that decompose ρ , i.e., $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$.

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The criteria above guarantee that E_{inv} is an entanglement monotone under LOCC [14]. However, we will explicitly show here that E_{inv} is also monotone under the more general class of separable operations (see the upper bound in Theorem 1). These measures capture genuine multipartite entanglement and are in general dimension dependent, i.e., may not exist for some choices of the local dimensions d_i . The measures are therefore dependent on the space the states are embedded in, as we will see below. An example is Wootters' concurrence of two qubits [15] and its higher dimension generalization called G concurrence [16]. In particular, these are the only SL-invariant measures for the bipartite case, with $d_1 = d_2$. No SL-invariant measure exists for $d_1 \neq d_2$, hence a simple embedding of a 2×2 state into a 2×3 state is not measurable. For multipartite systems there is more than one measure, and their construction is based on SL-invariant polynomials [14,17]. An SL-invariant polynomial $f(|\psi\rangle\langle\psi|)$ is a polynomial in the components of the state $|\psi\rangle$, invariant under the action of the group G , i.e., $f(g|\psi\rangle\langle\psi|g^\dagger) = f(|\psi\rangle\langle\psi|)$, for all $g \in G$. Then $E_{\text{inv}}(|\psi\rangle\langle\psi|) = |f_k(|\psi\rangle\langle\psi|)|^{1/k}$ is an example of an SL-invariant measure, where f_k is a homogeneous SL-invariant polynomial of degree k . In particular, for three qubits the square root of the 3-tangle [18] is the simplest multipartite SL-invariant measure.

In the next Lemma we prove that the convex roof extension on an SL-invariant measure satisfies the SL-invariance and homogeneity properties, similarly to the pure state case. This is the most important technical result of our paper and is crucial for the proof of our main result, Theorem 1. This important Lemma was overlooked by the community, a partial version being stated before in Ref. [7], with inequality instead of equality and valid only for bipartite separable operations.

Lemma 1. Any entanglement measure defined as a convex roof extension of an SL-invariant pure state entanglement measure remains SL invariant and homogeneous of degree 1, i.e.,

(i) $E_{\text{inv}}(g\rho g^\dagger) = E_{\text{inv}}(\rho)$, for all $g \in G$ and all $\rho \in \mathcal{B}(\mathcal{H})$ and

(ii) $E_{\text{inv}}(r\rho) = rE_{\text{inv}}(\rho)$, for all non-negative r and all $\rho \in \mathcal{B}(\mathcal{H})$.

Proof. Proof of (i): Let $g \in G$ and let $\{|\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|\}$ be an optimal un-normalized ensemble decomposition of ρ , i.e.,

$$E_{\text{inv}}(\rho) = \sum_i E_{\text{inv}}(|\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|). \quad (2)$$

Then $\{g|\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|g^\dagger\}$ must be some ensemble decomposition of the positive operator $g\rho g^\dagger$, in general not the optimal one. Hence

$$\begin{aligned} E_{\text{inv}}(g\rho g^\dagger) &\leq \sum_i E_{\text{inv}}(g|\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|g^\dagger) \\ &= \sum_i E_{\text{inv}}(|\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|) = E_{\text{inv}}(\rho). \end{aligned} \quad (3)$$

Let $\{|\tilde{\phi}_k\rangle\langle\tilde{\phi}_k|\}$ be an optimal un-normalized ensemble decomposition of $g\rho g^\dagger$, i.e.,

$$E_{\text{inv}}(g\rho g^\dagger) = \sum_k E_{\text{inv}}(|\tilde{\phi}_k\rangle\langle\tilde{\phi}_k|). \quad (4)$$

Note that $\{g^{-1}|\tilde{\phi}_k\rangle\langle\tilde{\phi}_k|g^{-1\dagger}\}$ is some ensemble decomposition of ρ , again not necessarily the optimal one, where g^{-1} is the inverse of g (the inverse is guaranteed to exist since g has determinant 1). Then

$$\begin{aligned} E_{\text{inv}}(\rho) &\leq \sum_k E_{\text{inv}}(g^{-1}|\tilde{\phi}_k\rangle\langle\tilde{\phi}_k|g^{-1\dagger}) \\ &= \sum_k E_{\text{inv}}(|\tilde{\phi}_k\rangle\langle\tilde{\phi}_k|) = E_{\text{inv}}(g\rho g^\dagger). \end{aligned} \quad (5)$$

Now part (i) follows from (3) and (5).

