Method for classifying multiqubit states via the rank of the coefficient matrix and its application to four-qubit states

Xiangrong Li[1] and Dafa Li[2,3]
[1]*Department of Mathematics, University of California, Irvine, California 92697-3875, USA*
[2]*Department of Mathematical Sciences, Tsinghua University, Beijing 100084 China*
[3]*Center for Quantum Information Science and Technology, Tsinghua National Laboratory for Information Science and Technology (TNList), Beijing 100084, China*

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We construct coefficient matrices of size 2^ℓ by $2^{n-\ell}$ associated with pure n-qubit states and prove the invariance of the ranks of the coefficient matrices under stochastic local operations and classical communication (SLOCC). The ranks give rise to a simple way of partitioning pure n-qubit states into inequivalent families and distinguishing degenerate families from one another under SLOCC. Moreover, the classification scheme via the ranks of coefficient matrices can be combined with other schemes to build a more refined classification scheme. To exemplify we classify the nine families of four qubits introduced by Verstraete *et al.* [Phys. Rev. A **65**, 052112 (2002)] further into inequivalent subfamilies via the ranks of coefficient matrices, and as a result, we find 28 genuinely entangled families and all the degenerate classes can be distinguished up to permutations of the four qubits. We also discuss the completeness of the classification of four qubits into nine families.

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I. INTRODUCTION

Quantum entanglement plays a crucial role in quantum information theory, with applications to quantum teleportation, quantum cryptography, and quantum computation [1]. The equivalence under stochastic local operations and classical communication (SLOCC) induces a natural partition of quantum states. The central task of SLOCC classification is to classify quantum states according to a criterion that is invariant under SLOCC.

SLOCC entanglement classification has been the subject of intensive study during the last decade [2–20]. For three qubits, there are six SLOCC equivalence classes of which two are genuinely entanglement classes: GHZ and W [2] and four degenerate classes can be distinguished by the local ranks (i.e., ranks of single-qubit reduced density matrices obtained by tracing out all but one qubit [2]). For four or more qubits, there are infinite SLOCC classes and it is highly desirable to partition the infinite classes into a finite number of families. The key lies in finding criteria to determine which family an arbitrary quantum state belongs to. In a pioneering work, Verstraete *et al.* [3] obtained nine SLOCC inequivalent families of four qubits using Lie group theory: G_{abcd}, L_{abc_2}, $L_{a_2b_2}$, L_{ab_3}, L_{a_4}, $L_{a_2 0_{3\oplus\bar{1}}}$, $L_{0_{5\oplus\bar{3}}}$, $L_{0_{7\oplus\bar{1}}}$, and $L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}}$. It is clear that some families obtained by Verstraete *et al.* [3] contain an infinite number of SLOCC classes and some contain both degenerate classes and genuinely entangled classes. It is of great importance to find a more refined partition of four-qubit states such that the degenerate classes are distinguished from the genuinely entangled families. Many other efforts have been devoted to the SLOCC entanglement classification of four qubits [5–13]. More recently, a few attempts have been made toward the generalization to a higher number of qubits, including odd n qubits [17], even n qubits [18], symmetric n qubits [14–16], and general n qubits [19,20].

This paper is organized as follows. We first construct coefficient matrices of size 2^ℓ by $2^{n-\ell}$ associated with pure n-qubit states and prove the invariance of the ranks of coefficient matrices under SLOCC in Sec. II. In Sec. III, we present a recursive formula which allows us to easily calculate the ranks of coefficient matrices of n-qubit biseparable states. We next show that the degenerate families of general n qubits are inequivalent to one another under SLOCC in Sec. IV. Section V is devoted to the classification of four qubits via the ranks of coefficient matrices. Section VI provides the discussion of the completeness of the nine families obtained by Verstraete *et al.* [3]. We finally conclude this paper in Sec. VII.

