Theory of refraction and reflection with partially coherent electromagnetic beams

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Laws of refraction and reflection of light are governed by the classic Fresnel formulas. These formulas are not applicable to partially coherent light. We develop a general theory of refraction and reflection of electromagnetic beams of any state of coherence. We find that coherence properties of such beams change, in general, on refraction and on reflection at a planar interface.

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I. INTRODUCTION

Refraction and reflection of light have been subjects of investigations for a very long time. Properties of refracted and reflected fields are usually studied by the use of the classic Fresnel formulas. Because these formulas have been developed for monochromatic plane waves, they do not apply when the light is partially coherent. In this paper, we present a theory of refraction and reflection of electromagnetic beams of any state of spatial coherence.

Derivations of the classic Fresnel formulas for refraction and reflection based on Maxwell's equations can be found in standard textbooks of electromagnetic theory (see, for example, Ref. [1], Chap. 7). In the usual treatments, it is assumed that the incident, the refracted, and the transmitted electromagnetic fields are monochromatic plane waves. Such waves are necessarily spatially fully coherent. Hence, the Fresnel formulas do not provide any information regarding possible changes of the coherence properties of light, when it is refracted and reflected at an interface. In the present paper, we show that such changes do, in general, occur.

In Sec. II, we recall the classic Fresnel formulas and the basic aspects of coherence theory of classical electromagnetic beams. In Sec. III, we develop the theory of reflection and refraction of partially coherent electromagnetic beams. In Sec. IV we show by an example that coherence properties of an electromagnetic beam may change on refraction and on reflection at a planar interface that separates two media of different dielectric properties.

II. SUMMARY OF SOME BASIC RESULTS

We first recall the main results relating to reflection and refraction of monochromatic plane waves, and then discuss some results from coherence theory of electromagnetic beams.

A. Fresnel formulas for reflection and refraction of monochromatic plane waves

Let us consider two media of different dielectric and magnetic properties, characterized by permittivities and permeabilities ϵ, μ, and ϵ', μ', respectively. The refractive indices n and n' of the two media are given by the usual formulas (see, for example, Ref. [1], p. 303)

$$n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}, \quad n' = \sqrt{\frac{\epsilon'\mu'}{\epsilon_0\mu_0}}, \tag{1}$$

where $\epsilon_0 \approx 8.854 \times 10^{-12}$ F/m and $\mu_0 = 4\pi \times 10^{-7}$ H/m are the vacuum permittivity and the vacuum permeability, respectively. We assume that the two media are separated by a planar interface.

Suppose that a monochromatic plane wave is incident on the interface at an angle $\theta_{\mathbb{i}}$ of incidence (see Fig. 1). The electric field vector of the incident wave can be represented in the form

$$\mathbf{E}^{(\mathbb{i})}(\mathbf{r},t) = \mathbf{E}_0^{(\mathbb{i})} \exp[i(\mathbf{k}^{(\mathbb{i})} \cdot \mathbf{r} - \omega t)], \tag{2}$$

where $\mathbf{k}^{(\mathbb{i})}$ is the wave vector. As usual we refer to the plane that is defined by the wave vector $\mathbf{k}^{(\mathbb{i})}$ and the normal $\mathbf{n}$ to the surface of the interface as the plane of incidence. As is well known, the wave vectors of the refracted and the reflected fields also lie in this plane.

Since the incident wave is transverse, the electric field $\mathbf{E}^{(\mathbb{i})}$ has no component in the direction of propagation. Hence, $\mathbf{E}^{(\mathbb{i})}$ can be expressed as the sum of two mutually orthogonal components, $\mathbf{E}_{\mathrm{v}}^{(\mathbb{i})}$ and $\mathbf{E}_{\mathrm{h}}^{(\mathbb{i})}$, which are perpendicular and parallel, respectively, to the plane of incidence (see Fig.1). Consequently

$$\begin{aligned}\mathbf{E}^{(\mathbb{i})}(\mathbf{r},t) &= \mathbf{E}_{\mathrm{v}}^{(\mathbb{i})}(\mathbf{r},t) + \mathbf{E}_{\mathrm{h}}^{(\mathbb{i})}(\mathbf{r},t)\\ &= \left(\mathbf{E}_{0\mathrm{v}}^{(\mathbb{i})} + \mathbf{E}_{0\mathrm{h}}^{(\mathbb{i})}\right) \exp[i(\mathbf{k}^{(\mathbb{i})} \cdot \mathbf{r} - \omega t)].\end{aligned} \tag{3}$$

Let $\mathbf{E}^{(\mathbb{t})}(\mathbf{r},t)$ and $\mathbf{E}^{(\mathbb{r})}(\mathbf{r},t)$ be the electric field vectors of the transmitted and of the reflected plane waves, respectively. Each of them can also be uniquely decomposed along the directions perpendicular and parallel to the plane of incidence; i.e.,

$$\begin{aligned}\mathbf{E}^{(\mathbb{t})}(\mathbf{r},t) &= \mathbf{E}_{\mathrm{v}}^{(\mathbb{t})}(\mathbf{r},t) + \mathbf{E}_{\mathrm{h}}^{(\mathbb{t})}(\mathbf{r},t)\\ &= \left(\mathbf{E}_{0\mathrm{v}}^{(\mathbb{t})} + \mathbf{E}_{0\mathrm{h}}^{(\mathbb{t})}\right) \exp[i(\mathbf{k}^{(\mathbb{t})} \cdot \mathbf{r} - \omega t)],\end{aligned} \tag{4}$$

$$\begin{aligned}\mathbf{E}^{(\mathbb{r})}(\mathbf{r},t) &= \mathbf{E}_{\mathrm{v}}^{(\mathbb{r})}(\mathbf{r},t) + \mathbf{E}_{\mathrm{h}}^{(\mathbb{r})}(\mathbf{r},t)\\ &= \left(\mathbf{E}_{0\mathrm{v}}^{(\mathbb{r})} + \mathbf{E}_{0\mathrm{h}}^{(\mathbb{r})}\right) \exp[i(\mathbf{k}^{(\mathbb{r})} \cdot \mathbf{r} - \omega t)].\end{aligned} \tag{5}$$

Moduli of the wave vectors are given by the expressions (Ref. [1], Eq. (7.33))

$$|\mathbf{k}^{(\mathbb{t})}| = \omega\sqrt{\epsilon'\mu'}, \tag{6a}$$

$$|\mathbf{k}^{(\mathbb{i})}| = |\mathbf{k}^{(\mathbb{r})}| = \omega\sqrt{\epsilon\mu}. \tag{6b}$$

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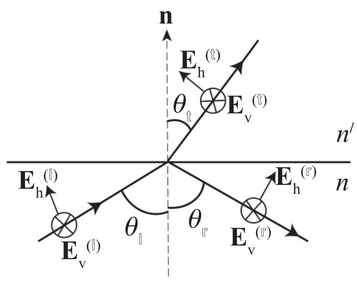

FIG. 1. The geometry relating to refraction and reflection of a monochromatic plane wave at an interface; h and v directions are chosen to be parallel and perpendicular to the plane of incidence.

The components of the transmitted and the reflected fields at the interface are related to the components of the incident field by the Fresnel formulas (Ref. [1], Eqs. (7.39) and (7.41))

$$E_{0\mathrm{v}}^{(\mathbb{t})} = \frac{2n\cos\theta_{\mathbb{i}}}{n\cos\theta_{\mathbb{i}} + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta_{\mathbb{i}}}} E_{0\mathrm{v}}^{(\mathbb{i})} \equiv T_{\mathrm{v}}(n,n',\theta_{\mathbb{i}})E_{0\mathrm{v}}^{(\mathbb{i})}, \tag{7a}$$

$$E_{0\mathrm{h}}^{(\mathbb{t})} = \frac{2nn'\cos\theta_{\mathbb{i}}}{\frac{\mu}{\mu'}n'^2\cos\theta_{\mathbb{i}} + n\sqrt{n'^2 - n^2\sin^2\theta_{\mathbb{i}}}} E_{0\mathrm{h}}^{(\mathbb{i})} \equiv T_{\mathrm{h}}(n,n',\theta_{\mathbb{i}})E_{0\mathrm{h}}^{(\mathbb{i})}, \tag{7b}$$

$$E_{0\mathrm{v}}^{(\mathbb{r})} = \frac{n\cos\theta_{\mathbb{i}} - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta_{\mathbb{i}}}}{n\cos\theta_{\mathbb{i}} + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta_{\mathbb{i}}}} E_{0\mathrm{v}}^{(\mathbb{i})} \equiv R_{\mathrm{v}}(n,n',\theta_{\mathbb{i}})E_{0\mathrm{v}}^{(\mathbb{i})}, \tag{7c}$$

$$E_{0\mathrm{h}}^{(\mathbb{r})} = \frac{\frac{\mu}{\mu'}n'^2\cos\theta_{\mathbb{i}} - n\sqrt{n'^2 - n^2\sin^2\theta_{\mathbb{i}}}}{\frac{\mu}{\mu'}n'^2\cos\theta_{\mathbb{i}} + n\sqrt{n'^2 - n^2\sin^2\theta_{\mathbb{i}}}} E_{0\mathrm{h}}^{(\mathbb{i})} \equiv R_{\mathrm{h}}(n,n',\theta_{\mathbb{i}})E_{0\mathrm{h}}^{(\mathbb{i})}. \tag{7d}$$

B. Elements of coherence theory of stochastic electromagnetic beams

In the optical and higher-frequency ranges of the electromagnetic spectrum, the concept of monochromaticity is an idealization that is not encountered in practice. Electromagnetic fields generated by even the best lasers always contain some random fluctuations, for example, due to spontaneous emission or due to mechanical vibrations of the mirrors at the ends of the laser cavities. If these fluctuations are statistically stationary, the field can be represented, at each frequency ω, by an ensemble $\{\mathbf{E}(\mathbf{r},\omega)\}$ of monochromatic realizations (see, for example, Ref. [2], Sec. 4.1). When the field is beamlike, one can neglect the field components along the direction of propagation. Hence, each member of the ensemble of the electric field can be represented in terms of two mutually orthogonal components, which are perpendicular to the direction of propagation. As in the previous section, we label them by the subscripts v and h. The second-order correlation properties of such a field [3] may be characterized by a 2×2 correlation matrix $\overleftrightarrow{W}(\mathbf{r}_1,\mathbf{r}_2;\omega)$, known as the cross-spectral density matrix (CSDM). It is defined at a pair of points specified by position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, by the formula (Ref. [2], Chap. 9)

$$\overleftrightarrow{W}(\mathbf{r}_1,\mathbf{r}_2;\omega) = \begin{pmatrix} \langle E_{\mathrm{v}}^*(\mathbf{r}_1;\omega)E_{\mathrm{v}}(\mathbf{r}_2;\omega)\rangle & \langle E_{\mathrm{v}}^*(\mathbf{r}_1;\omega)E_{\mathrm{h}}(\mathbf{r}_2;\omega)\rangle \\ \langle E_{\mathrm{h}}^*(\mathbf{r}_1;\omega)E_{\mathrm{v}}(\mathbf{r}_2;\omega)\rangle & \langle E_{\mathrm{h}}^*(\mathbf{r}_1;\omega)E_{\mathrm{h}}(\mathbf{r}_2;\omega)\rangle \end{pmatrix}. \tag{8}$$