Proof of (ii): This follows at once, since if $\{|\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|\}$ is an optimal ensemble decomposition of a density operator ρ , then $\{r|\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|\}$ will automatically be an optimal decomposition of the operator $r\rho$. Then the result follows from the homogeneity of $E_{\text{inv}}(\cdot)$ for pure states. ■

Definition 2. Let Λ be a completely positive trace preserving (CPTP) multipartite separable operation acting on $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) \otimes \dots \otimes \mathcal{B}(\mathcal{H}_n)$, with operator-sum representation given by Kraus operators $\{K_m^{(1)} \otimes K_m^{(2)} \otimes \dots \otimes K_m^{(n)}\}_m$, with $\sum_m K_m^{(1)\dagger} K_m^{(1)} \otimes \dots \otimes K_m^{(n)\dagger} K_m^{(n)} = I \otimes \dots \otimes I$. We define the *entanglement resilience factor* (ERF) of Λ to be

$$\mathcal{F}(\Lambda) := \min_m \sum_m |\det(K_m^{(1)})|^{2/d_1} \dots |\det(K_m^{(n)})|^{2/d_n}, \quad (6)$$

where the minimum is taken over all separable Kraus decompositions $\{K_m^{(1)} \otimes K_m^{(2)} \otimes \dots \otimes K_m^{(n)}\}_m$ of Λ .

Lemma 2. $\mathcal{F}(\Lambda) \leq 1$, with equality only if Λ is a separable random unitary channel, i.e., all Kraus operators are proportional to unitaries.

Proof. From the closure condition we have

$$\sum_m K_m^{(1)\dagger} K_m^{(1)} \otimes \dots \otimes K_m^{(n)\dagger} K_m^{(n)} = I \otimes \dots \otimes I. \quad (7)$$

Taking the determinant on both sides and applying the Minkowski's inequality for a sum of positive (semidefinite) operators (see p. 47 of Ref. [19]) yields

$$\sum_m |\det(K_m^{(1)})|^{2/d_1} \dots |\det(K_m^{(n)})|^{2/d_n} \leq 1, \quad (8)$$

where we have used $\det(AA^\dagger) = |\det(A)|^2$ and $\det(A \otimes B) = \det(A)^b \det(B)^a$ for A and B square matrices of dimension $a \times a$ and $b \times b$, respectively. The result then follows at once from Eqs. (6) and (8). ■

The ERF was first introduced in Ref. [10] for the particular case of one-sided operations of the form $\Lambda^{(1)} \otimes I \otimes \dots \otimes I$, where its computation reduced to an optimization problem due to the unitary freedom in the Kraus representation of a quantum operation [5]. However, in our case, it is not at all clear whether a separable operation admits more than one separable representation (up to a relabeling of the Kraus operators), excluding the trivial cases where the operation is a tensor product of independent channels $\Lambda = \Lambda^{(1)} \otimes \dots \otimes \Lambda^{(n)}$. For this latter case

$$\mathcal{F}(\Lambda^{(1)} \otimes \dots \otimes \Lambda^{(n)}) \leq \prod_k \mathcal{F}(\Lambda^{(k)}), \quad (9)$$

since among all possible separable representations of Λ there are Kraus representations of the form $\{K_{j_1}^{(1)} \otimes \dots \otimes K_{j_n}^{(n)}\}_{j_1, \dots, j_n}$, with $\sum_{j_k} K_{j_k}^{(k)\dagger} K_{j_k}^{(k)} = I$ for any individual party

k , and this implies that the minimization (6) can split into n individual parts, which proves (9).

The following Theorem constitutes our central result.

