Angular momentum conservation in partially coherent wave fields

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Angular momentum in electromagnetic wave systems is an important yet difficult subject to deal with, both classically and quantum mechanically. We investigate the angular momentum of partially coherent electromagnetic wave fields and establish the general formalism for angular momentum conservation, based on the Maxwell stress tensor for partially coherent sources and fields in the space-frequency domain. This formalism is applied to study the angular momentum properties of several classes of partially coherent electromagnetic beams possessing circulation.

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I. INTRODUCTION

Energy, momentum, and angular momentum are arguably the most fundamental properties in physics, and these basic physical properties are as important in electromagnetism and optics as they are in mechanics. It is now been known for some time that electromagnetic waves carry angular momentum in addition to momentum and energy: In the 1930s, it was demonstrated experimentally by Beth that a beam of circularly polarized light can produce a measurable torque in a macroscopic system [1]. Even earlier than this, Poynting provided a theoretical description of the angular momentum of circularly polarized light [2].

More recently, much attention has been paid to the orbital angular momentum of light, associated with the spatial phase distribution of the electromagnetic wave. This interest seems to have been sparked by a pair of theoretical papers that appeared in 1992, by Allen *et al.* [3] and by van Enk and Nienhuis [4]. Since then, a large body of research, both theoretical and experimental, has been dedicated to the study of optical angular momentum (see, for instance, Ref. [5]). This research has recently taken on greater importance with the development of optical tweezers and spanners in which microscopic particles are trapped, manipulated, and rotated by tightly focused beams of light. Beams carrying spin or orbital angular momentum have been shown to apply a torque to absorptive microscopic particles [6,7], and such torque has been applied to the development of devices such as "microoptomechanical pumps" [8].

The studies to date, however, have primarily been concerned with the angular momentum properties of spatially coherent beams of light. Though partially coherent beams have been extensively studied and conservation laws for energy [9–11] and momentum [12–14] have been formulated, the laws relating to the angular momentum of such beams have not been given similar attention. The exception is a pair of intriguing papers that appeared in the past decade, addressing the angular momentum of light using coherence theory in the space-time domain [15,16]. Perhaps due to the complexity of the results, however, it seems that little use has been made of them.

In this article we establish the formalism for the angular momentum of partially coherent fields in the space-frequency domain, and adapt this formalism to study the angular momentum properties of some model partially coherent fields. This research is justified in large part by the observation that partially coherent fields are often preferable in optical applications to their fully coherent counterparts (see, for instance, Ref. [17]). In Sec. II, we determine the definition of angular momentum of a partially coherent electromagnetic field in the space-frequency domain and derive the formulas relating to the conservation of angular momentum for such fields. In Sec. III, we consider the implications of these results for angular momentum in paraxial partially coherent beams. In Sec. IV, we investigate the angular momentum properties of several classes of partially coherent beams known to possess circulation, and demonstrate that these classes possess fundamentally different angular momentum properties. Section V presents concluding remarks.

The angular momentum of light has led to a number of conceptual difficulties, both in classical and quantum physics. These difficulties arise in part from the lack of a gauge-invariant definition of angular momentum of light when both spin and orbital angular momentum are considered [18], as well as from the the dual intrinsic and extrinsic nature of orbital angular momentum [19], among others. We will attempt to be clear as to where these difficulties lie within our own calculations, and propose solutions or explanations whenever possible.

II. CONSERVATION OF ANGULAR MOMENTUM

We consider a region of space that contains time-fluctuating electric and magnetic fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$. The fields are assumed to be statistically stationary, at least in the wide sense (Sec. 2.2 of Ref. [20]). We begin by following the derivation by Jackson (Sec. 6.8 of Ref. [21]) for the Maxwell stress tensor of deterministic electromagnetic fields and the appropriately modified stress tensor for partially coherent fields considered in Ref. [13].

The mechanical torque $\mathbf{T}_{\text{mech}}$ applied to a single charged particle with respect to the origin of the coordinate system can be expressed in Gaussian units via the Lorentz force as

$$\mathbf{T}_{\mathrm{mech}} = \frac{d\mathbf{L}_{\mathrm{mech}}}{dt} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \left[q\mathbf{E} + q\frac{\mathbf{v}}{c} \times \mathbf{B} \right]. \qquad (1)$$

In this formula $\mathbf{r}$ represents the position vector, q represents the charge and $\mathbf{v}$ the velocity of the particle, and c is the speed of light in vacuum.

