Quantifying non-Markovianity via correlations

Shunlong Luo,* Shuangshuang Fu, and Hongting Song
Academy of Mathematics and Systems Science, Chinese Academy of Sciences, 100190 Beijing, People's Republic of China
(Received 20 August 2012; published 8 October 2012)

In the study of open quantum systems, memory effects are usually ignored, and this leads to dynamical semigroups and Markovian dynamics. However, in practice, non-Markovian dynamics is the rule rather than the exception. With the recent emergence of quantum information theory, there is a flurry of investigations of non-Markovian dynamics, and several significant measures for non-Markovianity are introduced from various perspectives such as deviation from divisibility, information exchange between a system and its environment, or entanglement with the environment. In this work, by exploiting the correlations flow between a system and an arbitrary ancillary, we propose a considerably intuitive measure for non-Markovianity by use of correlations as quantified by the quantum mutual information rather than entanglement. The fundamental properties, physical significance, and differences and relations with existing measures for non-Markovianity are elucidated. The measure captures quite directly and deeply the characteristics of non-Markovianity from the perspective of information. A simplified version based on Jamiołkowski-Choi isomorphism which encodes operations via bipartite states and does not involve any optimization is also proposed.

DOI: 10.1103/PhysRevA.86.044101 PACS number(s): 03.65.Yz, 03.65.Ta, 42.50.Lc

I. INTRODUCTION

Quantum dynamics is usually classified into Markovian or non-Markovian according to the absence or presence of memory effects. While realistic dynamics usually exhibits memory effects, in the theoretical investigations, the Markovian approximations are often adopted, and quantum dynamics is often mathematically described by a quantum dynamical semigroup, or equivalently, by a solution of the underlying master equation of the Lindblad type [1,2]. This constitutes the main content of the present theory of open quantum systems [3–6]. Such a situation is not because the non-Markovian dynamics is not important, but rather is essentially due to the fact that, mathematically, it is usually difficult, if not totally intractable, to treat non-Markovian dynamics in general. However, since non-Markovian dynamics is responsible for a wide variety of physically interesting effects, in recent years it is attracting increasing attention in both theory and practice [7–23], in particular in the context of quantum information processing, due to their ability in regaining lost information and recovering coherence for quantum technologies. Non-Markovianity can be served as a resource in certain quantum protocols, and can be exploited for quantum metrology and quantum key distribution [16,17].

Although both Markovian and non-Markovian dynamics in the classical setting are well defined, soundly constructed, and widely studied [24], their quantum extensions remain elusive and subtle. Various, even controversial in some sense, non-Markovian criteria have been proposed in recent years, and several measures for non-Markovianity are introduced based on different considerations such as semigroups, divisibility, or backflow of information: (i) In a rather abstract and general framework, Wolf *et al.* identified non-Markovianity with the breakdown of the semigroup property of the dynamical maps, and introduced a measure for non-Markovianity in terms of the deviation of the logarithm of dynamical maps from the canonical Lindblad generators [7]. (ii) Rivas *et al.* proposed two measures for non-Markovianity based on dynamical divisibility (which is a generalization of the semigroup property) and entanglement with environment, respectively [8]. (iii) Based on the trace distance which quantifies the distinguishability of quantum states, Breuer *et al.* suggested a measure for non-Markovianity in terms of the increasing of distinguishability between different evolving states, which may be interpreted as recovery of lost information (the flow of information from the environment back to the open system) [9]. (iv) Lu *et al.* quantified non-Markovianity in terms of the quantum Fisher information flow [10]. (v) By use of the fidelity between a dynamically evolved state and its earlier time state, Rajagopal *et al.* proposed a signature for non-Markovianity, which may also be formulated in terms of the Bures distance [11]. (vi) Hou *et al.* introduced an alternative measure for non-Markovianity in terms of the deviation from divisibility, as quantified by the negative eigenvalues of the transition maps [12]. (v) For continuous-variable open quantum systems, Vasile *et al.* initiated the study of a measure for Gaussian non-Markovianity by use of fidelity rather than the trace distance in assessing distinguishability and information backflow [13].