II. THE INVARIANCE OF THE RANKS OF COEFFICIENT MATRICES

Let $|\psi\rangle_{1\cdots n} = \sum_{i=0}^{2^n-1} a_i|i\rangle$ be an n-qubit pure state. We associate with the state $|\psi\rangle_{1\cdots n}$ a 2^ℓ by the $2^{n-\ell}$ coefficient matrix $C_{1\cdots\ell,(\ell+1)\cdots n}(|\psi\rangle_{1\cdots n})$ whose entries are the coefficients $a_0,a_1,\ldots,a_{2^n-1}$ of the state $|\psi\rangle_{1\cdots n}$ arranged in ascending lexicographical order. To illustrate, we list $C_{1\cdots\ell,(\ell+1)\cdots n}(|\psi\rangle_{1\cdots n})$ below as

$$\begin{pmatrix} a_{\underbrace{0\cdots0}_{\ell}\underbrace{0\cdots0}_{n-\ell}} & \cdots & a_{\underbrace{0\cdots0}_{\ell}\underbrace{1\cdots1}_{n-\ell}} \\ a_{\underbrace{0\cdots1}_{\ell}\underbrace{0\cdots0}_{n-\ell}} & \cdots & a_{\underbrace{0\cdots1}_{\ell}\underbrace{1\cdots1}_{n-\ell}} \\ \vdots & \vdots & \vdots \\ a_{\underbrace{1\cdots1}_{\ell}\underbrace{0\cdots0}_{n-\ell}} & \cdots & a_{\underbrace{1\cdots1}_{\ell}\underbrace{1\cdots1}_{n-\ell}} \end{pmatrix}. \tag{1}$$

In the binary form of the coefficient matrix in Eq. (1), bits 1 to ℓ and $\ell+1$ to n are referred to as the row bits and column bits, respectively. If $\ell = 0$, $C_{\emptyset,1\cdots n}(|\psi\rangle_{1\cdots n})$ reduces to the row vector $(a_0,\ldots,a_{2^n-1})$, and if $\ell = n$, $C_{1\cdots n,\emptyset}(|\psi\rangle_{1\cdots n})$ reduces to the column vector $(a_0,\ldots,a_{2^n-1})^T$.

Let $\{q_1,q_2,\ldots,q_n\}$ be a permutation of $\{1,2,\ldots,n\}$. Let $C_{q_1\cdots q_\ell,q_{\ell+1}\cdots q_n}(|\psi\rangle_{1\cdots n})$ be the $2^\ell \times 2^{n-\ell}$ coefficient matrix of the state $|\psi\rangle_{1\cdots n}$, which is constructed from the coefficient matrix $C_{12\cdots\ell,\ell+1\cdots n}$ in Eq. (1) by taking the corresponding permutation. Here $q_1,\ldots,q_\ell$ are the row bits and $q_{\ell+1},\ldots,q_n$ are the column bits. Indeed, we only need to specify the row bits, as the column bits would simply be the rest of the bits. In the sequel, we will omit the subscripts $q_{\ell+1},\ldots,q_n$ and simply write $C_{q_1\cdots q_\ell}$, whenever the column bits are clear from the context.

It is known that two n-qubit pure states $|\psi\rangle_{1\cdots n}$ and $|\psi'\rangle_{1\cdots n}$ are equivalent to each other under SLOCC if and only if there are local invertible operators $\mathcal{A}_1$, $\mathcal{A}_2$, ..., and $\mathcal{A}_n$ such that [2]

$$|\psi'\rangle_{1\cdots n} = \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_n |\psi\rangle_{1\cdots n}. \tag{2}$$