If we consider a continuous distribution of charges and currents, we can replace the charge q by the charge density $\rho(\mathbf{r},t)$ and the quantity $q\mathbf{v}$ by the current density $\mathbf{J}(\mathbf{r},t)$. Integrating over a volume V, bounded by a surface S, containing the charges and currents gives the total mechanical torque about the origin of the coordinate system as

$$\mathbf{T}_{\mathrm{mech}} = \int \mathbf{r}' \times \left\{ \rho(\mathbf{r}',t)\mathbf{E}(\mathbf{r}',t) + \frac{1}{c}[\mathbf{J}(\mathbf{r}',t) \times \mathbf{B}(\mathbf{r}',t)] \right\} d^3r'. \qquad (2)$$

It is worth noting at this point that we will restrict our discussion exclusively to the microscopic set of Maxwell's equations. The proper form of the momentum formulas in the macroscopic set (employing $\mathbf{D}$ and $\mathbf{H}$) has been debated for many years in what is known as the Abraham-Minkowski controversy [22]. In short, there is a question of the proper manner in which one separates the momentum of the medium from the momentum of the electromagnetic field; new solutions to this controversy are still being proposed [23]. By restricting our discussion to microscopic fields and sources, we avoid the controversy without losing generality; however, the results then require additional assumptions to apply them to dynamic problems in material media.

Continuing on, Eq. (2) may be written entirely in terms of field quantities by the use of Maxwell's equations:

$$\rho(\mathbf{r},t) = \frac{1}{4\pi}\nabla \cdot \mathbf{E}(\mathbf{r},t), \qquad (3)$$

$$\mathbf{J}(\mathbf{r},t) = \frac{c}{4\pi}\left[\nabla \times \mathbf{B}(\mathbf{r},t) - \frac{1}{c}\frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t} \right]. \qquad (4)$$

The bracketed part of the integrand of Eq. (2) can be manipulated to the form

$$\begin{aligned} \rho\mathbf{E} + \frac{1}{c}\mathbf{J} \times \mathbf{B} &= \frac{1}{4\pi}\left[\mathbf{E}(\nabla \cdot \mathbf{E}) + \left(\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \right) \times \mathbf{B} \right] \\ &= \frac{1}{4\pi}[\mathbf{E}(\nabla \cdot \mathbf{E}) + \mathbf{B}(\nabla \cdot \mathbf{B}) - \mathbf{E} \times (\nabla \times \mathbf{E}) \\ &\quad - \mathbf{B} \times (\nabla \times \mathbf{B})] - \frac{1}{4\pi}\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}). \qquad (5) \end{aligned}$$

where we have, for the moment, suppressed the functional dependencies for brevity.

The mechanical torque then becomes

$$\mathbf{T}_{\mathrm{mech}} = \int \mathbf{r}' \times \left\{ \mathbf{Q} - \frac{\partial}{\partial t}\left(\frac{1}{4\pi c}\mathbf{E} \times \mathbf{B} \right) \right\} d^3r', \qquad (6)$$

where

$$\mathbf{Q} = \frac{1}{4\pi}[\mathbf{E}(\nabla \cdot \mathbf{E}) + \mathbf{B}(\nabla \cdot \mathbf{B}) - \mathbf{E} \times (\nabla \times \mathbf{E}) - \mathbf{B} \times (\nabla \times \mathbf{B})]. \qquad (7)$$

It will be convenient for our discussion of partial coherence to restrict our attention momentarily to monochromatic fields and sources, e.g., $\mathbf{J}(\mathbf{r},t) \equiv \mathbf{J}(\mathbf{r},\omega)e^{-i\omega t}$; we may then write the cycle-averaged mechanical torque as

$$\mathbf{T}_{\mathrm{mech}} = \mathrm{Re}\left[\int \mathbf{r}' \times \left\{ \mathbf{Q} - \frac{\partial}{\partial t}\left(\frac{1}{8\pi c}\mathbf{E} \times \mathbf{B}^* \right) \right\} d^3r' \right], \qquad (8)$$

with $\mathbf{Q}$ now given as

$$\begin{aligned} \mathbf{Q} = \frac{1}{8\pi}[&\mathbf{E}^*(\nabla \cdot \mathbf{E}) + \mathbf{B}(\nabla \cdot \mathbf{B}^*) - \mathbf{E} \times (\nabla \times \mathbf{E}^*) \\ &- \mathbf{B}^* \times (\nabla \times \mathbf{B})]. \qquad (9) \end{aligned}$$

At this point, our calculation is exact and has suffered from no approximations. We now follow the standard approach and note that the second term in the integrand of Eq. (8) naturally appears to represent the time derivative of the angular momentum due to the fields, which we define as

$$\mathbf{L}_{\mathrm{f}} = \frac{1}{8\pi c}\mathrm{Re}\left\{ \int \mathbf{r}' \times (\mathbf{E} \times \mathbf{B}^*) d^3r' \right\}. \qquad (10)$$

Therefore the total rate of change of angular momentum of the combined system of fields and charges within the volume of integration can be written as

$$\mathbf{T}_{\mathrm{tot}} = \frac{\partial}{\partial t}[\mathbf{L}_{\mathrm{f}} + \mathbf{L}_{\mathrm{mech}}] = \mathrm{Re}\left\{ \int [\mathbf{r}' \times \mathbf{Q}(\mathbf{r}')] d^3r' \right\}. \qquad (11)$$

It is perhaps worth noting that it is somewhat inaccurate to call this quantity the total "torque," since this term is typically used to describe forces on matter, whereas $\mathbf{T}_{\mathrm{tot}}$ also includes the rate of change of the angular momentum of the fields. To keep the notation simple, we will still refer to this quantity as the "total torque" and use the symbol $\mathbf{T}_{\mathrm{tot}}$, with the understanding that we are referring to the "total rate of change of the angular momentum."