Here the asterisk denotes the complex conjugate and the angle brackets denote the ensemble average. Apart from a proportionality factor that depends on the choice of units, one can express the spectral density $S(\mathbf{r},\omega)$, at a point $\mathbf{r}$, at frequency ω by the formula (Ref. [2], Chap. 9, Eq. (2))

$$S(\mathbf{r},\omega) \equiv \mathrm{Tr}\,\overleftrightarrow{W}(\mathbf{r},\mathbf{r};\omega), \tag{9}$$

where Tr denotes the trace of the matrix. The spectral degree of coherence of the beam, i.e., the spatial degree of coherence in the frequency domain, is defined by the formula (Ref. [2], Chap. 9, Eq. (8))

$$\eta(\mathbf{r}_1,\mathbf{r}_2;\omega) \equiv \frac{\mathrm{Tr}\,\overleftrightarrow{W}(\mathbf{r}_1,\mathbf{r}_2;\omega)}{\sqrt{S(\mathbf{r}_1;\omega)}\sqrt{S(\mathbf{r}_2;\omega)}}. \tag{10}$$

If one performs a Young's interference experiment with the pinholes located at positions $\mathbf{r}_1$ and $\mathbf{r}_2$, the visibility of the fringes, at frequency ω, can be shown to be equal to $|\eta(\mathbf{r}_1,\mathbf{r}_2;\omega)|$. It can be readily shown that $0 \leqslant |\eta(\mathbf{r}_1,\mathbf{r}_2;\omega)| \leqslant 1$. When $|\eta(\mathbf{r}_1,\mathbf{r}_2;\omega)| = 1$, i.e., when the fringe visibility has the maximum possible value, the beam is said to be spatially completely coherent, at frequency ω, at the pair of points $(\mathbf{r}_1,\mathbf{r}_2)$. In the other extreme case, when $\eta(\mathbf{r}_1,\mathbf{r}_2;\omega) = 0$, the beam is said to be spatially incoherent at the two points, at that frequency. In any intermediate case $(0 < |\eta(\mathbf{r}_1,\mathbf{r}_2;\omega)| < 1)$, the beam is said to be partially coherent at the two points, and at that frequency.

The electric field associated with an optical beam is not measurable. On the other hand, the correlation functions that appear in Eqs. (8)–(10) are measurable quantities. The usefulness of coherence theory comes largely from the fact that the theory is formulated in terms of measurable correlation functions, rather than in terms of nonmeasurable fields (in this context see Ref. [4]).

There are other definitions of the degree of coherence of electromagnetic beams. One of them was introduced in Ref. [5]. There have been some discussions relating to this definition [6,7]. However, for the problem addressed in this paper, it does not matter which of the definitions one employs [8].

C. The Fresnel formulas in matrix representation

It is usually convenient to represent the electric field vector as a column matrix, i.e., in the form

$$\mathbf{E}(\mathbf{r},\omega) = \begin{pmatrix} E_{\mathrm{v}}(\mathbf{r},\omega) \\ E_{\mathrm{h}}(\mathbf{r},\omega) \end{pmatrix} = (E_{\mathrm{v}}(\mathbf{r},\omega)\ E_{\mathrm{h}}(\mathbf{r},\omega))^T, \tag{11}$$

where the superscript T denotes transpose of the matrix. In this notation Eq. (8) can be expressed in the form

$$\overleftrightarrow{W}(\mathbf{r}_1,\mathbf{r}_2;\omega) = \langle \mathbf{E}^*(\mathbf{r}_1;\omega)\cdot\mathbf{E}^T(\mathbf{r}_2;\omega)\rangle, \tag{12}$$

where the dot $(\cdot)$ denotes matrix multiplication.

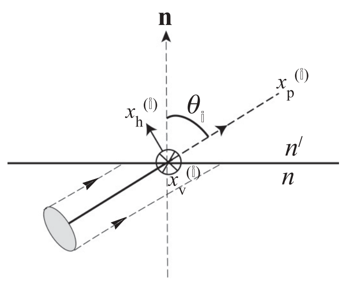

FIG. 2. The $(x_{\mathrm{v}}^{(\mathrm{i})}, x_{\mathrm{h}}^{(\mathrm{i})}, x_{\mathrm{p}}^{(\mathrm{i})})$ coordinate system.

The usual Fresnel formulas (7), at each frequency ω, can be written in the matrix form as

$$\mathbf{E}_0^{(\mathrm{t})}(\omega) = \overleftrightarrow{\mathrm{T}}(n,n',\theta_{\mathrm{i}}) \cdot \mathbf{E}_0^{(\mathrm{i})}(\omega), \tag{13a}$$

$$\mathbf{E}_0^{(\mathrm{r})}(\omega) = \overleftrightarrow{\mathrm{R}}(n,n',\theta_{\mathrm{i}}) \cdot \mathbf{E}_0^{(\mathrm{i})}(\omega), \tag{13b}$$

where

$$\overleftrightarrow{\mathrm{T}}(n,n',\theta_{\mathrm{i}}) = \begin{pmatrix} T_{\mathrm{v}}(n,n',\theta_{\mathrm{i}}) & 0 \\ 0 & T_{\mathrm{h}}(n,n',\theta_{\mathrm{i}}) \end{pmatrix}, \tag{14a}$$

$$\overleftrightarrow{\mathrm{R}}(n,n',\theta_{\mathrm{i}}) = \begin{pmatrix} R_{\mathrm{v}}(n,n',\theta_{\mathrm{i}}) & 0 \\ 0 & R_{\mathrm{h}}(n,n',\theta_{\mathrm{i}}) \end{pmatrix}. \tag{14b}$$

The matrix representation of the Fresnel formulas turns out to be particularly useful in connection with the theory of refraction and the reflection with partially coherent electromagnetic beams.

III. REFRACTION AND REFLECTION OF PARTIALLY COHERENT ELECTROMAGNETIC BEAMS

In this section we formulate a general theory of refraction and reflection with electromagnetic beams of any state of spatial coherence.

A. The main coordinate systems used

Suppose that a light beam is incident on the plane interface between two media. We assume that the beam axis makes an angle θ_{i} with the normal $\mathbf{n}$ to the interface (see Fig. 2). We label the beam axis by the symbol $x_{\mathrm{p}}^{(\mathrm{i})}$ and choose its positive direction along the direction of propagation of the beam. We choose two other axes along v and h directions (see Fig. 2) and denote them by $x_{\mathrm{v}}^{(\mathrm{i})}$ and $x_{\mathrm{h}}^{(\mathrm{i})}$, respectively. Thus, we have a coordinate system $(x_{\mathrm{v}}^{(\mathrm{i})}, x_{\mathrm{h}}^{(\mathrm{i})}, x_{\mathrm{p}}^{(\mathrm{i})})$, which we call an "incident-beam axis coordinate system." Similarly, we introduce transmitted-beam axis and reflected-beam axis coordinate systems, and denote them by $(x_{\mathrm{v}}^{(\mathrm{t})}, x_{\mathrm{h}}^{(\mathrm{t})}, x_{\mathrm{p}}^{(\mathrm{t})})$ and $(x_{\mathrm{v}}^{(\mathrm{r})}, x_{\mathrm{h}}^{(\mathrm{r})}, x_{\mathrm{p}}^{(\mathrm{r})})$, respectively. We assume that $x_{\mathrm{p}}^{(\mathrm{t})}$ and $x_{\mathrm{p}}^{(\mathrm{r})}$ lie in the plane formed by $x_{\mathrm{p}}^{(\mathrm{i})}$ and $\mathbf{n}$. If $x_{\mathrm{p}}^{(\mathrm{t})}$ and $x_{\mathrm{p}}^{(\mathrm{t})}$ make angles θ_{t} and θ_{r}, respectively, with $\mathbf{n}$, one has the relations

$$\theta_{\mathrm{i}} = \theta_{\mathrm{r}}, \tag{15a}$$

$$\sqrt{\mu\epsilon}\,\sin\theta_{\mathrm{i}} = \sqrt{\mu'\epsilon'}\,\sin\theta_{\mathrm{t}}. \tag{15b}$$

The $(x_{\mathrm{v}}^{(\mathrm{t})}, x_{\mathrm{h}}^{(\mathrm{t})}, x_{\mathrm{p}}^{(\mathrm{t})})$ coordinate system can be obtained from the $(x_{\mathrm{v}}^{(\mathrm{i})}, x_{\mathrm{h}}^{(\mathrm{i})}, x_{\mathrm{p}}^{(\mathrm{i})})$ coordinate system by a clockwise rotation through the angle $(\theta_{\mathrm{i}} - \theta_{\mathrm{t}})$ around the $x_{\mathrm{v}}^{(\mathrm{i})}$ axis (see Fig. 3). It is to be noted that the positive direction of $x_{\mathrm{v}}^{(\mathrm{i})}$ is pointing at a right angle into the plane of the figure. One then has

$$\begin{pmatrix} x_{\mathrm{v}}^{(\mathrm{t})} \\ x_{\mathrm{h}}^{(\mathrm{t})} \\ x_{\mathrm{p}}^{(\mathrm{t})} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_{\mathrm{i}} - \theta_{\mathrm{t}}) & -\sin(\theta_{\mathrm{i}} - \theta_{\mathrm{t}}) \\ 0 & \sin(\theta_{\mathrm{i}} - \theta_{\mathrm{t}}) & \cos(\theta_{\mathrm{i}} - \theta_{\mathrm{t}}) \end{pmatrix} \begin{pmatrix} x_{\mathrm{v}}^{(\mathrm{i})} \\ x_{\mathrm{h}}^{(\mathrm{i})} \\ x_{\mathrm{p}}^{(\mathrm{i})} \end{pmatrix}. \tag{16}$$

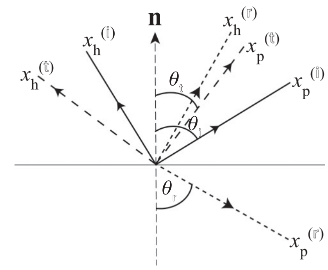

FIG. 3. The h and the p directions of the incident-, the refracted-, and the reflected-beam axis coordinate systems; the v direction for all three systems is the same, and points at a right angle into the plane of the figure.

The $(x_{\mathrm{v}}^{(\mathrm{r})}, x_{\mathrm{h}}^{(\mathrm{r})}, x_{\mathrm{p}}^{(\mathrm{r})})$ coordinate system can be obtained from the $(x_{\mathrm{v}}^{(\mathrm{i})}, x_{\mathrm{h}}^{(\mathrm{i})}, x_{\mathrm{p}}^{(\mathrm{i})})$ coordinate system by a counterclockwise rotation through the angle $(\pi - 2\theta_{\mathrm{i}})$ around the $x_{\mathrm{v}}^{(\mathrm{i})}$ axis (see Fig. 3). Hence, the two coordinate systems are related by the formula

$$\begin{pmatrix} x_{\mathrm{v}}^{(\mathrm{r})} \\ x_{\mathrm{h}}^{(\mathrm{r})} \\ x_{\mathrm{p}}^{(\mathrm{r})} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\cos 2\theta_{\mathrm{i}} & \sin 2\theta_{\mathrm{i}} \\ 0 & -\sin 2\theta_{\mathrm{i}} & -\cos 2\theta_{\mathrm{i}} \end{pmatrix} \begin{pmatrix} x_{\mathrm{v}}^{(\mathrm{i})} \\ x_{\mathrm{h}}^{(\mathrm{i})} \\ x_{\mathrm{p}}^{(\mathrm{i})} \end{pmatrix}. \tag{17}$$

B. Angular spectrum representation of the incident beam

As already mentioned in Sec. II B, for a partially coherent electromagnetic beam, the associated electric field vector is a random quantity, which may be represented by a statistical ensemble $\{\mathbf{E}(\mathbf{r},\omega)\}$ of realizations in the frequency domain. The simplest types of correlation properties of such beams are characterized by the so-called *cross-spectral density matrix* (CSDM), defined by Eq. (8).