In terms of coefficient matrices, it can be verified that the following result holds: For any two SLOCC equivalent n-qubit pure states $|\psi\rangle_{1\cdots n}$ and $|\psi'\rangle_{1\cdots n}$, their coefficient matrices $C_{q_1\cdots q_\ell}$ satisfy the equation:

$$C_{q_1\cdots q_\ell}(|\psi'\rangle_{1\cdots n}) = \left(\mathcal{A}_{q_1} \otimes \cdots \otimes \mathcal{A}_{q_\ell}\right) C_{q_1\cdots q_\ell}(|\psi\rangle_{1\cdots n}) \left(\mathcal{A}_{q_{\ell+1}} \otimes \cdots \otimes \mathcal{A}_{q_n}\right)^T, \tag{3}$$

where $\mathcal{A}_1$,$\mathcal{A}_2$, ..., and $\mathcal{A}_n$ are the local operators in Eq. (2). Conversely, if there are local invertible operators $\mathcal{A}_1$,$\mathcal{A}_2$, ..., and $\mathcal{A}_n$ such that Eq. (3) holds true for some $C_{q_1\cdots q_\ell}$, then $|\psi\rangle_{1\cdots n}$ and $|\psi'\rangle_{1\cdots n}$ are equivalent under SLOCC.

It immediately follows from Eq. (3) that the rank of any coefficient matrix of an n-qubit pure state is invariant under SLOCC. This leads to the following theorem.

Theorem 1. If two n-qubit pure states are SLOCC equivalent then their coefficient matrices $C_{q_1\cdots q_\ell}$ given above have the same rank.

Restated in the contrapositive the theorem reads: If two coefficient matrices $C_{q_1\cdots q_\ell}$ associated with two n-qubit pure states differ in their ranks, then the two states belong necessarily to different SLOCC classes.

Coefficient matrices constructed above turn out to be closely related to reduced density matrices. We let $\rho_{12\cdots n}(|\psi\rangle_{1\cdots n}) = |\psi\rangle_{1\cdots n\,1\cdots n}\langle\psi|$ be the density matrix of an n-qubit pure state $|\psi\rangle_{1\cdots n}$, and we let $\rho_{q_1\cdots q_\ell}$ be the ℓ-qubit reduced density matrix obtained from $\rho_{12\cdots n}$ by tracing out $n-\ell$ qubits. As has been previously noted for bipartite systems of dimensions $d \times d$, a reduced density matrix has a full rank factorization in terms of the corresponding coefficient matrix and its conjugate transpose [22]. This factorization also holds for n-qubit states [23]:

$$\rho_{q_1\cdots q_\ell}(|\psi\rangle_{1\cdots n}) = C_{q_1\cdots q_\ell}(|\psi\rangle_{1\cdots n}) C^\dagger_{q_1\cdots q_\ell}(|\psi\rangle_{1\cdots n}), \tag{4}$$

where $C^\dagger$ is the conjugate transpose of C. An important relationship between reduced density matrices and SLOCC polynomial invariants can be obtained by taking the determinants of both sides of Eq. (4) for even n and for $\ell = n/2$, yielding

$$\det \rho_{q_1\cdots q_{n/2}}(|\psi\rangle_{1\cdots n}) = \left|\det C_{q_1\cdots q_{n/2}}(|\psi\rangle_{1\cdots n})\right|^2. \tag{5}$$

Here $\det C_{q_1\cdots q_{n/2}}(|\psi\rangle_{1\cdots n})$ is a SLOCC polynomial invariant of degree $2^{n/2}$ for even n qubits and its absolute value can be used as an entanglement measure [24]. Thus we have the following:

Theorem 2. For even n-qubit pure states, the determinants of $n/2$-qubit reduced density matrices are the squares of the SLOCC polynomial invariants of degree $2^{n/2}$, with the absolute values of the latter quantifying $n/2$-qubit entanglement of the even n-qubit states after tracing out the other $n/2$ qubits.

As an example, when $n=4$ we have $\det\rho_{12} = |L|^2$, $\det\rho_{13} = |M|^2$, and $\det\rho_{14} = |N|^2$, where L, M, and N are polynomial invariants of degree 4 [25]. When $n=6$, there are 10 three-qubit reduced density matrices and 10 polynomial invariants of degree 8: $D_6^1,\ldots,D_6^{10}$ [24]. For reduced density matrix ρ_{123} and polynomial invariant D_6^1, we have $\det\rho_{123} = |D_6^1|^2$. Similar equations hold for other reduced density matrices and polynomial invariants with appropriate permutations of qubits.