In general, all these measures do not coincide exactly in revealing non-Markovianity [25–27], although there are many instances of coincidence [28]. A universal definition for quantum (non-)Markovian dynamics is still lacking. At present, we have several closely related, but conceptually different definitions (or conventions) for non-Markovianity.

Motivated by the idea of exploiting the correlations between the system and an *arbitrary* ancillary system (rather than only the purification counterpart of the system), we will introduce a conceptually simple, mathematically computable, and physically intuitive measure for non-Markovianity by use of quantum mutual information rather than entanglement. Its significance is illustrated through several examples.

The work is organized as follows. In Sec. II, a measure for non-Markovianity based on correlations flow is introduced. We illustrate the concept by making comparison with other measures in Sec. III. We conclude with discussion in Sec. IV.

*luosl@amt.ac.cn

II. NON-MARKOVIANITY VIA CORRELATIONS

The precise setup is as follows. Consider a quantum system with Hilbert space H and state space $S(H)$ (the set of all density operators). Let $\Lambda = \{\Lambda_t\}$ be a quantum dynamical evolution described by a family of quantum operations (channels, or linear, completely positive, trace preserving maps) Λ_t on $S(H)$. Let H^a be an *arbitrary* ancillary system which may be correlated with the system H. Thus the "system + ancillary" space is $H \otimes H^a$. In this context, recall that the total correlations in a bipartite state ρ^{sa} shared between the system and the ancillary is quantified by the quantum mutual information

$$I(\rho^{sa}) := S(\rho^s) + S(\rho^a) - S(\rho^{sa}).$$

Here $\rho^s = \mathrm{tr}_a \rho^{sa}$, $\rho^a = \mathrm{tr}_s \rho^{sa}$ are the reduced states for the system and ancillary, respectively, and $S(\rho^s) := -\mathrm{tr}\rho^s \log_2 \rho^s$ is the von Neumann entropy. The quantum mutual information is an essential measure for total correlations [29–32].

If the dynamics $\{\Lambda_t\}$ is Markovian in the sense that

$$\Lambda_t = \Lambda_{t,r}\Lambda_r, \quad r \leqslant t$$

for some operations $\Lambda_{t,r}$, then for any bipartite state ρ^{sa} on the composite system $H \otimes H^a$, put

$$\rho_t^{sa} := (\Lambda_t \otimes \mathbb{1})\rho^{sa},$$

where $\mathbb{1}$ is the identity operation on the ancillary state space $S(H^a)$, we have

$$I\left(\rho_t^{sa}\right) = I((\Lambda_{t,r}\Lambda_r \otimes \mathbb{1})\rho^{sa}) = I((\Lambda_{t,r} \otimes \mathbb{1})\rho_r^{sa}) \leqslant I\left(\rho_r^{sa}\right)$$

due to the monotonicity of the quantum mutual information under local operations. Consequently, the quantum mutual information $I(\rho_t^{sa})$ is a monotonically decreasing function of $t \geqslant 0$, which implies that $\frac{d}{dt}I(\rho_t^{sa}) \leqslant 0$ for any Markovian dynamics. Any violation of this monotonicity (i.e., $\frac{d}{dt}I(\rho_t^{sa}) > 0$) is an indication for non-Markovianinty of the dynamics $\{\Lambda_t\}$. From this, we may introduce a measure for non-Markovianity as follows:

$$\mathcal{N}(\Lambda) := \sup_{\rho^{sa}} \int_{(d/dt)I(\rho_t^{sa})>0} \frac{d}{dt} I\left(\rho_t^{sa}\right) dt.$$

Here $\rho_t^{sa} := (\Lambda_t \otimes \mathbb{1})\rho^{sa}$ and the sup is over all bipartite states ρ^{sa} on $H \otimes H^a$, with H^a an arbitrary ancillary space.