We recall that for an electromagnetic beam one can neglect, for each member of the statistical ensemble, the field component along the direction of propagation, i.e., along the beam axis. Each of the other two components, which are normal to the direction of propagation, can be expressed in the following form using the angular spectrum representation (see, for example, Ref. [9], Sec. 3.2; [10]):

$$\begin{aligned} E_l^{(\mathrm{i})}(\mathbf{r},\omega) = &\iint_{p^2+q^2<|\mathbf{k}^{(\mathrm{i})}|^2} A_l^{(\mathrm{i})}(p,q;\omega)\exp\left[i\left(px_{\mathrm{v}}^{(\mathrm{i})} + qx_{\mathrm{h}}^{(\mathrm{i})}\right.\right. \\ &\left.\left.+wx_{\mathrm{p}}^{(\mathrm{i})}\right)\right]dp\,dq + \iint_{p^2+q^2>|\mathbf{k}^{(\mathrm{i})}|^2} A_l^{(\mathrm{i})}(p,q;\omega) \\ &\times \exp\left[i\left(px_{\mathrm{v}}^{(\mathrm{i})} + qx_{\mathrm{h}}^{(\mathrm{i})}\right)\right]\exp\left[-(p^2+q^2\right. \\ &\left.-|\mathbf{k}^{(\mathrm{i})}|^2)x_{\mathrm{p}}^{(\mathrm{i})}\right]dp\,dq, \end{aligned} \tag{18}$$

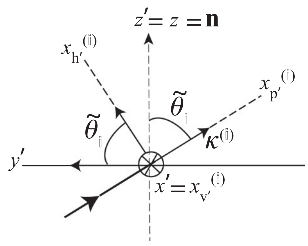

FIG. 4. The notation used for descriptions of the $(x_{v'}^{(\mathbb{i})},x_{h'}^{(\mathbb{i})},x_{p'}^{(\mathbb{i})})$ and the (x',y',z') coordinate systems.

where $l = \mathrm{v,h}$; $\mathbf{r} \equiv (x_{\mathrm{v}}^{(\mathbb{i})},x_{\mathrm{h}}^{(\mathbb{i})},x_{\mathrm{p}}^{(\mathbb{i})})$; and $w = \sqrt{|\mathbf{k}^{(\mathbb{i})}|^2 - p^2 - q^2}$. It is to be kept in mind that both $E_l^{(\mathbb{i})}(\mathbf{r},\omega)$ and $A_l^{(\mathbb{i})}(p,q;\omega)$ are random quantities, being members of suitably constructed statistical ensembles. The first term on the right-hand side of Eq. (18) represents a contribution from plane waves, each of which has field components with amplitudes $A_l^{(\mathbb{i})}(p,q;\omega)$. The second term is a contribution from evanescent waves. Since the incident field is assumed to be beamlike, one has (see Ref. [9], Sec. 5.6)

$$\left|A_l^{(\mathbb{i})}(p,q;\omega)\right| \approx 0, \quad \text{unless} \quad p^2 + q^2 \ll |\mathbf{k}^{(\mathbb{i})}|^2. \tag{19}$$

Equation (19) shows that one may neglect contributions from the evanescent waves in Eq. (18), and one then obtains for $E_l^{(\mathbb{i})}(\mathbf{r},\omega)$ the expression

$$E_l^{(\mathbb{i})}(\mathbf{r},\omega) \approx \iint_{p^2+q^2\ll|\mathbf{k}^{(\mathbb{i})}|^2} A_l^{(\mathbb{i})}(p,q;\omega) \times \exp\left[i\left(px_{\mathrm{v}}^{(\mathbb{i})} + qx_{\mathrm{h}}^{(\mathbb{i})} + wx_{\mathrm{p}}^{(\mathbb{i})}\right)\right]dp\,dq. \tag{20}$$

Clearly, each plane-wave component with amplitude $\mathbf{A}^{(\mathbb{i})}(p,q;\omega)$ propagates along a direction specified by the unit vector $\boldsymbol{\kappa}^{(\mathbb{i})}$, which, in the $(x_{\mathrm{v}}^{(\mathbb{i})},x_{\mathrm{h}}^{(\mathbb{i})},x_{\mathrm{p}}^{(\mathbb{i})})$ coordinate system, is given by

$$\boldsymbol{\kappa}^{(\mathbb{i})} \equiv \left(\frac{p}{|\mathbf{k}^{(\mathbb{i})}|},\frac{q}{|\mathbf{k}^{(\mathbb{i})}|},\sqrt{1-\frac{p^2+q^2}{|\mathbf{k}^{(\mathbb{i})}|^2}}\right). \tag{21}$$

C. Generalization of Fresnel formulas for transmission

The normal $\mathbf{n}$ to the interface and each unit vector $\boldsymbol{\kappa}^{(\mathbb{i})}$ form a plane that is different from the plane formed by $\mathbf{n}$ and $x_{\mathrm{p}}^{(\mathbb{i})}$. We introduce an axis $x_{\mathrm{p}'}^{(\mathbb{i})}$, whose positive direction is along $\boldsymbol{\kappa}^{(\mathbb{i})}$. We choose two other mutually perpendicular axes $x_{\mathrm{v}'}^{(\mathbb{i})}$ and $x_{\mathrm{h}'}^{(\mathbb{i})}$, which are perpendicular and parallel, respectively, to the plane formed by $\mathbf{n}$ and $x_{\mathrm{p}'}^{(\mathbb{i})}$ (see Fig. 4). The components of the electric field vector associated with the plane wave propagating along $\boldsymbol{\kappa}^{(\mathbb{i})}$ can be decomposed along the $x_{\mathrm{v}'}^{(\mathbb{i})}$ and $x_{\mathrm{h}'}^{(\mathbb{i})}$ directions and are denoted by $A_{\mathrm{v}'}^{(\mathbb{i})}(p,q;\omega)$ and $A_{\mathrm{h}'}^{(\mathbb{i})}(p,q;\omega)$, respectively. They are related to $A_{\mathrm{v}}^{(\mathbb{i})}(p,q;\omega)$ and $A_{\mathrm{h}}^{(\mathbb{i})}(p,q;\omega)$ by the formula (see Appendix A)

$$\begin{pmatrix} A_{\mathrm{v}'}^{(\mathbb{i})}(p,q;\omega) \\ A_{\mathrm{h}'}^{(\mathbb{i})}(p,q;\omega) \end{pmatrix} = \overleftrightarrow{\mathscr{U}}^{(\mathbb{i})}(p,q;\theta_{\mathbb{i}},\omega)\cdot\begin{pmatrix} A_{\mathrm{v}}^{(\mathbb{i})}(p,q;\omega) \\ A_{\mathrm{h}}^{(\mathbb{i})}(p,q;\omega) \end{pmatrix}, \tag{22}$$

where

$$\overleftrightarrow{\mathscr{U}}^{(\mathbb{i})} = \begin{pmatrix} \cos\alpha & \sin\alpha\cos\theta_{\mathbb{i}} \\ -\sin\alpha\cos\theta_{\mathbb{i}} & \cos\alpha\cos\tilde{\theta}_{\mathbb{i}}\cos\theta_{\mathbb{i}} + \sin\tilde{\theta}_{\mathbb{i}}\sin\theta_{\mathbb{i}} \end{pmatrix}, \tag{23}$$

and

$$\alpha = \tan^{-1}\left(-\frac{p}{q\cos\theta_{\mathbb{i}} - w\sin\theta_{\mathbb{i}}}\right). \tag{24}$$

The quantity $\tilde{\theta}_{\mathbb{i}}(p,q)$ in Eq. (23) denotes the angle between $\boldsymbol{\kappa}^{(\mathbb{i})}$ and $\mathbf{n}$ (see Fig. 4). Expressions for $\cos\tilde{\theta}_{\mathbb{i}}$ and $\sin\tilde{\theta}_{\mathbb{i}}$ can be obtained by evaluating the quantities $\boldsymbol{\kappa}^{(\mathbb{i})}\cdot\mathbf{n}$ and $|\boldsymbol{\kappa}^{(\mathbb{i})}\times\mathbf{n}|$, respectively (see Appendix B). In general, they are functions of both p and q. However, in the case under consideration, they may be approximated by functions of q only and are given by the formulas (see Appendix B)

$$\cos[\tilde{\theta}_{\mathbb{i}}(p,q)] \approx \cos[\tilde{\theta}_{\mathbb{i}}(q)] = \cos\theta_{\mathbb{i}} + \frac{q}{|\mathbf{k}^{(\mathbb{i})}|}\sin\theta_{\mathbb{i}}, \tag{25a}$$

$$\sin[\tilde{\theta}_{\mathbb{i}}(p,q)] \approx \sin[\tilde{\theta}_{\mathbb{i}}(q)] = \sin\theta_{\mathbb{i}} - \frac{q}{|\mathbf{k}^{(\mathbb{i})}|}\cos\theta_{\mathbb{i}}. \tag{25b}$$

Because $A_{\mathrm{v}}^{(\mathbb{i})}(p,q;\omega)$ and $A_{\mathrm{h}}^{(\mathbb{i})}(p,q;\omega)$ denote amplitudes of the field associated with plane waves, the field generated by their transmission can be obtained by direct application of Eqs. (7). They are given by the formula

$$A_{l'}^{(\mathbb{t})} = T_{l'}(n,n';p,q)A_{l'}^{(\mathbb{i})}(p,q;\omega), \tag{26}$$

where $l' = \mathrm{v}',\mathrm{h}'$, and

$$T_{\mathrm{v}'}(n,n';p,q) = \frac{2n\cos\tilde{\theta}_{\mathbb{i}}(p,q)}{n\cos\tilde{\theta}_{\mathbb{i}}(p,q) + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\tilde{\theta}_{\mathbb{i}}(p,q)}}, \tag{27a}$$

$$T_{\mathrm{h}'}(n,n';p,q) = \frac{2nn'\cos\tilde{\theta}_{\mathbb{i}}(p,q)}{\frac{\mu}{\mu'}n'^2\cos\tilde{\theta}_{\mathbb{i}}(p,q) + n\sqrt{n'^2 - n^2\sin^2\tilde{\theta}_{\mathbb{i}}(p,q)}}. \tag{27b}$$

If we introduce the matrix

$$\overleftrightarrow{\mathscr{T}}(n,n';p,q) = \begin{pmatrix} T_{\mathrm{v}'}(n,n';p,q) & 0 \\ 0 & T_{\mathrm{h}'}(n,n';p,q) \end{pmatrix}, \tag{28}$$

we may express Eq. (26) as

$$\begin{pmatrix} A_{\mathrm{v}'}^{(\mathbb{t})}(\tilde{p},\tilde{q};\omega) \\ A_{\mathrm{h}'}^{(\mathbb{t})}(\tilde{p},\tilde{q};\omega) \end{pmatrix} = \overleftrightarrow{\mathscr{T}}(n,n';p,q)\cdot\begin{pmatrix} A_{\mathrm{v}'}^{(\mathbb{i})}(p,q;\omega) \\ A_{\mathrm{h}'}^{(\mathbb{i})}(p,q;\omega) \end{pmatrix}. \tag{29}$$