Theorem 1 (Entanglement evolution). Let Λ be a multipartite separable operation with operator-sum representation given by Kraus operators $\{K_m^{(1)} \otimes K_m^{(2)} \otimes \dots \otimes K_m^{(n)}\}_m$, with $\sum_m K_m^{(1)\dagger} K_m^{(1)} \otimes \dots \otimes K_m^{(n)\dagger} K_m^{(n)} = I \otimes \dots \otimes I$, let $E_{\text{inv}}(\cdot)$ be an SL-invariant entanglement measure, and let $\rho \in \mathcal{B}(\mathcal{H})$ be an entangled input state with $E_{\text{inv}}(\rho) \neq 0$. Then the average output entanglement is independent of the input state and of the entanglement measure, and is given by

$$\frac{\sum_m p_m E_{\text{inv}}(\sigma_m)}{E_{\text{inv}}(\rho)} = \sum_m |\det(K_m^{(1)})|^{2/d_1} \dots |\det(K_m^{(n)})|^{2/d_n} \leq 1, \quad (10)$$

with equality if and only if Λ is a separable random unitary channel [20]. Here $p_m \sigma_m = (K_m^{(1)} \otimes \dots \otimes K_m^{(n)}) \rho (K_m^{(1)} \otimes \dots \otimes K_m^{(n)})^\dagger$ and $\sum_m p_m = 1$.

In particular, note that the upper bound in Eq. (10) explicitly shows that E_{inv} is an entanglement monotone under separable operations, as we claimed in the remarks following Definition 2.

Proof. We start by noting that

$$(K_m^{(1)} \otimes \dots \otimes K_m^{(n)}) \rho (K_m^{(1)} \otimes \dots \otimes K_m^{(n)})^\dagger = p_m \sigma_m. \quad (11)$$

For the Kraus operators that have nonvanishing determinant (i.e., are invertible), we have

$$\begin{aligned} & |\det(K_m^{(1)})|^{2/d_1} \dots |\det(K_m^{(n)})|^{2/d_n} \\ & \times \left[\frac{K_m^{(1)} \otimes \dots \otimes K_m^{(n)}}{(\det K_m^{(1)})^{1/d_1} \dots (\det K_m^{(n)})^{1/d_n}} \right] (\rho) \\ & \times \left[\frac{K_m^{(1)} \otimes \dots \otimes K_m^{(n)}}{(\det K_m^{(1)})^{1/d_1} \dots (\det K_m^{(n)})^{1/d_n}} \right]^\dagger = p_m \sigma_m. \quad (12) \end{aligned}$$

Note that the quantity in square brackets has a determinant of magnitude one. Use now the homogeneity and SL-invariance property of any convex roof SL-invariant measure (see Lemma 1) and apply E_{inv} to Eq. (12) to get

$$|\det(K_m^{(1)})|^{2/d_1} \dots |\det(K_m^{(n)})|^{2/d_n} E_{\text{inv}}(\rho) = p_m E_{\text{inv}}(\sigma_m). \quad (13)$$

The Kraus operators that have determinant zero produce states with $p_m E_{\text{inv}}(\sigma_m) = 0$, see Lemma 1 of Ref. [10], and hence (13) is generally valid for all Kraus operators. Now summing (13) over m yields the desired equality. The upper bound of 1 follows again from Minkowski's inequality for a sum of positive semidefinite operators (see the proof of Lemma 2). ■

Operationally Theorem 1 can be thought of in terms of a separable measurement repeated multiple times, each time the outcome m and the resulting state σ_m being recorded and at the end the output entanglement being averaged over all outcomes. Theorem 1 implies that the ratio between the average output entanglement and the initial entanglement can be computed *explicitly* as a function of the operator-sum representation of the separable operation and has the following properties: (i) it

is independent of the input state and (ii) it is independent of the entanglement measure, as long as the latter is SL invariant.

In Theorem 1 we considered only input states with nonvanishing entanglement, since whenever $E_{\text{inv}}(\rho) = 0$ the average output entanglement must also be zero, hence such nonentangled states ρ are not of interest.

The following Corollary then follows.

Corollary 1. The ERF is bounded by

$$\frac{E_{\text{inv}}[\Lambda(\rho)]}{E_{\text{inv}}(\rho)} \leq \mathcal{F}[\Lambda] \leq \frac{\sum_m p_m E_{\text{inv}}(\sigma_m)}{E_{\text{inv}}(\rho)}. \quad (14)$$

Proof. The inequality $\mathcal{F}[\Lambda] \leq [\sum_m p_m E_{\text{inv}}(\sigma_m)]/E_{\text{inv}}(\rho)$ follows from the definition of the ERF as a minimum over all Kraus decompositions, see (6), and the equality (13) proved above.