While the above measure is fundamental, its evaluation is complicated due to the formidable optimization. Fortunately, in practice we may use the following simplified version as a significant substitute, which is intuitive in its own right: We take $H^a = H$ and let $\rho^{sa} = |\Psi\rangle\langle\Psi|$ be *any* maximally correlated pure state between the system and the ancillary H^a, then we come to an alternative measure for non-Markovianity as follows:

$$\mathcal{N}_0(\Lambda) := \int_{(d/dt)I(\rho_t^{sa})>0} \frac{d}{dt} I\left(\rho_t^{sa}\right) dt.$$

Here $\rho_t^{sa} := (\Lambda_t \otimes \mathbb{1})|\Psi\rangle\langle\Psi|$, and in general, $\mathcal{N}_0(\Lambda)$ is independent of the choice of $|\Psi\rangle$. This procedure for obtaining a simple measure for non-Markovianity has a twofold meaning: First, mathematically, the correspondence between an operation and a bipartite state as stipulated by the above equation is precisely the Jamiołkowski-Choi isomorphism [33,34], which implies that we could exploit the correlations structure in a bipartite state in order to study an operation. Second, physically, any mixed state of a system can be viewed as the reduced state of a higher-dimensional *pure* state, and the system is correlated (both classically and quantum mechanically) with an ancillary, and thus the action of a quantum operation on the system state can be studied via the correlations between the system and the ancillary. The measure $\mathcal{N}_0(\Lambda)$ for non-Markovianity can be rather straightforwardly evaluated for both discrete and continuous variable systems.

III. COMPARISON

In order to make comparison with other non-Markovian approaches, let us first recall two celebrated non-Markovian measures:

(1) As defined by Breuer, Laine, and Piilo (BLP) [9], a dynamical evolution $\Lambda = \{\Lambda_t\}$ is Markovian in the sense of decreasing distinguishability of evolving states if $\frac{1}{2}\mathrm{tr}|\Lambda_t(\rho - \tau)|$ is a monotonically decreasing function of $t \geqslant 0$ for any states ρ and τ. Since the trace distance is fundamentally related to distinguishability of quantum states [35], non-Markovianity in this sense indicates the increasing of distinguishability, and thus may be interpreted as information recovery as opposed to the information loss (memory loss) in Markovian dynamics. The associated measure for non-Markovianity is then defined as

$$\mathcal{N}_{\mathrm{BLP}}(\Lambda) := \sup_{\rho,\tau} \int_{(d/dt)\mathrm{tr}|\Lambda_t(\rho-\tau)|>0} \frac{1}{2}\frac{d}{dt}\mathrm{tr}|\Lambda_t(\rho - \tau)| dt.$$

(2) According to the approach of Rivas, Huelga, and Plenio (RHP) [8], a dynamical evolution $\Lambda = \{\Lambda_t\}$ is Markovian in the sense of divisibility if there exist quantum operations $\Lambda_{t,r}$ such that $\Lambda_t = \Lambda_{t,r}\Lambda_r$ for all $0 \leqslant r \leqslant t$. This clearly includes the dynamical semigroups as a particular case, and thus is more general than the dynamics described by the master equation with canonical Lindblad generators [1,2]. In this context, let $\rho^{sa} = |\Psi\rangle\langle\Psi|$ be a maximally entangled pure state between the system and an ancillary, then for $\epsilon > 0$, $\mathrm{tr}|(\Lambda_{t+\epsilon,t} \otimes \mathbb{1})\rho^{sa}| = 1$ if and only if $\Lambda_{t+\epsilon,t}$ is completely positive (noting that $\Lambda_{t+\epsilon,t}$ is trace preserving), and $\mathrm{tr}|(\Lambda_{t+\epsilon,t} \otimes \mathbb{1})\rho^{sa}| > 1$ otherwise. The RHP measure for non-Markovianity is defined as [8]

$$\mathcal{N}_{\mathrm{RHP}}(\Lambda) := \int_0^\infty \lim_{\epsilon\to 0} \frac{\mathrm{tr}|(\Lambda_{t+\epsilon,t} \otimes \mathbb{1})\rho^{sa}| - 1}{\epsilon} dt.$$