If the angle of incidence is smaller than the critical angle, the transmitted plane-wave field components will combine to generate the electric field of the transmitted beam [11]. In the angular spectrum representation, the electric field components of the transmitted beam are then given by the formula

$$E_l^{(\mathbb{t})}(\mathbf{r},\omega) = \iint_{\tilde{p}^2+\tilde{q}^2\ll|\mathbf{k}^{(\mathbb{t})}|^2} A_l^{(\mathbb{t})}(\tilde{p},\tilde{q};\omega) \times \exp\left[i\left(\tilde{p}x_{\mathrm{v}}^{(\mathbb{t})} + \tilde{q}x_{\mathrm{h}}^{(\mathbb{t})} + \tilde{w}x_{\mathrm{p}}^{(\mathbb{t})}\right)\right]d\tilde{p}\,d\tilde{q}, \tag{30}$$

where (see Appendix D)

$$\tilde{p} = p, \quad \tilde{q} = \frac{\cos\theta_{\mathfrak{i}}}{\cos\theta_{\mathfrak{t}}} q, \tag{31}$$

and $\tilde{w} = \sqrt{|\mathbf{k}^{(\mathfrak{t})}|^2 - \tilde{p}^2 - \tilde{q}^2}$. The components $A_l^{(\mathfrak{t})}$ are related to the components $A_{l'}^{(\mathfrak{t})}$ used in Eqs. (26), by the formula (see Appendix E)

$$\begin{pmatrix} A_{\mathrm{v}}^{(\mathfrak{t})}(\tilde{p},\tilde{q};\omega) \\ A_{\mathrm{h}}^{(\mathfrak{t})}(\tilde{p},\tilde{q};\omega) \end{pmatrix} = \overleftrightarrow{\mathcal{U}}^{(\mathfrak{t})\dagger} \cdot \begin{pmatrix} A_{\mathrm{v}'}^{(\mathfrak{t})}(p,q;\omega) \\ A_{\mathrm{h}'}^{(\mathfrak{t})}(p,q;\omega) \end{pmatrix}, \tag{32}$$

where

$$\overleftrightarrow{\mathcal{U}}^{(\mathfrak{t})} = \begin{pmatrix} \cos\alpha & \sin\alpha\cos\theta_{\mathfrak{t}} \\ -\sin\alpha\cos\theta_{\mathfrak{t}} & \cos\alpha\cos\tilde{\theta}_{\mathfrak{t}}\cos\theta_{\mathfrak{t}} + \sin\tilde{\theta}_{\mathfrak{t}}\sin\theta_{\mathfrak{t}} \end{pmatrix}, \tag{33}$$

and α is given by Eq. (24). Using Eqs. (22), (29), and (32), one can express $A_l^{(\mathfrak{t})}$ in Eq. (30) in terms of $A_l^{(\mathfrak{i})}$ used in Eq. (20); one then finds that

$$\begin{pmatrix} A_{\mathrm{v}}^{(\mathfrak{t})} \\ A_{\mathrm{h}}^{(\mathfrak{t})} \end{pmatrix} = \overleftrightarrow{\mathcal{U}}^{(\mathfrak{t})\dagger} \cdot \overleftrightarrow{\mathcal{T}} \cdot \overleftrightarrow{\mathcal{U}}^{(\mathfrak{i})} \begin{pmatrix} A_{\mathrm{v}}^{(\mathfrak{i})} \\ A_{\mathrm{h}}^{(\mathfrak{i})} \end{pmatrix}. \tag{34}$$

Equation (34) may be rewritten in the following compact form by representing the column matrices by vectors:

$$\mathbf{A}^{(\mathfrak{t})} = (\overleftrightarrow{\mathcal{U}}^{(\mathfrak{t})\dagger} \cdot \overleftrightarrow{\mathcal{T}} \cdot \overleftrightarrow{\mathcal{U}}^{(\mathfrak{i})}) \cdot \mathbf{A}^{(\mathfrak{i})}. \tag{35}$$

With the help of the results that we just derived, we now determine the elements of the CSDM of the incident and of the transmitted beams.

Using Eqs. (8) and (20), one finds that the elements of the CSDM of the incident beam may be expressed in the form

$$\begin{aligned} W_{lm}^{(\mathfrak{i})}(\mathbf{r},\mathbf{r}';\omega) &= \iiiint dp\, dq\, dp'\, dq' \langle A_l^{(\mathfrak{i})*}(p,q;\omega) A_m^{(\mathfrak{i})}(p',q';\omega)\rangle \\ &\quad \times \exp\left[i\left\{\left(p' x_{\mathrm{v}'}^{(\mathfrak{i})} + q' x_{\mathrm{h}'}^{(\mathfrak{i})} + w' x_{\mathrm{p}'}^{(\mathfrak{i})}\right)\right.\right. \\ &\quad \left.\left. - \left(p x_{\mathrm{v}}^{(\mathfrak{i})} + q x_{\mathrm{h}}^{(\mathfrak{i})} + w x_{\mathrm{p}}^{(\mathfrak{i})}\right)\right\}\right], \end{aligned} \tag{36}$$

where $l = \mathrm{h,v}$ and $m = \mathrm{h,v}$, and the integrations extend over the domains $p^2 + q^2 \ll |\mathbf{k}^{(\mathfrak{i})}|^2$ and $p'^2 + q'^2 \ll |\mathbf{k}^{(\mathfrak{i})}|^2$. Let us now introduce a matrix $\overleftrightarrow{W}_A^{(\mathfrak{i})}(p,q,p',q';\omega)$, whose elements are given by the quantities $\langle A_l^{(\mathfrak{i})*}(p,q;\omega) A_m^{(\mathfrak{i})}(p',q';\omega)\rangle$; i.e., one has

$$\overleftrightarrow{W}_A^{(\mathfrak{i})}(p,q,p',q';\omega) = \langle \mathbf{A}^{(\mathfrak{i})*}(p,q;\omega) \cdot \mathbf{A}^{(\mathfrak{i})T}(p',q';\omega)\rangle. \tag{37}$$

Similarly one finds from Eqs. (8) and (30) that

$$\begin{aligned} W_{lm}^{(\mathfrak{t})}(\mathbf{r},\mathbf{r}';\omega) &= \iiiint d\tilde{p}\, d\tilde{q}\, d\tilde{p}'\, d\tilde{q}' \left\{ \overleftrightarrow{W}_A^{(\mathfrak{t})}(\tilde{p},\tilde{q},\tilde{p}',\tilde{q}';\omega)\right\}_{lm} \\ &\quad \times \exp\left[i\left\{\left(\tilde{p}' x_{\mathrm{v}'}^{(\mathfrak{t})} + \tilde{q}' x_{\mathrm{h}'}^{(\mathfrak{t})} + \tilde{w}' x_{\mathrm{p}'}^{(\mathfrak{t})}\right)\right.\right. \\ &\quad \left.\left. - \left(\tilde{p} x_{\mathrm{v}}^{(\mathfrak{t})} + \tilde{q} x_{\mathrm{h}}^{(\mathfrak{t})} + \tilde{w} x_{\mathrm{p}}^{(\mathfrak{t})}\right)\right\}\right], \end{aligned} \tag{38}$$

where $l = \mathrm{h,v}$, $m = \mathrm{h,v}$, and

$$\overleftrightarrow{W}_A^{(\mathfrak{t})}(\tilde{p},\tilde{q},\tilde{p}',\tilde{q}';\omega) = \langle \mathbf{A}^{(\mathfrak{t})*}(\tilde{p},\tilde{q};\omega) \cdot \mathbf{A}^{(\mathfrak{t})T}(\tilde{p}',\tilde{q}';\omega)\rangle. \tag{39}$$

Using Eqs. (35), (37), and (39), one readily finds that the matrices $\overleftrightarrow{W}_A^{(\mathfrak{t})}$ and $\overleftrightarrow{W}_A^{(\mathfrak{i})}$ are related by the formula

$$\begin{aligned} \overleftrightarrow{W}_A^{(\mathfrak{t})}(\tilde{p},\tilde{q},\tilde{p}',\tilde{q}';\omega) &= \overleftrightarrow{\mathcal{U}}_{\mathcal{T}}^{*}(p,q;n,n') \cdot \overleftrightarrow{W}_A^{(\mathfrak{i})}(p,q,p',q';\omega) \\ &\quad \cdot \overleftrightarrow{\mathcal{U}}_{\mathcal{T}}^{T}(p',q';n,n'), \end{aligned} \tag{40}$$

where $\tilde{p},\tilde{q}$, etc., are related to p,q, etc., by Eq. (31), and

$$\begin{aligned} \overleftrightarrow{\mathcal{U}}_{\mathcal{T}}(p,q;n,n') &= \{\overleftrightarrow{\mathcal{U}}^{(\mathfrak{t})}(p,q)\}^{\dagger} \cdot \overleftrightarrow{\mathcal{T}}(p,q;n,n') \\ &\quad \cdot \overleftrightarrow{\mathcal{U}}^{(\mathfrak{i})}(p,q). \end{aligned} \tag{41}$$

On substituting the expression for $\overleftrightarrow{W}_A^{(\mathfrak{t})}$ from Eq. (40) into Eq. (38), and on using Eq. (31), one can express the CSDM of the transmitted beam in terms of the parameters of the incident beam, and in terms of the parameters relating to properties of the two media.

From formulas (38)–(41) it is evident that the correlation properties of a partially coherent beam will, in general, change on transmission of the beam into another medium. In Sec. IV we illustrate this fact by an example.