Finally, the inequality $E_{\text{inv}}(\Lambda(\rho))/E_{\text{inv}}(\rho) \leq \mathcal{F}[\Lambda]$ can be proved by noting that $E_{\text{inv}}(\Lambda(\rho))/E_{\text{inv}}(\rho)$ is independent of the Kraus representation of the channel. We can therefore choose the Kraus representation that achieves the minimum in Eq. (6), $\{\tilde{K}_m^{(1)} \otimes \tilde{K}_m^{(2)} \otimes \dots \otimes \tilde{K}_m^{(n)}\}_m$, which yields

$$\mathcal{F}[\Lambda] = \sum_m |\det(\tilde{K}_m^{(1)})|^{2/d_1} \dots |\det(\tilde{K}_m^{(n)})|^{2/d_n}. \quad (15)$$

But

$$\sum_m |\det(\tilde{K}_m^{(1)})|^{2/d_1} \dots |\det(\tilde{K}_m^{(n)})|^{2/d_n} = \frac{\sum_m p_m E_{\text{inv}}(\sigma_m)}{E_{\text{inv}}(\rho)}, \quad (16)$$

as we just showed. Now, we use the convexity of E_{inv} (we remind the reader that the measure E_{inv} is defined on mixed states via a convex roof extension [see part (iii) of Definition 1], to get

$$\frac{E_{\text{inv}}(\Lambda(\rho))}{E_{\text{inv}}(\rho)} \leq \frac{\sum_m p_m E_{\text{inv}}(\sigma_m)}{E_{\text{inv}}(\rho)} = \mathcal{F}[\Lambda], \quad (17)$$

and this concludes the proof. ■

In particular, for one-sided operations, $\mathcal{F}[\Lambda]$ can be computed explicitly [10] and, additionally, for one-sided operations and pure state inputs, the first inequality of Eq. (14) is actually an equality, $E_{\text{inv}}[\Lambda(|\psi\rangle\langle\psi|)]/E_{\text{inv}}(|\psi\rangle\langle\psi|) = \mathcal{F}[\Lambda]$ [10]. However, in the most general case of multipartite separable operations, this is no longer true. Consider, for example, a bipartite separable operation Λ with an operator-sum representation given by $\{\sqrt{p}I \otimes I, \sqrt{1-p}X \otimes X\}$, where $0 < p < 1$ and $X = |0\rangle\langle 1| + |1\rangle\langle 0|$ is the usual bit-flip Pauli operator. In this case it is not hard to see that there are no other separable representations of Λ (any two Kraus representations are related by a unitary matrix, and it follows at once that a different representation of Λ will not be separable). The ERF $\mathcal{F}[\Lambda]$ can then be computed explicitly, since there is no minimization to be done, and equals 1. The maximally entangled state $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ is invariant under Λ , hence $E_{\text{inv}}[\Lambda(|\psi\rangle\langle\psi|)]/E_{\text{inv}}(|\psi\rangle\langle\psi|) = \mathcal{F}[\Lambda] = 1$ and the bound is saturated. But, in general, an arbitrary bipartite state $|\phi\rangle$ is not invariant under Λ and has $E_{\text{inv}}[\Lambda(|\phi\rangle\langle\phi|)]/E_{\text{inv}}(|\phi\rangle\langle\phi|) < 1$, as one can easily check using the closed-form expression for the bipartite concurrence [15].

Entanglement breaking. Interestingly, the capability of a channel to destroy entanglement is another quantity that is also independent of the initial state, as we will explain below. This is closely related to the work of Ref. [21], in which the authors define correlation measures (classical and quantum) based on how much “noise” one has to “inject” by local operations into an entangled quantum state to make it separable, in the limit of many copies and asymptotically vanishing errors. We now point out that such a measure cannot be defined in the zero-error single copy case, since the quantity of interest is state independent. The capability of a channel to destroy entanglement is quantified as follows.