Since $\mathcal{N}_{\mathrm{RHP}}(\Lambda) = 0$ implies that $\mathcal{N}_{\mathrm{BLP}}(\Lambda) = 0$, any divisible dynamics is Markovian according to BLP. But the converse is not true in general [25]. Both of these measures are difficult to evaluate: The former involves a formidable optimization over a pair of density operators, the latter involves the transition map $\Lambda_{t+\epsilon,t}$ which in general cannot be computed. It is remarkable that RHP also proposed another measure for non-Markovianity based on the consideration of entanglement with an ancillary system [8]. However, since an entanglement measure is usually very difficult to evaluate by itself, and furthermore there is no compelling reason for using entanglement rather than general correlations, this alternative measure has attracted little attention, and needs further investigation. Now, we compare the measure $\mathcal{N}_0(\Lambda)$

with existing ones for revealing non-Markovianity through several prototypical examples.

Example 1. Consider the dynamics $\rho_t = \Lambda_t \rho_0$ on a qubit system described by the differential equation

$$\frac{d}{dt}\rho_t = \gamma(t)(\sigma_z \rho_t \sigma_z - \rho_t).$$

Here σ_z is a Pauli spin operator. If we denote the initial state of the open quantum system as $\rho_0 = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$, then the dynamics can be expressed as

$$\rho_t = \Lambda_t \rho_0 = \begin{pmatrix} \alpha & \beta f(t) \\ \gamma f(t) & \delta \end{pmatrix},$$

with $f(t) := \exp[-2\int_0^t \gamma(\tau)d\tau]$. After simple calculations, we get

$$\rho_t^{sa} = (\Lambda_t \otimes \mathbb{1})|\Psi\rangle\langle\Psi| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & f(t) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ f(t) & 0 & 0 & 1 \end{pmatrix},$$

where $|\Psi\rangle$ is the canonical maximally entangled state for the "system + ancillary." Therefore

$$I(\rho_t^{sa}) = 2 + \frac{1+f(t)}{2}\log_2\frac{1+f(t)}{2} + \frac{1-f(t)}{2}\log_2\frac{1-f(t)}{2},$$

and

$$\frac{d}{dt}I(\rho_t^{sa}) = -\gamma(t)f(t)\log_2\frac{1+f(t)}{1-f(t)}.$$

From the above expression and noting that $f(t) > 0$, we see clearly that $\frac{d}{dt}I(\rho_t^{sa}) > 0$ is equivalent to $\gamma(t) < 0$, and

$$\mathcal{N}_0(\Lambda) = -\int_{\gamma(t)<0}\gamma(t)f(t)\log_2\frac{1+f(t)}{1-f(t)}dt.$$

From Ref. [26], we know that

$$\mathcal{N}_{\rm BLP}(\Lambda) = -2\int_{\gamma(t)<0}\gamma(t)f(t)dt,$$

and from Ref. [9], we know that

$$\mathcal{N}_{\rm RHP}(\Lambda) = -2\int_{\gamma(t)<0}\gamma(t)dt.$$

In this example, the three measures detect non-Markovianity in an equivalent way, i.e., all lead to the same criterion $\gamma(t) < 0$.