D. Generalization of Fresnel formulas for reflection

One may follow a procedure similar to that used in Sec. III C to formulate the theory of reflection of partially coherent beams. In the $(x_{\mathrm{v}'}^{(\mathfrak{i})}, x_{\mathrm{h}'}^{(\mathfrak{i})}, x_{\mathrm{p}'}^{(\mathfrak{i})})$ coordinate system introduced in Sec. III C, the plane-wave components $A_{l'}^{(\mathfrak{r})}$ generated by reflection of the components $A_{l'}^{(\mathfrak{i})}$ are evidently given by the formula

$$A_{l'}^{(\mathfrak{r})} = R_{l'}(n,n';p,q) A_{l'}^{(\mathfrak{i})}(p,q;\omega), \tag{42}$$

where $l' = \mathrm{v}',\mathrm{h}'$, and

$$R_{\mathrm{v}'}(n,n';p,q) = \frac{n\cos\tilde{\theta}_{\mathfrak{i}}(p,q) - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\tilde{\theta}_{\mathfrak{i}}(p,q)}}{n\cos\tilde{\theta}_{\mathfrak{i}}(p,q) + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\tilde{\theta}_{\mathfrak{i}}(p,q)}}, \tag{43a}$$

$$R_{\mathrm{h}'}(n,n';p,q) = \frac{\frac{\mu}{\mu'}n'^2\cos\tilde{\theta}_{\mathfrak{i}}(p,q) - n\sqrt{n'^2 - n^2\sin^2\tilde{\theta}_{\mathfrak{i}}(p,q)}}{\frac{\mu}{\mu'}n'^2\cos\tilde{\theta}_{\mathfrak{i}}(p,q) + n\sqrt{n'^2 - n^2\sin^2\tilde{\theta}_{\mathfrak{i}}(p,q)}}. \tag{43b}$$

Let us introduce the matrix

$$\overleftrightarrow{\mathcal{R}}(n,n';p,q) = \begin{pmatrix} R_{\mathrm{v}'}(n,n';p,q) & 0 \\ 0 & R_{\mathrm{h}'}(n,n';p,q) \end{pmatrix}. \tag{44}$$

One can then rewrite Eq. (42) in the form

$$\begin{pmatrix} A_{\mathrm{v}'}^{(\mathfrak{r})}(\tilde{p},\tilde{q};\omega) \\ A_{\mathrm{h}'}^{(\mathfrak{r})}(\tilde{p},\tilde{q};\omega) \end{pmatrix} = \overleftrightarrow{\mathcal{R}}(n,n';p,q) \cdot \begin{pmatrix} A_{\mathrm{v}'}^{(\mathfrak{i})}(p,q;\omega) \\ A_{\mathrm{h}'}^{(\mathfrak{i})}(p,q;\omega) \end{pmatrix}. \tag{45}$$

The electric field components $A_{\mathrm{v}'}^{(\mathfrak{r})}$ and $A_{\mathrm{h}'}^{(\mathfrak{r})}$ may be expressed in the $(x_{\mathrm{v}}^{(\mathfrak{r})}, x_{\mathrm{h}}^{(\mathfrak{r})}, x_{\mathrm{p}}^{(\mathfrak{r})})$ coordinate system by the formula

(see Appendix E)

$$\begin{pmatrix} A_{\mathrm{v}}^{(\mathbb{r})} \\ A_{\mathrm{h}}^{(\mathbb{t})} \end{pmatrix} = \overleftrightarrow{\mathscr{U}}^{(\mathbb{r})\dagger} \cdot \begin{pmatrix} A_{\mathrm{v}'}^{(\mathbb{r})} \\ A_{\mathrm{h}'}^{(\mathbb{r})} \end{pmatrix}, \tag{46}$$

where

$$\overleftrightarrow{\mathscr{U}}^{(\mathbb{r})} = \begin{pmatrix} \cos\alpha & -\sin\alpha\cos\theta_{\mathbb{i}} \\ \sin\alpha\cos\theta_{\mathbb{i}} & \cos\alpha\cos\tilde{\theta}_{\mathbb{i}}\cos\theta_{\mathbb{i}} + \sin\tilde{\theta}_{\mathbb{i}}\sin\theta_{\mathbb{i}} \end{pmatrix}, \tag{47}$$

and α is given by Eq. (24). Using Eqs. (22), (45), and (46), one has

$$\begin{pmatrix} A_{\mathrm{v}}^{(\mathbb{r})} \\ A_{\mathrm{h}}^{(\mathbb{r})} \end{pmatrix} = \overleftrightarrow{\mathscr{U}}^{(\mathbb{r})\dagger} \cdot \overleftrightarrow{\mathcal{R}} \cdot \overleftrightarrow{\mathscr{U}}^{(\mathbb{i})} \begin{pmatrix} A_{\mathrm{v}}^{(\mathbb{i})} \\ A_{\mathrm{h}}^{(\mathbb{i})} \end{pmatrix}. \tag{48}$$

On representing the column matrices by vectors, one can rewrite Eq. (48) in the compact form

$$\mathbf{A}^{(\mathbb{r})} = (\overleftrightarrow{\mathscr{U}}^{(\mathbb{r})\dagger} \cdot \overleftrightarrow{\mathcal{R}} \cdot \overleftrightarrow{\mathscr{U}}^{(\mathbb{i})}) \cdot \mathbf{A}^{(\mathbb{i})}. \tag{49}$$

The field components $A_l^{(\mathbb{r})}$ will combine to generate the electric vector of the transmitted field. Hence, in the $(x_{\mathrm{v}}^{(\mathbb{r})}, x_{\mathrm{h}}^{(\mathbb{r})}, x_{\mathrm{p}}^{(\mathbb{r})})$ coordinate system, one can express the angular spectrum of the reflected electric field in the form (noting that $|\mathbf{k}^{(\mathbb{r})}| = |\mathbf{k}^{(\mathbb{i})}|$)

$$E_l^{(\mathbb{r})}(\mathbf{r},\omega) = \iint_{\bar{p}^2+\bar{q}^2 \ll |\mathbf{k}^{(\mathbb{i})}|^2} A_l^{(\mathbb{r})}(\bar{p},\bar{q};\omega) \times \exp\left[i\left(\bar{p}x_{\mathrm{v}}^{(\mathbb{r})} + \bar{q}x_{\mathrm{h}}^{(\mathbb{r})} + \bar{w}x_{\mathrm{p}}^{(\mathbb{r})}\right)\right] d\bar{p}\, d\bar{q}, \tag{50}$$

where (see Appendix F)

$$\bar{p} = p, \quad \bar{q} = -q, \tag{51}$$

implying that $\bar{w} = \sqrt{|\mathbf{k}^{(\mathbb{i})}|^2 - \bar{p}^2 - \bar{q}^2} = w$.

Following the same procedure as that we used in Sec. III C, one can express the CSDM of the reflected beam at a pair of points $(\mathbf{r},\mathbf{r}')$ at the interface by the formula

$$\begin{aligned} W_{lm}^{(\mathbb{r})}(\mathbf{r},\mathbf{r}';\omega) &= \iiiint d\bar{p}\, d\bar{q}\, d\bar{p}'\, d\bar{q}' \left\{ \overleftrightarrow{W}_A^{(\mathbb{r})}(\bar{p},\bar{q},\bar{p}',\bar{q}';\omega) \right\}_{lm} \\ &\quad \times \exp\left[i\left\{ \left(\bar{p}'x_{\mathrm{v}'}^{(\mathbb{r})} + \bar{q}'x_{\mathrm{h}'}^{(\mathbb{r})} + \bar{w}'x_{\mathrm{p}'}^{(\mathbb{r})}\right) - \left(\bar{p}x_{\mathrm{v}}^{(\mathbb{r})} + \bar{q}x_{\mathrm{h}}^{(\mathbb{r})} \right.\right.\right. \\ &\quad \left.\left.\left. + \bar{w}x_{\mathrm{p}}^{(\mathbb{r})}\right)\right\}\right], \end{aligned} \tag{52}$$

where $l = \mathrm{h,v}$, $m = \mathrm{h,v}$; the integrations are carried out throughout the regions $\bar{p}^2 + \bar{q}^2 \ll |\mathbf{k}^{(\mathbb{i})}|^2$, $\bar{p}'^2 + \bar{q}'^2 \ll |\mathbf{k}^{(\mathbb{i})}|^2$, and

$$\overleftrightarrow{W}_A^{(\mathbb{r})}(\bar{p},\bar{q},\bar{p}',\bar{q}';\omega) = \langle \mathbf{A}^{(\mathbb{r})*}(\bar{p},\bar{q};\omega) \cdot \mathbf{A}^{(\mathbb{r})T}(\bar{p}',\bar{q}';\omega) \rangle. \tag{53}$$

Using Eqs. (37), (49), and (53), one immediately finds that the matrices $\overleftrightarrow{W}_A^{(\mathbb{r})}$ and $\overleftrightarrow{W}_A^{(\mathbb{i})}$ are related by the formula

$$\overleftrightarrow{W}_A^{(\mathbb{r})}(\bar{p},\bar{q},\bar{p}',\bar{q}';\omega) = \overleftrightarrow{\mathscr{U}}_{\mathcal{R}}^{*}(p,q;n,n') \cdot \overleftrightarrow{W}_A^{(\mathbb{i})}(p,q,p',q';\omega) \cdot \overleftrightarrow{\mathscr{U}}_{\mathcal{R}}^{T}(p',q';n,n'), \tag{54}$$

where

$$\overleftrightarrow{\mathscr{U}}_{\mathcal{R}}(p,q;n,n') = \{\overleftrightarrow{\mathscr{U}}^{(\mathbb{r})}(p,q)\}^{\dagger} \cdot \overleftrightarrow{\mathcal{R}}(p,q;n,n') \cdot \overleftrightarrow{\mathscr{U}}^{(\mathbb{i})}(p,q). \tag{55}$$

Using Eqs. (51), (52), and (54), one can determine the CSDM of the reflected beam, and one can then study the changes in the coherence properties of the field produced on reflection.

IV. CHANGE OF THE COHERENCE PROPERTIES OF A LIGHT BEAM ON REFRACTION AND REFLECTION

Let us first note that the following conditions hold for any optical beam whose CSDM is given by Eq. (36):

$$w \approx |\mathbf{k}^{(\mathbb{i})}|\left(1 - \frac{p^2+q^2}{2|\mathbf{k}^{(\mathbb{i})}|^2}\right), \quad w' \approx |\mathbf{k}^{(\mathbb{i})}|\left(1 - \frac{p'^2+q'^2}{2|\mathbf{k}^{(\mathbb{i})}|^2}\right). \tag{56}$$

On substituting from Eq. (56) into Eq. (36), making the changes of variables $p \to -p$, $q \to -q$, then introducing the two-dimensional vectors $\mathbf{f} \equiv (p,q)$ and $\mathbf{f}' \equiv (p',q')$, and finally using Eq. (37), one obtains the following expression for the matrix elements $W_{lm}^{(\mathbb{i})}$ [12]:

$$\begin{aligned} W_{lm}^{(\mathbb{i})}(\mathbf{r},\mathbf{r}';\omega) &= \exp\left[i|\mathbf{k}^{(\mathbb{i})}|\left(x_{\mathrm{p}}^{\prime(\mathbb{i})} - x_{\mathrm{p}}^{(\mathbb{i})}\right)\right] \iint d^2f\, d^2f' \\ &\quad \left\{\overleftrightarrow{W}_A^{(\mathbb{i})}(-\mathbf{f},\mathbf{f}';\omega)\right\}_{lm} \exp\left[\frac{i}{2|\mathbf{k}^{(\mathbb{i})}|}\left\{|\mathbf{f}|^2 x_{\mathrm{p}}^{(\mathbb{i})} - |\mathbf{f}'|^2 x_{\mathrm{p}}^{\prime(\mathbb{i})}\right\}\right] \\ &\quad \times \exp[i\{\mathbf{f}\cdot\boldsymbol{\rho}^{(\mathbb{i})} + \mathbf{f}'\cdot\boldsymbol{\rho}^{\prime(\mathbb{i})}\}]. \end{aligned} \tag{57}$$

It can readily be shown from Eq. (57) that $\overleftrightarrow{W}_A^{(\mathbb{i})}(-\mathbf{f},\mathbf{f}';\omega)$ is just the Fourier transform of the CSDM of the incident beam at the source plane, i.e., that (cf. Ref. [9], Sec. 5.6)

$$\overleftrightarrow{W}_A^{(\mathbb{i})}(-\mathbf{f},\mathbf{f}';\omega) = \left(\frac{1}{2\pi}\right)^4 \iint d^2\rho_0 d^2\rho_0' \, \overleftrightarrow{W}^{(\mathbb{i})}(\boldsymbol{\rho}_0,\boldsymbol{\rho}_0';\omega) \times \exp[-i(\mathbf{f}\cdot\boldsymbol{\rho}_0 + \mathbf{f}'\cdot\boldsymbol{\rho}_0')], \tag{58}$$

where $\boldsymbol{\rho}_0$ and $\boldsymbol{\rho}_0'$ are two-dimensional position vectors representing a pair of points on the source plane. For a given source, $\overleftrightarrow{W}^{(\mathbb{i})}(\boldsymbol{\rho}_0,\boldsymbol{\rho}_0';\omega)$ will be known. Using Eqs. (57) and (58), one can then determine $\overleftrightarrow{W}^{(\mathbb{i})}(\mathbf{r},\mathbf{r}';\omega)$ and, consequently, the coherence properties of the incident beam at the interface of the two media. From Eqs. (38), (40), (41), and (58), one can also determine the CSDM of the transmitted beam at the interface, and one can, therefore, study its coherence properties. Similarly, from Eqs. (52), (54), (55), and (58) one obtains an expression for the CSDM of the reflected beam at the interface, and one can then study its coherence properties also.