To express $\mathbf{Q}$ more clearly, we will employ a tensor notation, i.e.,

$$[\mathbf{E}^*(\nabla \cdot \mathbf{E}) - \mathbf{E} \times (\nabla \times \mathbf{E}^*)]_i = \partial_j \left(E_i^* E_j - \tfrac{1}{2}\delta_{ij}\mathbf{E}^* \cdot \mathbf{E} \right), \qquad (12)$$

with $\partial_i \equiv \partial/\partial x_i$, so that we have

$$\begin{aligned} &\{\mathbf{r} \times [\mathbf{E}^*(\nabla \cdot \mathbf{E}) - \mathbf{E} \times (\nabla \times \mathbf{E}^*)]\}_i \\ &\quad = \epsilon_{ijk} r_j \partial_l \left(E_k^* E_l - \tfrac{1}{2}\delta_{kl}\mathbf{E}^* \cdot \mathbf{E} \right), \\ &\{\mathbf{r} \times [\mathbf{B}(\nabla \cdot \mathbf{B}^*) - \mathbf{B}^* \times (\nabla \times \mathbf{B})]\}_i \\ &\quad = \epsilon_{ijk} r_j \partial_l \left(B_k B_l^* - \tfrac{1}{2}\delta_{kl}\mathbf{B}^* \cdot \mathbf{B} \right). \end{aligned}$$

Throughout the remainder of this article we will use the Einstein summation convention unless specified otherwise.

With these relations, we have

$$\begin{aligned} [\mathbf{T}_{\mathrm{tot}}]_i &= \mathrm{Re}\left[\int \epsilon_{ijk} r_j' \partial_l' \left\{ \frac{1}{8\pi}\left[E_k^* E_l + B_k B_l^* \right.\right.\right. \\ &\quad \left.\left.\left. - \frac{1}{2}(\mathbf{E}^* \cdot \mathbf{E} + \mathbf{B}^* \cdot \mathbf{B})\delta_{kl} \right] \right\} d^3r' \right] \\ &= \mathrm{Re}\left\{ \int \epsilon_{ijk} r_j' \partial_l' N_{kl} d^3r' \right\}, \qquad (13) \end{aligned}$$

where N_{kl} is defined as

$$N_{kl} \equiv \frac{1}{8\pi}\left[E_k^* E_l + B_k B_l^* - \frac{1}{2}(\mathbf{E}^*\cdot\mathbf{E} + \mathbf{B}^*\cdot\mathbf{B})\delta_{kl}\right]. \tag{14}$$

Due to the antisymmetry of the Levi-Civita tensor ϵ_{ijk} and the symmetry of N_{kl}, we may write

$$\mathrm{Re}\{r_j'\partial_l'(\epsilon_{ijk}N_{kl})\} = \mathrm{Re}\{\partial_l'(\epsilon_{ijk}r_j'N_{kl})\}. \tag{15}$$

Returning to a vector formalism, we may then write

$$\mathbf{T}_{\mathrm{tot}} = \mathrm{Re}\left\{\int \nabla'\cdot(\mathbf{r}'\times\mathbf{N})d^3r'\right\} \tag{16}$$

which may be converted to a surface integral using the divergence theorem,

$$\mathbf{T}_{\mathrm{tot}} = \mathrm{Re}\left[\oint_S \mathbf{r}'\times\mathbf{N}\cdot\hat{\mathbf{n}}da'\right], \tag{17}$$

with $\hat{\mathbf{n}}$ being the normal to the closed surface S.

Equation (17) is conventionally interpreted as a conservation law for angular momentum; a change of the total angular momentum in a closed volume is directly connected to a net flow of angular momentum through the surface bounding the volume. The tensor

$$\mathbf{M}(\mathbf{r}') \equiv \mathbf{r}'\times\mathbf{N}(\mathbf{r}') \tag{18}$$

is known as the *angular momentum flux density* of the field, whereas the quantity

$$\mathbf{l}_{\mathrm{f}}(\mathbf{r}') \equiv \frac{1}{8\pi c}\mathbf{r}'\times[\mathbf{E}^*(\mathbf{r}')\times\mathbf{B}(\mathbf{r}')]$$

is known as the *angular momentum density* of the field. These two quantities will be discussed in more detail in the next section.

These quantities are, as written, applicable to a system of monochromatic fields and sources. However, they can be straightforwardly converted to expressions for statistically stationary partially coherent fields by using the theory of optical coherence in the space-frequency domain.