Example 2. Consider the dynamics described by the time-local master equation

$$\frac{d}{dt}\rho_t = -\frac{i}{2}s(t)[\sigma_+\sigma_-,\rho_t] + \gamma(t)\left(\sigma_-\rho_t\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-,\rho_t\}\right),$$

with $s(t) = -2{\rm Im}\frac{\dot{G}(t)}{G(t)}$ (imaginary part), $\gamma(t) = -2{\rm Re}\frac{\dot{G}(t)}{G(t)}$ (real part), and the function $G(t)$ is the solution of the integrodifferential equation

$$\frac{d}{dt}G(t) = -\int_0^t dt_1 f(t-t_1)G(t_1)$$

corresponding to the initial condition $G(0) = 1$, the kernel $f(t-t_1)$ represents a certain two-point correlation function

$$f(t-t_1) = \sum_k |g_k|^2 e^{i(\omega_0-\omega_k)(t-t_1)}.$$

Further calculations show that

$$\rho_t = \begin{pmatrix} \alpha + (1-|G(t)|^2)\delta & G(t)^*\beta \\ G(t)\gamma & |G(t)|^2\delta \end{pmatrix}.$$

Therefore, we have

$$\rho_t^{sa} = (\Lambda_t \otimes \mathbb{1})|\Psi\rangle\langle\Psi| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & G(t)^* \\ 0 & 1-|G(t)|^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ G(t) & 0 & 0 & |G(t)|^2 \end{pmatrix},$$

and

$$\frac{d}{dt}I(\rho_t^{sa}) = |G(t)|\log_2\frac{[1+|G(t)|^2][2-|G(t)|^2]}{|G(t)|^2[1-|G(t)|^2]}\frac{d}{dt}|G(t)|.$$

It is clear from the above expression that $\frac{d}{dt}I(\rho_t^{sa}) > 0$ is equivalent to $\frac{d}{dt}|G(t)| > 0$. This detects the non-Markovianity in a similar fashion as BLP [9].

We have also treated several other examples, and find that our measure detects the same non-Markovianity range as BLP dose. However, from a general point of view, the measure $\mathcal{N}(\Lambda)$ is different from other measures, and it is desirable to illustrate the difference and links through simple examples. We leave this as problems to the interested readers.

IV. SUMMARY

Based on essential features of non-Markovianity, we have introduced a figure of merit for non-Markovianity from the informational perspective. When particularized to a simple yet fundamental setting, the measure does not involve any optimization and can be evaluated rather straightforwardly. Its significance and power are illustrated through prototypical examples, and intrinsic relations with other approaches are elucidated.

From a conceptual and intuitive point of view, the measure $\mathcal{N}(\Lambda)$ for non-Markovianity captures fully the informational aspect of the dynamics Λ. We have only explored some elementary and preliminary features of this measure, and it is desirable to further characterize and investigate its structure, properties, applications, and relations with other measures for non-Markovianity.

Since the central theme of modern quantum technologies is the control of quantum coherence and quantum correlations, which usually suffer from decoherence and decay, but non-Markovianity entails new features of quantum effects related to decoherence, dissipation, decay, and relaxation, non-Markovianity may be exploited for the benefits of quantum technologies. Our approach may be of theoretical interest in such an endeavor of fighting decoherence and information loss.

ACKNOWLEDGMENTS

This work was supported by the Science Fund for Creative Research Groups, National Natural Science Foundation of China, Grant No. 10721101, the National Center for Mathematics and Interdisciplinary Sciences, Chinese Academy of Sciences, Grant No. Y029152K51.