As an example, let us consider a light beam generated by a so-called Gaussian Schell-model source. The elements of the CSDM at the source plane are given by the formula (cf. Ref. [2], Sec. 9.4.2)

$$W_{lm}^{(\mathbb{i})}(\boldsymbol{\rho}_0,\boldsymbol{\rho}_0';\omega) = A_l A_m B_{lm} \exp\left[-\frac{\rho_0^2+\rho_0'^2}{4\sigma^2}\right] \times \exp\left[-\frac{(\boldsymbol{\rho}_0'-\boldsymbol{\rho}_0)^2}{2\delta^2}\right], \tag{59}$$

where the parameters A_l, B_{lm}, σ, and δ are independent of position. However, they cannot be chosen arbitrarily, and, in our case, the following relations must hold (see, for example,

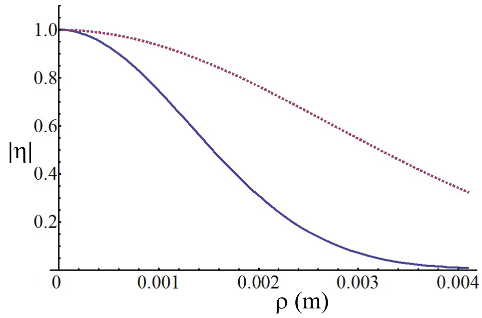

FIG. 5. (Color online) Change of the coherence properties on refraction: the modulus $|\eta|$ of the degree of coherence of the incident (solid line) and of the transmitted (dotted line) beams are plotted as functions of $\rho = |\mathbf{r}' - \mathbf{r}|$, at $\omega \approx 3.2 \times 10^{15}\ \text{s}^{-1}$, for values of the parameters $\theta_{\mathbb{i}} = 40^\circ$, $\delta = 0.001$ m, $\sigma = 0.01$ m, $A_{\rm v}/A_{\rm h} = 1$, and $B_{\rm hv} = 9/16$.

Refs. [13,14]):

$$B_{lm} = B^*_{ml}, \quad B_{lm} = 1 \quad \text{when} \quad l = m,$$
$$|B_{lm}| \leqslant 1 \quad \text{when} \quad l \neq m. \tag{60}$$

Suppose now that the beam generated by such a source propagates initially in air (refractive index $n \approx 1$) and is then incident on a planar surface of a flint glass slab (refractive index $n' = 1.62$). We choose the center of the source to be 1 m away from the point where the incident-beam axis intersects the interface. We calculate the degree of coherence of the incident, of the transmitted, and of the reflected beams at a pair of points $(\mathbf{r},\mathbf{r}')$ at the interface, at a frequency $\omega \approx 3.2 \times 10^{15}\ \text{s}^{-1}$, when the parameters are chosen as follows: $\delta = 0.001$ m, $\sigma = 0.01$ m, $A_{\rm h} = A_{\rm v} = 1$ (in suitable units), and $B_{\rm hv} = 9/16$ (in suitable units). We choose the point $\mathbf{r}$ as the point of intersection of the incident-beam axis with the interface, and we choose $\mathbf{r}'$ to represent a variable point along the negative y axis (see Fig. 7). We assume $|\mathbf{r}' - \mathbf{r}| = \rho$.

The moduli of the degree of coherence of the incident and of the transmitted beams are plotted in Fig. 5 as functions of ρ. The figure shows that the state of coherence of the light beam has changed on refraction. Figure 6 shows the difference between the moduli of the degrees of coherence of the incident and of the reflected beams.

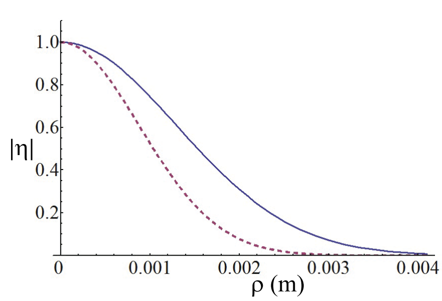

FIG. 6. (Color online) Change of the coherence properties on reflection: the modulus $|\eta|$ of the degree of coherence of the incident (solid line) and of the reflected (dotted line) beams are plotted as functions of $\rho = |\mathbf{r}' - \mathbf{r}|$, for the same choice of parameters as used in Fig. 5.

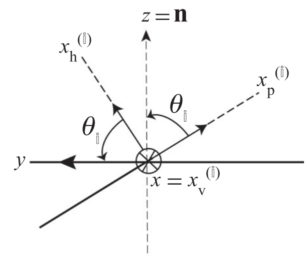

FIG. 7. The geometry relating to the relationship between the $(x_{\rm v}^{(\mathbb{i})}, x_{\rm h}^{(\mathbb{i})}, x_{\rm p}^{(\mathbb{i})})$ and the (x,y,z) coordinate systems.

V. SUMMARY

We have developed a theory of refraction and reflection with partially coherent electromagnetic beams. Application of the theory shows that if a partially coherent light beam is refracted or reflected at a surface separating two media, its coherence properties, in general, change. This fact is illustrated by an example in Figs. 5 and 6.

ACKNOWLEDGMENTS

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APPENDIX A: DERIVATION OF EQS. (22)–(24)

The derivation makes use of the fact that the coordinate systems $(x_{\rm v}^{(\mathbb{i})}, x_{\rm h}^{(\mathbb{i})}, x_{\rm p}^{(\mathbb{i})})$ and $(x_{\rm v'}^{(\mathbb{i})}, x_{\rm h'}^{(\mathbb{i})}, x_{\rm p'}^{(\mathbb{i})})$ can be related by rotations around three different axes, making it possible to introduce a unitary transformation matrix $\overleftrightarrow{U}^{(\mathbb{i})}(p,q;\theta_{\mathbb{i}},\omega)$, which is the product of three rotation matrices. Since the quantities $A_l^{(\mathbb{i})}(p,q;\omega)$ are components of a vector (tensor of rank one), their transformation laws are governed by the same unitary matrix.

Let us introduce a coordinate system (x,y,z), with the x axis chosen along the $x_{\rm v}^{(\mathbb{i})}$ axis, the z axis chosen along $\mathbf{n}$, and the y axis chosen following the right-hand rule (see. Fig. 7). We refer to (x,y,z) as the "interface coordinate system for the incident beam axis." One can readily see that the $(x_{\rm v}^{(\mathbb{i})}, x_{\rm h}^{(\mathbb{i})}, x_{\rm p}^{(\mathbb{i})})$ coordinate system can be obtained from the (x,y,z) system by a clockwise rotation through an angle $\theta_{\mathbb{i}}$ around the $x_{\rm v}^{(\mathbb{i})}$ axis (see Fig. 7). The coordinates are, therefore, related by the formula

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{\mathbb{i}} & -\sin\theta_{\mathbb{i}} \\ 0 & \sin\theta_{\mathbb{i}} & \cos\theta_{\mathbb{i}} \end{pmatrix} \begin{pmatrix} x_{\rm v}^{(\mathbb{i})} \\ x_{\rm h}^{(\mathbb{i})} \\ x_{\rm p}^{(\mathbb{i})} \end{pmatrix}. \tag{A1}$$

We denote the transformation matrix in Eq. (A1) by $\overleftrightarrow{U}_1^{(\mathbb{i})}(\theta_{\mathbb{i}})$. It depends on the orientation of the axis of the incident beam.

Similarly, one can introduce another interface coordinate system (x',y',z') associated with the wave vector $\boldsymbol{\kappa}^{(\mathbb{i})}$. In this case, we choose the x axis along the $x_{\rm v'}^{(\mathbb{i})}$ axis, the z axis again along $\mathbf{n}$, and the y axis in accordance with the right-hand rule (see Fig. 4). Hence, the coordinate systems $(x_{\rm v'}^{(\mathbb{i})}, x_{\rm h'}^{(\mathbb{i})}, x_{\rm p'}^{(\mathbb{i})})$ and

(x',y',z') are related by the formula

$$\begin{pmatrix} x_{v'}^{(\mathbb{i})} \\ x_{h'}^{(\mathbb{i})} \\ x_{p'}^{(\mathbb{i})} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\tilde{\theta}_{\mathbb{i}} & \sin\tilde{\theta}_{\mathbb{i}} \\ 0 & -\sin\tilde{\theta}_{\mathbb{i}} & \cos\tilde{\theta}_{\mathbb{i}} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}, \tag{A2}$$

where $\tilde{\theta}_{\mathbb{i}}$ is the angle between $\boldsymbol{\kappa}^{(\mathbb{i})}$ and $\mathbf{n}$. Expressions for $\cos\tilde{\theta}_{\mathbb{i}}$ and $\sin\tilde{\theta}_{\mathbb{i}}$ are derived in Appendix B. We denote the transformation matrix in Eq. (A2) by $\overleftrightarrow{U}_3^{(\mathbb{i})}(p,q;\omega)$.

We note that the coordinate system (x',y',z') can be obtained by counterclockwise rotation through an angle α, say, around the z axis of the (x,y,z) coordinate system (see Fig. 8), where (see Appendix C)

$$\alpha = \tan^{-1}\left(-\frac{p}{q\cos\theta_{\mathbb{i}} - w\sin\theta_{\mathbb{i}}}\right). \tag{A3}$$

Hence,

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{A4}$$

We denote the transformation matrix in Eq. (A4) by $\overleftrightarrow{U}_2^{(\mathbb{i})}(p,q)$.