Traditionally, the coherence properties of an electromagnetic field have been described using the complex correlation tensors of the electric and magnetic fields of the form

$$\Gamma_{ij}^E(\mathbf{r}_1,\mathbf{r}_2,\tau) \equiv \langle E_i^*(\mathbf{r}_1,t)E_j(\mathbf{r}_2,t+\tau)\rangle, \tag{19}$$

$$\Gamma_{ij}^B(\mathbf{r}_1,\mathbf{r}_2,\tau) \equiv \langle B_i^*(\mathbf{r}_1,t)B_j(\mathbf{r}_2,t+\tau)\rangle, \tag{20}$$

where the angle brackets denote time or ensemble averaging. These quantities, however, satisfy hyperbolic differential equations and can be difficult to interpret and derive. Modern coherence theory typically involves studying the frequency-domain counterparts to these correlation functions, known as the cross-spectral density tensors.

The complex correlation tensors are related to the cross-spectral density tensors in the frequency domain by a Fourier transform, i.e.,

$$W_{ij}^E(\mathbf{r}_1,\mathbf{r}_2,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma_{ij}^E(\mathbf{r}_1,\mathbf{r}_2,\tau)e^{-i\omega\tau}d\tau. \tag{21}$$

However, it has been shown that one can also derive the mathematical form of the cross-spectral density tensor directly from an average over an ensemble of monochromatic realizations of the wave field (Sec. 4.1 of Ref. [24]), so that

$$W_{ij}^E(\mathbf{r}_1,\mathbf{r}_2,\omega) = \langle E_i^*(\mathbf{r}_1,\omega)E_j(\mathbf{r}_2,\omega)\rangle_\omega,$$

where $\langle\cdots\rangle_\omega$ represents an average over said realizations. It follows from Eq. (21) that the cross-spectral density tensor has dimensions of [electric field]^2/[frequency], and the torque derived from it will therefore be the average torque per unit frequency, to be labeled $\boldsymbol{\tau}_{\mathrm{tot}}$.

We may readily take the ensemble average of Eq. (17) to find this average torque per unit frequency,

$$\langle\boldsymbol{\tau}_{\mathrm{tot}}(\omega)\rangle = \mathrm{Re}\left[\oint_S\langle\mathbf{M}\rangle\cdot\hat{\mathbf{n}}da'\right], \tag{22}$$

where the angular momentum flux density may be written in the form

$$\langle M_{iq}(\mathbf{r},\omega)\rangle = \frac{1}{8\pi}\epsilon_{ijk}r_j\Big\{\big[W_{kq}^E(\mathbf{r},\mathbf{r},\omega) + W_{kq}^B(\mathbf{r},\mathbf{r},\omega)\big] - \frac{1}{2}\delta_{kq}\big[W_{pp}^E(\mathbf{r},\mathbf{r},\omega) + W_{pp}^B(\mathbf{r},\mathbf{r},\omega)\big]\Big\}. \tag{23}$$

Furthermore, the angular momentum density $\boldsymbol{\lambda}_f$ of the field per unit frequency may be written as

$$\langle\boldsymbol{\lambda}_f(\mathbf{r},\omega)\rangle = \frac{1}{8\pi c}\mathbf{r}\times\langle\mathbf{E}^*(\mathbf{r},\omega)\times\mathbf{B}(\mathbf{r},\omega)\rangle. \tag{24}$$

To clarify these formulas, we use the monochromatic form of Faraday's law to write all magnetic fields as electric fields, i.e.,

$$\nabla\times\mathbf{E} = i\frac{\omega}{c}\mathbf{B}.$$

We may then express the angular momentum density as

$$\langle\boldsymbol{\lambda}_f(\mathbf{r},\omega)\rangle_i = -\frac{i}{8\pi\omega}\big[\epsilon_{ijk}r_l\partial_j' W_{lk}^E(\mathbf{r},\mathbf{r}',\omega) - \epsilon_{ljk}r_l\partial_j' W_{ik}^E(\mathbf{r},\mathbf{r}',\omega)\big]_{\mathbf{r}'=\mathbf{r}}. \tag{25}$$

Furthermore, the angular momentum flux density may be written as

$$\langle M_{iq}(\mathbf{r},\omega)\rangle = \frac{1}{8\pi}\epsilon_{ijk}r_j\left\{W_{kq}^E(\mathbf{r},\mathbf{r}',\omega) + \frac{1}{k^2}\epsilon_{klm}\epsilon_{qrt}\frac{\partial}{\partial x_l}\frac{\partial}{\partial x_r'}W_{mt}^E(\mathbf{r},\mathbf{r}',\omega) - \frac{1}{2}\delta_{kq}\left[W_{pp}^E(\mathbf{r},\mathbf{r}',\omega) + \frac{1}{k^2}\epsilon_{prs}\epsilon_{puv}\frac{\partial}{\partial x_r}\frac{\partial}{\partial x_u'}W_{sv}^E(\mathbf{r},\mathbf{r}',\omega)\right]\right\}_{\mathbf{r}=\mathbf{r}'}. \tag{26}$$

This expression can be simplified, albeit not shortened, using a standard identity for the product of Levi-Civita tensors:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}. \tag{27}$$