Definition 3. A CPTP map $\Phi : \mathcal{B}(\mathcal{H}_d) \longrightarrow \mathcal{B}(\mathcal{H}_d)$ is called *partial- r entanglement breaking* [22] if

$$\text{SN}[(\Phi \otimes I)\rho] \leq r, \quad \forall \rho \in \mathcal{B}(\mathcal{H}_d \otimes \mathcal{H}_d). \quad (18)$$

It generalizes the usual notion of entanglement breaking channels [23], the latter having $r = 1$. Here $\text{SN}(\sigma)$ denotes the Schmidt number [24] of the bipartite density operator σ , defined as

$$\text{SN}(\rho) := \min_{\{p_i, |\psi_i\rangle\}} \left\{ \max_i \text{SR}(|\psi_i\rangle) \right\}, \quad (19)$$

where the minimum is taken over all possible pure state ensemble decompositions of ρ and $\text{SR}(|\psi_i\rangle)$ denotes the Schmidt rank on $|\psi_i\rangle$, i.e., the number of nonzero Schmidt coefficients of $|\psi_i\rangle$. A state σ is separable if and only if $\text{SN}(\sigma) = 1$.

The following proposition was recently proved in Ref. [25] (see Proposition 8).

Proposition 1. A CPTP map $\Phi : \mathcal{B}(\mathcal{H}_d) \longrightarrow \mathcal{B}(\mathcal{H}_d)$ is *partial- r entanglement breaking* if and only if there exist a full Schmidt rank pure state $|\psi\rangle \in \mathcal{H}_d \otimes \mathcal{H}_d$ (i.e., having all Schmidt coefficients strictly positive) for which

$$\text{SN}[(\Phi \otimes I)|\psi\rangle\langle\psi|] = r. \quad (20)$$

In other words, if a channel breaks entanglement for one (maximal Schmidt number) state, then it must break entanglement for *all* states; that is, its “disentangling power” is an intrinsic property of the channel independent of the input state, as long as the input has maximal Schmidt number (this corresponds to the case of nonzero entanglement as quantified by any SL-invariant measure). This means that one cannot use the approach of Ref. [21] to quantify entanglement (or correlations) as the total amount of local noise one has to inject in a state to make it separable, since in a single copy zero-error regime this quantity is state independent.

Conclusions and open questions. We studied the evolution of mixed-state entanglement under local decoherence. Our

main result is summarized by Theorem 1, in which we proved that multipartite mixed-state entanglement evolution under separable operations is measure and state independent and is solely a function of the channel. This significantly extends the results in Refs. [7–9] applicable only for the restricted case of bipartite separable operations, and also generalizes the previous work of Ref. [10], where only one-sided operations of the form $\Lambda^{(1)} \otimes I \otimes \dots \otimes I$ were considered. We derived a closed-form expression for the average entanglement evolution under the most general class of separable channels, and our results are automatically valid for LOCC since the latter is a proper subset of the former.

An interesting fact is that our equality in Theorem 1 depends on the Kraus representation of the separable operation. Given such a separable operation, is there more than one separable representation of it (aside from a permutation of the Kraus operators)? A complete solution to this problem will allow an explicit evaluation of the ERF $\mathcal{F}(\Lambda)$ and provide a nontrivial upper bound on the entanglement evolution of arbitrary mixed states, Corollary 1. We observed numerically using nonlinear optimization methods that there are many cases of separable operations that do not admit more than one nontrivial Kraus representation, but this is far from a rigorous proof.

Our results are valid for SL-invariant entanglement measures based on convex roof extension and are not applicable in general to arbitrary entanglement measures. However, it follows at once that the convex roof extension is an upper bound on *any* other convex measure that coincides with the former on pure states, i.e., both give the same result on a pure state. In this case Theorem 1 provides a useful upper bound on the average entanglement evolution, i.e., the equality in Eq. (10) becomes an inequality.

We discussed the disentangling power of a quantum channel and pointed out that it exhibits a similar state invariance as our average entanglement ratio in Theorem 1. This forbids a straightforward extension of Ref. [21] to the single copy zero-error regime. An interesting generalization will be to define a disentangling power of a separable operation, instead on a single-sided channel, and characterize the class of partial- r entanglement breaking such operations. This approach may prove useful in extending [21] to the single copy zero-error regime, since one does not expect the disentangling power to be state invariant anymore.

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