Combining Eqs. (A1), (A2), and (A4), one obtains the following law of transformation from the coordinate system $(x_v^{(\mathbb{i})},x_h^{(\mathbb{i})},x_p^{(\mathbb{i})})$ to the coordinate system $(x_{v'}^{(\mathbb{i})},x_{h'}^{(\mathbb{i})},x_{p'}^{(\mathbb{i})})$:

$$\begin{pmatrix} x_{v'}^{(\mathbb{i})} \\ x_{h'}^{(\mathbb{i})} \\ x_{p'}^{(\mathbb{i})} \end{pmatrix} = \overleftrightarrow{U}^{(\mathbb{i})}(p,q;\theta_{\mathbb{i}},\omega)\cdot \begin{pmatrix} x_{v}^{(\mathbb{i})} \\ x_{h}^{(\mathbb{i})} \\ x_{p}^{(\mathbb{i})} \end{pmatrix}, \tag{A5}$$

with

$$\begin{aligned} \overleftrightarrow{U}^{(\mathbb{i})}(p,q;\theta_{\mathbb{i}},\omega) &= \overleftrightarrow{U}_3^{(\mathbb{i})}(p,q;\omega)\cdot\overleftrightarrow{U}_2^{(\mathbb{i})}(p,q)\cdot\overleftrightarrow{U}_1^{(\mathbb{i})}(\theta_{\mathbb{i}}) \\ &= \begin{pmatrix} \cos\alpha & \sin\alpha\cos\theta_{\mathbb{i}} & -\sin\alpha\sin\theta_{\mathbb{i}} \\ -\sin\alpha\cos\tilde{\theta}_{\mathbb{i}} & \cos\alpha\cos\tilde{\theta}_{\mathbb{i}}\cos\theta_{\mathbb{i}} + \sin\tilde{\theta}_{\mathbb{i}}\sin\theta_{\mathbb{i}} & \sin\tilde{\theta}_{\mathbb{i}}\cos\theta_{\mathbb{i}} - \cos\alpha\cos\tilde{\theta}_{\mathbb{i}}\sin\theta_{\mathbb{i}} \\ \sin\alpha\sin\tilde{\theta}_{\mathbb{i}} & \cos\tilde{\theta}_{\mathbb{i}}\sin\theta_{\mathbb{i}} - \cos\alpha\sin\tilde{\theta}_{\mathbb{i}}\cos\theta_{\mathbb{i}} & \cos\tilde{\theta}_{\mathbb{i}}\cos\theta_{\mathbb{i}} + \cos\alpha\sin\tilde{\theta}_{\mathbb{i}}\sin\theta_{\mathbb{i}} \end{pmatrix}. \end{aligned} \tag{A6}$$

$\overleftrightarrow{U}^{(\mathbb{i})}$ is a unitary matrix, and, therefore, the inverse transformation matrix is given by $\{\overleftrightarrow{U}^{(\mathbb{i})}\}^{-1} = \{\overleftrightarrow{U}^{(\mathbb{i})}\}^{\dagger}$.

Transformation of the field components $A_l^{(\mathbb{i})}(p,q;\omega)$ is governed by the equations

$$\begin{pmatrix} A_{v'}^{(\mathbb{i})}(p,q;\omega) \\ A_{h'}^{(\mathbb{i})}(p,q;\omega) \\ A_{p'}^{(\mathbb{i})}(p,q;\omega) \end{pmatrix} = \overleftrightarrow{U}^{(\mathbb{i})}(p,q;\theta_{\mathbb{i}},\omega)\cdot \begin{pmatrix} A_{v}^{(\mathbb{i})}(p,q;\omega) \\ A_{h}^{(\mathbb{i})}(p,q;\omega) \\ A_{p}^{(\mathbb{i})}(p,q;\omega) \end{pmatrix}, \tag{A7a}$$

$$\begin{pmatrix} A_{v}^{(\mathbb{i})}(p,q;\omega) \\ A_{h}^{(\mathbb{i})}(p,q;\omega) \\ A_{p}^{(\mathbb{i})}(p,q;\omega) \end{pmatrix} = \{\overleftrightarrow{U}^{(\mathbb{i})}(p,q;\theta_{\mathbb{i}},\omega)\}^{\dagger}\cdot \begin{pmatrix} A_{v'}^{(\mathbb{i})}(p,q;\omega) \\ A_{h'}^{(\mathbb{i})}(p,q;\omega) \\ A_{p'}^{(\mathbb{i})}(p,q;\omega) \end{pmatrix}. \tag{A7b}$$

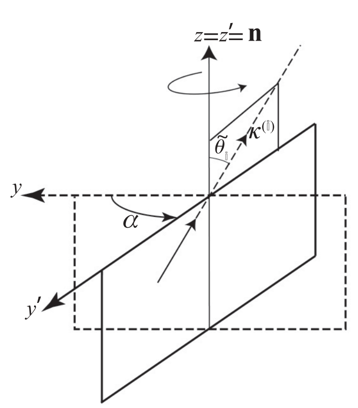

FIG. 8. The transformation from the (x,y,z) to the (x',y',z') coordinate system.

We note that $A_{p'}^{(\mathbb{i})}(p,q;\omega)=0$, and because of the beamlike nature of the incident field, one can assume that $A_{p}^{(\mathbb{i})}(p,q;\omega)\approx 0$. Hence, the transformation equation (A7a) reduces to

$$A_{v'}^{(\mathbb{i})} = A_{v}^{(\mathbb{i})}\cos\alpha + A_{h}^{(\mathbb{i})}\sin\alpha\cos\theta_{\mathbb{i}}, \tag{A8a}$$

$$\begin{aligned} A_{h'}^{(\mathbb{i})} &= -A_{v}^{(\mathbb{i})}\sin\alpha\cos\tilde{\theta}_{\mathbb{i}} \\ &\quad + A_{h}^{(\mathbb{i})}(\cos\alpha\cos\tilde{\theta}_{\mathbb{i}}\cos\theta_{\mathbb{i}} + \sin\tilde{\theta}_{\mathbb{i}}\sin\theta_{\mathbb{i}}), \end{aligned} \tag{A8b}$$

where, for the sake of brevity, we have not shown the dependence on the various variables and parameters. Similarly Eq. (A7b) reduces to

$$A_{v}^{(\mathbb{i})} = A_{v'}^{(\mathbb{i})}\cos\alpha - A_{h'}^{(\mathbb{i})}\sin\alpha\cos\theta_{\mathbb{i}}, \tag{A9a}$$

$$\begin{aligned} A_{h}^{(\mathbb{i})} &= A_{v'}^{(\mathbb{i})}\sin\alpha\cos\tilde{\theta}_{\mathbb{i}} \\ &\quad + A_{h'}^{(\mathbb{i})}(\cos\alpha\cos\tilde{\theta}_{\mathbb{i}}\cos\theta_{\mathbb{i}} + \sin\tilde{\theta}_{\mathbb{i}}\sin\theta_{\mathbb{i}}). \end{aligned} \tag{A9b}$$

We may now introduce the 2×2 matrix

$$\overleftrightarrow{\mathscr{U}}^{(\mathbb{i})} = \begin{pmatrix} \cos\alpha & \sin\alpha\cos\theta_{\mathbb{i}} \\ -\sin\alpha\cos\theta_{\mathbb{i}} & \cos\alpha\cos\tilde{\theta}_{\mathbb{i}}\cos\theta_{\mathbb{i}} + \sin\tilde{\theta}_{\mathbb{i}}\sin\theta_{\mathbb{i}} \end{pmatrix}. \tag{A10}$$

The matrices $\overleftrightarrow{\mathscr{U}}^{(\mathbb{i})}$ and $\{\overleftrightarrow{\mathscr{U}}^{(\mathbb{i})}\}^{\dagger}$ govern the transformation equations represented by Eqs. (A8) and (A9), respectively. The matrix $\overleftrightarrow{\mathscr{U}}^{(\mathbb{i})}$ is not, in general, unitary. However, for a paraxial beam both α and $(\theta_{\mathbb{i}}-\tilde{\theta}_{\mathbb{i}})$ will be small enough to assume that $\cos(\theta_{\mathbb{i}}-\tilde{\theta}_{\mathbb{i}})\approx 1$, $\cos\alpha\approx 1$, and $\sin\alpha\approx\alpha\to 0$. With this approximation, one has $\overleftrightarrow{\mathscr{U}}^{(\mathbb{i})}\cdot\{\overleftrightarrow{\mathscr{U}}^{(\mathbb{i})}\}^{\dagger}\approx 1$. Although this approximation is not essential, it is likely to be useful for rough estimates in some cases.

APPENDIX B: DERIVATION OF EQ. (25)

In terms of the incident-beam coordinate system $(x_{\rm v}^{(\mathbb{i})},x_{\rm h}^{(\mathbb{i})},x_{\rm p}^{(\mathbb{i})})$, the normal to the surface of the interface is represented by the unit vector $\mathbf{n} = (0, \sin\theta_{\mathbb{i}}, \cos\theta_{\mathbb{i}})$. Using expression (21) for $\boldsymbol{\kappa}^{(\mathbb{i})}$, one can now show that

$$\cos[\tilde{\theta}_{\mathbb{i}}(p,q)] = \boldsymbol{\kappa}^{(\mathbb{i})}\cdot\mathbf{n} = \frac{q}{|\mathbf{k}^{(\mathbb{i})}|}\sin\theta_{\mathbb{i}} + \sqrt{1-\frac{p^2+q^2}{|\mathbf{k}^{(\mathbb{i})}|^2}}\cos\theta_{\mathbb{i}}. \tag{B1}$$

Using the beam condition (19), one has

$$\sqrt{1-\frac{p^2+q^2}{|\mathbf{k}^{(\mathbb{i})}|^2}} \approx 1-\frac{p^2+q^2}{2|\mathbf{k}^{(\mathbb{i})}|^2}. \tag{B2}$$

Using Eqs. (B1) and (B2) one finds that

$$\cos[\tilde{\theta}_{\mathbb{i}}(p,q)] \approx \frac{q}{|\mathbf{k}^{(\mathbb{i})}|}\sin\theta_{\mathbb{i}} + \left(1-\frac{p^2+q^2}{2|\mathbf{k}^{(\mathbb{i})}|^2}\right)\cos\theta_{\mathbb{i}}. \tag{B3}$$

If one neglects second-order terms, one has

$$\cos[\tilde{\theta}_{\mathbb{i}}(p,q)] \approx \cos\theta_{\mathbb{i}} + \frac{q}{|\mathbf{k}^{(\mathbb{i})}|}\sin\theta_{\mathbb{i}} \equiv \cos[\tilde{\theta}_{\mathbb{i}}(q)]. \tag{B4}$$

From Eq. (B4), one can readily show by applying the beam condition that

$$\sin[\tilde{\theta}_{\mathbb{i}}(p,q)] \approx \sin\theta_{\mathbb{i}} - \frac{q}{|\mathbf{k}^{(\mathbb{i})}|}\cos\theta_{\mathbb{i}} \equiv \sin[\tilde{\theta}_{\mathbb{i}}(q)]. \tag{B5}$$

APPENDIX C: EXPRESSION FOR THE ANGLE α

In terms of the $(x_{\rm v}^{(\mathbb{i})},x_{\rm h}^{(\mathbb{i})},x_{\rm p}^{(\mathbb{i})})$ coordinate system, the unit vector $\boldsymbol{\kappa}^{(\mathbb{i})}$ is given by the components [cf. Eq. (21)]

$$\boldsymbol{\kappa}^{(\mathbb{i})} \equiv \left(\kappa_{\rm v}^{(\mathbb{i})},\kappa_{\rm h}^{(\mathbb{i})},\kappa_{\rm p}^{(\mathbb{i})}\right) = \left(\frac{p}{|\mathbf{k}^{(\mathbb{i})}|},\frac{q}{|\mathbf{k}^{(\mathbb{i})}|},\frac{w}{|\mathbf{k}^{(\mathbb{i})}|}\right). \tag{C1}$$

In the $(x_{\rm v'}^{(\mathbb{i})},x_{\rm h'}^{(\mathbb{i})},x_{\rm p'}^{(\mathbb{i})})$ coordinate system, this unit vector has the components

$$\boldsymbol{\kappa}^{(\mathbb{i})} \equiv \left(\kappa_{\rm v'}^{(\mathbb{i})},\kappa_{\rm h'}^{(\mathbb{i})},\kappa_{\rm p'}^{(\mathbb{i})}\right) = (0,0,1). \tag{C2}$$