The angular momentum flux density then takes on the form

$$\langle M_{iq}(\mathbf{r},\omega)\rangle = \frac{1}{8\pi}\left\{r_j\left[\epsilon_{ijk}W^E_{kq} - \frac{1}{2}\epsilon_{ijq}W^E_{pp}\right] + \frac{1}{k^2}r_m\epsilon_{qrt}\left[\partial_i\partial'_r W^E_{mt} - \partial_m\partial'_r W^E_{it}\right] + \frac{1}{2k^2}r_j\epsilon_{ijq}\left[\partial_v\partial'_u W^E_{uv} - \partial_u\partial'_u W^E_{vv}\right]\right\}\Bigg|_{\mathbf{r}'=\mathbf{r}}, \quad (28)$$

where we have again suppressed the arguments of the functions for brevity.

The expressions for angular momentum density and flux density, on substitution back into Eq. (22), may be used to determine the rate of change of angular momentum within a volume of partially coherent fields and charges at a given frequency ω. The total torque on the volume can be readily determined by a straightforward integration of Eq. (22) over all frequencies, i.e.,

$$\langle \mathbf{T}_{\text{tot}}\rangle = \int_0^\infty \langle \boldsymbol{\tau}_{\text{tot}}(\omega)\rangle d\omega. \quad (29)$$

For problems involving quasimonochromatic fields and sources of bandwidth $\Delta\omega$, it will usually be sufficient to evaluate $\langle \boldsymbol{\tau}_{\text{tot}}(\omega)\rangle$ at the center frequency of oscillation. Multiplied by the bandwidth, this quantity will be a good approximation to the total rate of change of angular momentum of the system.

Equations (26) and (29) already lead to an important conclusion relating to the angular momentum of partially coherent fields. Because the flux density and density depend on the derivatives of $W^E_{ij}(\mathbf{r},\mathbf{r}',\omega)$, it follows that they depend on the spatial correlations of the wave field as well as the intensity. In general, one would expect that changes to the state of coherence will result in changes to the angular momentum, and we will in fact see that this is the case.

III. ANGULAR MOMENTUM OF PARAXIAL ELECTROMAGNETIC PARTIALLY COHERENT BEAMS

One of the most natural uses of the previous formalism is its application to the study of the angular momentum of paraxial partially coherent beams. It was shown some time ago [25] that the angular momentum flux density of such a beam is the proper quantity for characterizing the angular momentum of a beam, and that the spin and orbital components can be decoupled in both the paraxial and nonparaxial regimes. In this section we derive the angular momentum flux density for paraxial partially coherent beams entirely in terms of the electric cross-spectral density tensor.

We will consider the z component of the angular momentum flowing across a surface of constant z, i.e., the quantity $\langle M_{zz}(\mathbf{r},\omega)\rangle$. Using the properties of the Levi-Civita tensor, we may readily simplify Eq. (28) to the form

$$\langle M_{zz}(\mathbf{r},\omega)\rangle = \frac{1}{8\pi}\left\{r_j\epsilon_{zjk}W^E_{kz} + \frac{1}{k^2}r_m\epsilon_{zrt} \times \left[\partial_z\partial'_r W^E_{mt} - \partial_m\partial'_r W^E_{zt}\right]\right\}\Bigg|_{\mathbf{r}'=\mathbf{r}}, \quad (30)$$

where arguments on the right of the equation have been suppressed for brevity. In the paraxial limit, we may assume that the z component of the field is negligible, which results in further simplification:

$$\langle M_{zz}(\mathbf{r},\omega)\rangle = \frac{1}{8\pi k^2}r_m\epsilon_{zrt}\partial_z\partial'_r W^E_{mt}\big|_{\mathbf{r}'=\mathbf{r}}. \quad (31)$$

Furthermore, the z behavior of the electric field in the paraxial limit is approximately of the form $\exp[ikz]$, which implies that the z derivative in the above expression may be mapped to $\partial_z \to -ik$. We are then left with

$$\langle M_{zz}(\mathbf{r},\omega)\rangle = -\frac{i}{8\pi k}r_m\epsilon_{zrt}\partial'_r W^E_{mt}\big|_{\mathbf{r}'=\mathbf{r}}. \quad (32)$$

We are left with sums over m, r, and t, which can only take on values x and y; these summations leave us with

$$\langle M_{zz}(\mathbf{r},\omega)\rangle = -\frac{i}{8\pi k}\{x'\partial'_x W^E_{xy} - y'\partial'_y W^E_{yx} + y'\partial'_x W^E_{yy} - x'\partial'_y W^E_{xx}\}_{\mathbf{r}'=\mathbf{r}}. \quad (33)$$

Recalling that the primed derivatives are over the second argument of the tensor components, the first two terms may be rewritten by use of a generalization of the product rule of calculus. For example,

$$x\partial_x\left(W^E_{xy}\right) = x'\partial'_x W^E_{xy} + x\partial_x W^E_{xy}. \quad (34)$$