[1] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, J. Math. Phys. **17**, 821 (1976).
[2] G. Lindblad, Commun. Math. Phys. **48**, 119 (1976).
[3] E. B. Davies, *Quantum Theory of Open Systems* (Academic, New York, 1976).
[4] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002).
[5] R. Alicki and K. Lendi, *Quantum Dynamical Semigroups and Applications* (Springer, Berlin, 2007).
[6] Á. Rivas and S. F. Huelga, *Open Quantum Systems, An Introduction* (Springer, Heidelberg, 2012).
[7] M. M. Wolf, J. Eisert, T. S. Cubitt, and J. I. Cirac, Phys. Rev. Lett. **101**, 150402 (2008).
[8] Á. Rivas, S. F. Huelga, and M. B. Plenio, Phys. Rev. Lett. **105**, 050403 (2010).
[9] H.-P. Breuer, E.-M. Laine, and J. Piilo, Phys. Rev. Lett. **103**, 210401 (2009).
[10] X.-M. Lu, X. Wang, and C. P. Sun, Phys. Rev. A **82**, 042103 (2010).
[11] A. K. Rajagopal, A. R. Usha Devi, and R. W. Rendell, Phys. Rev. A **82**, 042107 (2010).
[12] S. C. Hou, X. X. Yi, S. X. Yu, and C. H. Oh, Phys. Rev. A **83**, 062115 (2011).
[13] R. Vasile, S. Maniscalco, M. G. A. Paris, H.-P. Breuer, and J. Piilo, Phys. Rev. A **84**, 052118 (2011).
[14] D. Chruściński and A. Kossakowski, Phys. Rev. Lett. **104**, 070406 (2010).
[15] Z. Y. Xu, W. L. Yang, and M. Feng, Phys. Rev. A **81**, 044105 (2010).
[16] A. W. Chin, S. F. Huelga, and M. B. Plenio, arXiv:1103.1219.
[17] R. Vasile, S. Olivares, M. G. A. Paris, and S. Maniscalco, Phys. Rev. A **83**, 042321 (2011).
[18] B.-H. Liu, L. Li, Y.-F. Huang, C.-F. Li, G.-C. Guo, E.-M. Laine, H.-P. Breuer, and J. Piilo, Nat. Phys. **7**, 931 (2011).
[19] Z. He, J. Zou, L. Li, and B. Shao, Phys. Rev. A **83**, 012108 (2011).
[20] P. Haikka, S. McEndoo, G. De Chiara, G. M. Palma, and S. Maniscalco, Phys. Rev. A **84**, 031602 (2011).
[21] P. Haikka, J. Goold, S. McEndoo, F. Plastina, and S. Maniscalco, Phys. Rev. A **85**, 060101(R) (2012).
[22] S. F. Huelga, Á. Rivas, and M. B. Plenio, Phys. Rev. Lett. **108**, 160402 (2012).
[23] E.-M. Laine, H.-P. Breuer, J. Piilo, C.-F. Li, and G.-C. Guo, Phys. Rev. Lett. **108**, 210402 (2012).
[24] W. Feller, *An Introduction to Probability Theory and its Applications* (John Wiley, New York, 1966).
[25] P. Haikka, J. D. Cresser, and S. Maniscalco, Phys. Rev. A **83**, 012112 (2011).
[26] B. Vacchini, A. Smirne, E.-M. Laine, J. Piilo, and H.-P. Breuer, New J. Phys. **13**, 093004 (2011).
[27] D. Chruściński, A. Kossakowski, and Á. Rivas, Phys. Rev. A **83**, 052128 (2011).
[28] H.-S. Zeng, N. Tang, Y.-P. Zheng, and G.-Y. Wang, Phys. Rev. A **84**, 032118 (2011).
[29] V. Vedral, Rev. Mod. Phys. **74**, 197 (2002).
[30] B. Groisman, S. Popescu, and A. Winter, Phys. Rev. A **72**, 032317 (2005).
[31] B. Schumacher and M. D. Westmoreland, Phys. Rev. A **74**, 042305 (2006).
[32] N. Li and S. Luo, Phys. Rev. A **76**, 032327 (2007).
[33] A. Jamiołkowski, Rep. Math. Phys. **3**, 275 (1972).
[34] M.-D. Choi, Linear Alg. Appl. **10**, 285 (1975).
[35] C. A. Fuchs and J. van de Graaf, IEEE Trans. Inf. Theory **45**, 1216 (1999).