Since $\boldsymbol{\kappa}^{(\mathbb{i})}$ is a vector, its components in the above two coordinate systems are related by the transformation law given in Eq. (A5). On using Eq. (A6), and the fact that $\kappa_{\rm v'}^{(\mathbb{i})} = 0$, one readily finds that

$$\alpha = \tan^{-1}\left(-\frac{p}{q\cos\theta_{\mathbb{i}} - w\sin\theta_{\mathbb{i}}}\right). \tag{C3}$$

APPENDIX D: DERIVATION OF EQ. (31)

Equation (30) shows that the transmitted field components $E_l^{(\mathbb{t})}$ are generated by superposition of plane-wave components with amplitudes $A_l^{(\mathbb{t})}$. It is evident from formula (30) that each plane-wave component propagates along a direction specified by a unit vector $\boldsymbol{\kappa}^{(\mathbb{t})}$ (say), which in the $(x_{\rm v}^{(\mathbb{t})},x_{\rm h}^{(\mathbb{t})},x_{\rm p}^{(\mathbb{t})})$ coordinate system is given by

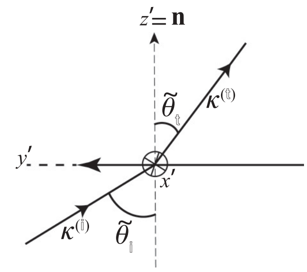

FIG. 9. The geometry showing that the vectors $\boldsymbol{\kappa}^{(\mathbb{i})}$, $\boldsymbol{\kappa}^{(\mathbb{t})}$, and $\mathbf{n}$ lie in the y'-z' plane.

$$\boldsymbol{\kappa}^{(\mathbb{t})} \equiv \left(\frac{\tilde{p}}{|\mathbf{k}^{(\mathbb{t})}|},\frac{\tilde{q}}{|\mathbf{k}^{(\mathbb{t})}|},\frac{\tilde{w}}{|\mathbf{k}^{(\mathbb{t})}|}\right), \tag{D1}$$

where $\tilde{w} = \sqrt{|\mathbf{k}^{(\mathbb{t})}|^2 - \tilde{p}^2 - \tilde{q}^2}$.

We note that $\boldsymbol{\kappa}^{(\mathbb{i})}$, $\boldsymbol{\kappa}^{(\mathbb{t})}$, and $\mathbf{n}$ lie on the y'-z' plane (see Fig. 9). Hence, in the (x',y',z') coordinate system, one has

$$\boldsymbol{\kappa}^{(\mathbb{i})} \equiv \left(\kappa_{x'}^{(\mathbb{i})},\kappa_{y'}^{(\mathbb{i})},\kappa_{z'}^{(\mathbb{i})}\right) = (0, -\sin\tilde{\theta}_{\mathbb{i}}, \cos\tilde{\theta}_{\mathbb{i}}), \tag{D2a}$$

$$\boldsymbol{\kappa}^{(\mathbb{t})} \equiv \left(\kappa_{x'}^{(\mathbb{t})},\kappa_{y'}^{(\mathbb{t})},\kappa_{z'}^{(\mathbb{t})}\right) = (0, -\sin\tilde{\theta}_{\mathbb{t}}, \cos\tilde{\theta}_{\mathbb{t}}). \tag{D2b}$$

Applying the transformation law (A4) of Appendix A, one now has

$$\kappa_x^{(\mathbb{i})} = \sin\alpha\sin\tilde{\theta}_{\mathbb{i}}, \quad \kappa_y^{(\mathbb{i})} = -\cos\alpha\sin\tilde{\theta}_{\mathbb{i}}, \tag{D3a}$$

$$\kappa_x^{(\mathbb{t})} = \sin\alpha\sin\tilde{\theta}_{\mathbb{t}}, \quad \kappa_y^{(\mathbb{t})} = -\cos\alpha\sin\tilde{\theta}_{\mathbb{t}}. \tag{D3b}$$

Snell's law of refraction implies that

$$\sin\tilde{\theta}_{\mathbb{t}} = \frac{|\mathbf{k}^{(\mathbb{i})}|}{|\mathbf{k}^{(\mathbb{t})}|}\sin\tilde{\theta}_{\mathbb{i}}. \tag{D4}$$

On using Eqs. (D3) and (D4) one readily obtains the following relations:

$$\kappa_x^{(\mathbb{t})} = \frac{|\mathbf{k}^{(\mathbb{i})}|}{|\mathbf{k}^{(\mathbb{t})}|}\kappa_x^{(\mathbb{i})}, \quad \kappa_y^{(\mathbb{t})} = \frac{|\mathbf{k}^{(\mathbb{i})}|}{|\mathbf{k}^{(\mathbb{t})}|}\kappa_y^{(\mathbb{i})}. \tag{D5}$$

Expressions for $\kappa_x^{(\mathbb{i})}$ and $\kappa_y^{(\mathbb{i})}$ can be obtained from Eqs. (21) and (A1) and are found to be

$$\kappa_x^{(\mathbb{i})} = \frac{p}{|\mathbf{k}^{(\mathbb{i})}|}, \quad \kappa_y^{(\mathbb{i})} = \frac{q}{|\mathbf{k}^{(\mathbb{i})}|}\cos\theta_{\mathbb{i}} - \sin\theta_{\mathbb{i}}, \tag{D6}$$

where, in obtaining the expression for $\kappa_y^{(\mathbb{i})}$, we used the fact that $p^2 + q^2 \ll |\mathbf{k}^{(\mathbb{i})}|^2$. In a similar way, one can obtain expressions for $\kappa_x^{(\mathbb{t})}$ and $\kappa_x^{(\mathbb{t})}$, using Eqs. (A1) and (D1). One then finds that

$$\kappa_x^{(\mathbb{t})} = \frac{\tilde{p}}{|\mathbf{k}^{(\mathbb{t})}|}, \quad \kappa_y^{(\mathbb{t})} = \frac{\tilde{q}}{|\mathbf{k}^{(\mathbb{t})}|} \cos\theta_{\mathbb{t}} - \sin\theta_{\mathbb{t}}. \tag{D7}$$

Finally, using Eqs. (6), (15b), (D5), (D6), and (D7), one readily obtains the formulas

$$\tilde{p} = p, \quad \tilde{q} = \frac{\cos\theta_{\mathbb{i}}}{\cos\theta_{\mathbb{t}}} q. \tag{D8}$$

APPENDIX E: DERIVATION OF EQS. (32), (33), (46), AND (47)

Since the quantities $A_l^{\mathbb{t}}(p,q;\omega)$ are components of a vector, they transform according to the law of transformation between the coordinate systems $(x_{\mathrm{v}}^{(\mathbb{t})},x_{\mathrm{h}}^{(\mathbb{t})},x_{\mathrm{p}}^{(\mathbb{t})})$ and $(x_{\mathrm{v'}}^{(\mathbb{t})},x_{\mathrm{h'}}^{(\mathbb{t})},x_{\mathrm{p'}}^{(\mathbb{t})})$. This transformation law is obtained by following the same procedure as used in Appendix A for obtaining Eqs. (A5) and (A6), except that $\theta_{\mathbb{i}}$ and $\theta_{\mathbb{t}}$ are replaced by $\tilde{\theta}_{\mathbb{i}}$ and $\tilde{\theta}_{\mathbb{t}}$, respectively. Since $\boldsymbol{\kappa}^{(\mathbb{t})}$ lies on the plane formed by $\boldsymbol{\kappa}^{(\mathbb{i})}$ and $\mathbf{n}$, α is given by expression (C3), obtained in Appendix C. Neglecting the components $A_{\mathrm{p}}^{(\mathbb{t})}$ and $A_{\mathrm{p'}}^{(\mathbb{t})}$, and following the same procedure as used in Appendix A, one readily obtains Eqs. (32) and (33).

Equations (46) and (47) are obtained in the same way as used in Appendix A for deriving Eqs. (A9) and (A10), and by replacing $\theta_{\mathbb{i}}$ and $\tilde{\theta}_{\mathbb{i}}$ by $(\pi - \theta_{\mathbb{i}})$ and $(\pi - \tilde{\theta}_{\mathbb{i}})$, respectively.

APPENDIX F: DERIVATION OF EQ. (51)

Using the same procedure that we employed in Appendix D, one can show that, in the (x,y,z) coordinate system,

$$\kappa_x^{(\mathbb{i})} = \kappa_x^{(\mathbb{r})} = \sin\alpha \sin\tilde{\theta}_{\mathbb{i}}, \tag{F1a}$$

$$\kappa_y^{(\mathbb{i})} = \kappa_y^{(\mathbb{r})} = -\cos\alpha \sin\tilde{\theta}_{\mathbb{i}}, \tag{F1b}$$

$$\kappa_z^{(\mathbb{i})} = -\kappa_z^{(\mathbb{r})} = \cos\tilde{\theta}_{\mathbb{i}}. \tag{F1c}$$

Using Eqs. (A1) and (F1), one can derive expressions for the components of $\boldsymbol{\kappa}^{(\mathbb{i})}$ in the $(x_{\mathrm{v}}^{(\mathbb{i})},x_{\mathrm{h}}^{(\mathbb{i})},x_{\mathrm{p}}^{(\mathbb{i})})$ coordinate system; comparing them with Eq. (21), one finds that

$$\frac{p}{|\boldsymbol{\kappa}^{(\mathbb{i})}|} = \sin\alpha \sin\tilde{\theta}_{\mathbb{i}}, \tag{F2a}$$

$$\frac{q}{|\boldsymbol{\kappa}^{(\mathbb{i})}|} = -\cos\alpha \sin\tilde{\theta}_{\mathbb{i}} \cos\theta_{\mathbb{i}} + \cos\tilde{\theta}_{\mathbb{i}} \sin\theta_{\mathbb{i}}. \tag{F2b}$$

In a similar way, transforming the components of $\boldsymbol{\kappa}^{(\mathbb{r})}$ in the $(x_{\mathrm{v}}^{(\mathbb{r})},x_{\mathrm{h}}^{(\mathbb{r})},x_{\mathrm{p}}^{(\mathbb{r})})$ coordinate system and using the conditions $\theta_{\mathbb{r}} = \theta_{\mathbb{i}}$, $\tilde{\theta}_{\mathbb{r}} = \tilde{\theta}_{\mathbb{i}}$, one obtains the formulas

$$\frac{\bar{p}}{|\boldsymbol{\kappa}^{(\mathbb{r})}|} = \sin\alpha \sin\tilde{\theta}_{\mathbb{i}}, \tag{F3a}$$

$$\frac{\bar{q}}{|\boldsymbol{\kappa}^{(\mathbb{r})}|} = \cos\alpha \sin\tilde{\theta}_{\mathbb{i}} \cos\theta_{\mathbb{i}} - \cos\tilde{\theta}_{\mathbb{i}} \sin\theta_{\mathbb{i}}. \tag{F3b}$$

Since $|\boldsymbol{\kappa}^{(\mathbb{i})}| = |\boldsymbol{\kappa}^{(\mathbb{r})}|$, one immediately finds from Eqs. (F2) and (F3) that

$$\bar{p} = p, \quad \bar{q} = -q. \tag{F4}$$

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[10] The formula is obtained as immediate generalization of the scalar formula given in Ref. [9], Sec. 5.6.

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