(It is to be remembered that $\mathbf{r}' = \mathbf{r}$ in the end of the calculation.) Our expression for the flux density takes on the form

$$\langle M_{zz}(\mathbf{r},\omega)\rangle = -\frac{i}{8\pi k}\{x\partial_x\left(W^E_{xy}\right) - y\partial_y\left(W^E_{yx}\right) + y\partial'_x W^E_{yy} - x\partial'_y W^E_{xx} - x\partial_x W^E_{xy} + y\partial_y W^E_{yx}\}_{\mathbf{r}'=\mathbf{r}}. \quad (35)$$

It is to be noted that the flux density appears in Eq. (22) in a surface integral over an infinite plane of constant z. Assuming the electromagnetic field is finite in extent, we may perform an integration by parts on the first two terms of Eq. (35); the flux density may then be interpreted to have the form

$$\langle M_{zz}(\mathbf{r},\omega)\rangle = -\frac{i}{8\pi k}\{-W^E_{xy} + W^E_{yx} + y\partial'_x W^E_{yy} - x\partial'_y W^E_{xx} - x\partial_x W^E_{xy} + y\partial_y W^E_{yx}\}_{\mathbf{r}'=\mathbf{r}}. \quad (36)$$

We now have the makings of a spin-orbit separation of angular momentum. The first two terms do not depend on the phase structure of the field and likely represent the spin contribution to the angular momentum. Noting that only the real part of M_{zz} appears in the conservation law, we may write

$$M_{\text{spin}} = \frac{1}{8\pi k}\text{Im}\left[W^E_{yx} - W^E_{xy}\right]. \quad (37)$$

The latter four terms, with their derivatives, appear to represent the orbital part of the angular momentum:

$$M_{\text{orbit}} = \frac{1}{8\pi k}\text{Im}\left[y\partial'_x W^E_{yy} - x\partial'_y W^E_{xx} - x\partial_x W^E_{xy} + y\partial_y W^E_{yx}\right]. \quad (38)$$

We can make this association more explicit by a final application of the paraxial approximation. Gauss' law in the paraxial limit takes on the form

$$\partial_x E_x + \partial_y E_y = 0. \tag{39}$$

With this, we may rewrite the orbital component of the angular momentum as

$$M_{\text{orbit}} = \frac{1}{8\pi k}\text{Im}\big[y\partial_x' W_{yy}^E - x\partial_y' W_{xx}^E + x\partial_y W_{yy}^E - y\partial_x W_{xx}^E\big]. \tag{40}$$

Converting this expression to polar coordinates, we find that

$$M_{\text{orbit}} = \frac{1}{8\pi k}\text{Im}\left[\frac{\partial}{\partial\phi}W_{yy}^E + \frac{\partial}{\partial\phi}W_{xx}^E\right]. \tag{41}$$

Equations (37) and (41) represent the separation of the angular momentum flux density of a paraxial partially coherent field into a spin and an orbital contribution. We can verify this interpretation by following the reasoning given in Ref. [25]. First, if we pass the beam through a waveplate that transforms $E_x \to E_x e^{i\psi_x}$ and $E_y \to E_y e^{i\psi_y}$, with ψ_x and ψ_y constant phases, the spin contribution will be altered but the orbital part will not. Second, if we pass the beam through a spiral phase plate that maps $E_x \to E_x e^{i\phi}$ and $E_y \to E_y e^{i\phi}$, the spin contribution will be unaffected but the orbital part will change.

IV. EXAMPLES

We now use the formalism developed in the previous section to investigate the angular momentum properties of a pair of model partially coherent fields that are known to possess circulation.

We begin by considering paraxial, uniformly unpolarized wave fields propagating in the z direction; such fields have, on average, no spin angular momentum. We may write the electric cross-spectral density tensor for such fields in dyadic form as

$$\mathbf{W}^E(\mathbf{r}_1,\mathbf{r}_2,\omega) = \hat{\mathbf{x}}\hat{\mathbf{x}}W(\mathbf{r}_1,\mathbf{r}_2,\omega) + \hat{\mathbf{y}}\hat{\mathbf{y}}W(\mathbf{r}_1,\mathbf{r}_2,\omega), \tag{42}$$

where $W(\mathbf{r}_1,\mathbf{r}_2,\omega)$ is a scalar cross-spectral density.

From Eq. (40), the angular momentum flux density of the beam is of the form

$$8\pi k M_{zz}(\mathbf{r},\omega) = \text{Im}\big\{y\partial_x' W_{yy}^E(\mathbf{r},\mathbf{r}',\omega) - x\partial_y' W_{xx}^E(\mathbf{r},\mathbf{r}',\omega) + x\partial_y W_{yy}^E(\mathbf{r},\mathbf{r}',\omega) - y\partial_x W_{xx}^E(\mathbf{r},\mathbf{r}',\omega)\big\}_{\mathbf{r}'=\mathbf{r}}. \tag{43}$$

There are two simple analytic classes of partially coherent beams that are known to possess "circulation" about the z axis. The first of these are so-called "twisted" Gaussian Schell-model beams [26], with cross-spectral density in the plane $z = 0$ of the form

$$W_{\text{twist}}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \frac{I_0}{\pi\sigma_s^2}e^{-(r_1^2+r_2^2)/2\sigma_s^2}e^{-(\mathbf{r}_1-\mathbf{r}_2)^2/2\sigma_g^2}e^{-iv(x_1y_2-y_1x_2)}. \tag{44}$$

Here σ_s is the overall width of the beam, while σ_g is the transverse correlation length of the beam. The parameter v is a "twist" parameter that represents the strength of beam circulation.

The second class of beams to be considered are partially coherent vortex beams derived by using a "beam wander" model [27], of the form

$$\begin{aligned} W_{\text{vortex}}(\mathbf{r}_1,\mathbf{r}_2,\omega) = {} & \frac{I_0}{2\pi\sigma_s^4[1+\gamma^4]/2}e^{-(r_1^2+r_2^2)/2\sigma_s^2}e^{-(\mathbf{r}_1-\mathbf{r}_2)^2/2\sigma_g^2} \\ & \times \{[\gamma^2(x_1+iy_1)+(x_1-x_2)+i(y_1-y_2)] \\ & \times [\gamma^2(x_2-iy_2)-(x_1-x_2)+i(y_1-y_2)] \\ & + 2\sigma_s^2\}, \end{aligned} \tag{45}$$

where $\gamma = \sigma_s/\sigma_g$. This cross-spectral density was derived by assuming that the central axis of a pure Laguerre-Gauss LG_0^1 laser mode wanders randomly in the transverse x-y plane; the spatial coherence of the beam decreases as the amount of wander increases. We have introduced the effective correlation length σ_g and the effective beam width σ_s to replace the parameters used to define the beam in earlier publications. This form of partially coherent vortex was demonstrated to be the generic form that appears in a linear optical system [28].

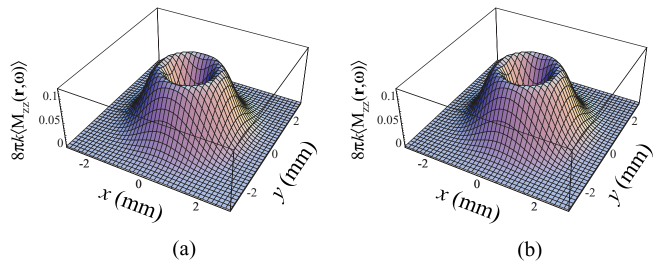

FIG. 1. (Color online) The angular momentum flux density of (a) an unpolarized twisted Gaussian Schell-model beam and (b) an unpolarized random vortex beam, with $\sigma_s = 1$ mm, $\sigma_g = 1$ mm, and $v = 1\ \text{mm}^{-2}$.

Examples of the angular momentum flux density for each class of circulating beam are shown in Fig. 1. The two figures can be seen to be almost identical, with the density increasing rapidly from the origin to a maximum on a circle, then decreasing rapidly beyond. However, the angular momentum flux density depends on the intensity of the wave field as well as its circulation; to remove intensity-dependent effects, we normalize the angular momentum flux by the field intensity, and evaluate the normalized density:

$$m_z(\mathbf{r},\omega) = \frac{8\pi k\langle M_{zz}(\mathbf{r},\omega)\rangle}{W(\mathbf{r},\mathbf{r},\omega)}. \tag{46}$$

The quantity m_z is plotted for a cross section of the field in Fig. 2. We can now see that the twisted Gaussian beam has a

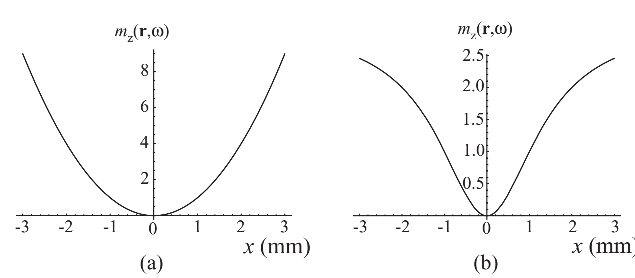

FIG. 2. Cross section of the normalized angular momentum flux density m_z of (a) an unpolarized twisted Gaussian Schell-model beam and (b) an unpolarized random vortex beam, with $\sigma_s = 1$ mm, $\sigma_g = 1$ mm, and $v = 1\ \text{mm}^{-2}$.

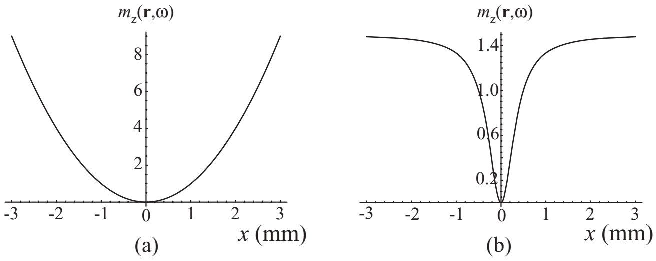

FIG. 3. Cross section of the normalized angular momentum flux density m_z of (a) an unpolarized twisted Gaussian Schell-model beam and (b) an unpolarized random vortex beam, with $\sigma_s = 1$ mm, $\sigma_g = 0.5$ mm, and $v = 1$ mm^{-2}.

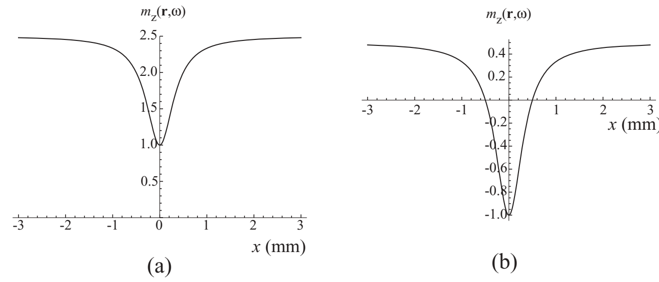

FIG. 4. Cross section of the normalized angular momentum flux density m_z of (a) a right-hand circularly polarized twisted Gaussian Schell-model beam and (b) a left-hand circularly polarized random vortex beam, with $\sigma_s = 1$ mm, $\sigma_g = 0.5$ mm, $P = 1$ and $v = 1$ mm^{-2}.

quadratically increasing m_z, while the vortex beam apparently has m_z initially quadratic but leveling off at larger x. To shed light on this behavior, m_z is plotted for the two beam classes at a lower state of coherence in Fig. 3. Now we can clearly see two different behaviors of the normalized flux density m_z for the vortex beam, a rapidly increasing quadratic region near the center of the beam and a constant region at distances far from the axis. In the limit $\sigma_g \to 0$, the quadratic region disappears entirely and the density becomes constant over all radii.

This behavior is consistent with the observation that partially coherent vortex beams behave like Rankine vortices [29], with a region of rigid body rotation near the center and a fluid-like rotation in the outer regions; this can be readily seen by simple arguments. A rigid body rotates with a constant angular frequency ω_r, and its velocity is related to its radial position from the center of rotation by $v = \omega_r r$. The flux density at any point will be then be proportional to $rv = \omega_r r^2$, and this r^2 dependence is what is seen in the interior of the partially coherent vortex beam. A fluid will have a circulation velocity that depends inversely upon the radial distance, i.e., $v = \alpha/r$, and therefore will have a flux density independent of position, $rv = \alpha$. This is seen in the outer region of the partially coherent vortex beam.

It is interesting to note that the twisted Gaussian beam has a fundamentally different behavior than the partially coherent vortex beam. The flux density for the twisted beam possesses an r^2 dependence for all radial positions, indicating that it has a "pure" rigid body rotational behavior. This result indicates that the properties of circulating partially coherent beams are richer than previously thought.

We may also consider the angular momentum of beams which are uniformly partially polarized. In matrix form, such beams have a cross-spectral density tensor of the form

$$\mathbf{W}^E(\mathbf{r}_1,\mathbf{r}_2,\omega) = (1-P)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} W(\mathbf{r}_1,\mathbf{r}_2,\omega) + 2P\begin{bmatrix} |a|^2 & a^*b \\ b^*a & |b|^2 \end{bmatrix} W(\mathbf{r}_1,\mathbf{r}_2,\omega), \tag{47}$$

where P represents the degree of polarization of the beam, with 0 representing unpolarized, and 1 representing complete polarization. The first term of the tensor is the unpolarized part of the field, while the second term represents the polarized part. Here we have adopted a form of the cross-spectral density density used in a number of studies; see, for instance, [30]. We take $a = 1\sqrt{2}$, $b = \pm i/\sqrt{2}$, the positive and negative signs representing right and left-hand circular polarizations, respectively.

Examples of the normalized angular momentum flux density for such beams are shown in Fig. 4. With right-hand circularly polarized light, the spin and orbit contributions are the same sign, and the spin contributes a uniform positive offset to the density. With left-hand circularly polarized light, the spin contribution is in opposition to the orbit, and the result is a region of negative angular momentum flux density in the center of the beam. Any offset between -1 and 1 can be achieved with an appropriate choice of P, the degree of polarization.

V. CONCLUSIONS

In this paper we have explicitly calculated the formulas relating to angular momentum and its conservation for partially coherent fields in the space-frequency representation. These formulas were applied to investigate the angular momentum flux density of partially coherent beams known to possess circulation. It was demonstrated that the partially coherent vortex beam possesses a Rankine vortex behavior in its angular momentum flux density, while the twisted Gaussian beam has a pure rigid body rotational behavior.

These results suggest that the angular momentum properties of partially coherent beams are richer than previously imagined, and that at least two distinct classes of partially coherent circulating beams exist. The general formalism introduced here may be applied to problems of optical trapping and spanning of microscopic particles, and variations in the partial coherence of the field may be used as an extra degree of freedom in the development of new trapping systems.

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