Remark 1. (i) The determinants of reduced density matrices are invariant under SLOCC. (ii) It is worth noting that Eq. (5) holds for bipartite systems of dimensions $d \times d$ as well [22].

As a particular case of Eq. (4), when $q_i = i$ we have $\rho_{1\cdots n}(|\psi\rangle_{1\cdots n}) = C_{1\cdots n}(|\psi\rangle_{1\cdots n}) C^\dagger_{1\cdots n}(|\psi\rangle_{1\cdots n})$. By virtue of Eq. (4), the rank of the ℓ-qubit reduced density matrix and the rank of the corresponding coefficient matrix are the same. In light of Theorem 1, we have the following result.

Corollary. The ranks of ℓ-qubit reduced density matrices obtained by tracing out $n-\ell$ qubits are invariant under SLOCC.

This is particularly true for the local ranks [2]. Note also that any complex matrix has a singular value decomposition, with the number of nonzero singular values equal to the rank of the matrix. This means that the number of nonzero singular values of any coefficient matrix of an n-qubit pure state is invariant under SLOCC.

III. A RECURSIVE FORMULA FOR THE RANKS OF *N*-QUBIT BISEPARABLE STATES

In principle, we can calculate the ranks of coefficient matrices for n-qubit biseparable pure states by direct calculations. However, in practice, this is rather cumbersome from the computational point of view, and as n becomes large, this might pose a serious problem. In order to avoid this difficulty, we propose a simple recursive formula for the ranks of n-qubit biseparable states.

Suppose that a biseparable n-qubit pure state $|\psi\rangle_{1\cdots n}$ is of the form $|\psi\rangle_{1\cdots n} = |\phi\rangle_{j_1\cdots j_k} \otimes |\varphi\rangle_{j_{k+1}\cdots j_n}$ with $|\phi\rangle_{j_1\cdots j_k}$ being a k-qubit state and $|\varphi\rangle_{j_{k+1}\cdots j_n}$ being an $(n-k)$-qubit state. We let $C_{q_1\cdots q_\ell}(|\psi\rangle_{1\cdots n})$ be the coefficient matrix associated with the state $|\psi\rangle_{1\cdots n}$. We let $C_{q_1^*\cdots q_s^*}(|\phi\rangle_{j_1\cdots j_k})$ be the 2^s by 2^{k-s} coefficient matrix associated with the k-qubit state $|\phi\rangle_{j_1\cdots j_k}$. Here $\{q_1^*,\ldots,q_s^*\} = \{q_1,\ldots,q_\ell\} \cap \{j_1,\ldots,j_k\}$ are the row bits, and by convention, the rest of the $k-s$ bits are the column bits. Moreover, we let $C_{q_1'\cdots q_t'}(|\varphi\rangle_{j_{k+1}\cdots j_n})$ be the 2^t by 2^{n-k-t} coefficient matrix associated with the $(n-k)$-qubit state $|\varphi\rangle_{j_{k+1}\cdots j_n}$. Here $\{q_1',\ldots,q_t'\} = \{q_1,\ldots,q_\ell\} \cap \{j_{k+1},\ldots,j_n\}$ are the row bits, and by convention, the rest of the $n-k-t$ bits are the column bits. It can be verified that

$$C_{q_1\cdots q_\ell}\left(|\phi\rangle_{j_1\cdots j_k} \otimes |\varphi\rangle_{j_{k+1}\cdots j_n}\right) = C_{q_1^*\cdots q_s^*}\left(|\phi\rangle_{j_1\cdots j_k}\right) \otimes C_{q_1'\cdots q_t'}\left(|\varphi\rangle_{j_{k+1}\cdots j_n}\right). \tag{6}$$

In view of the fact that the rank of the Kronecker product of two matrices is the product of their ranks, we arrive at the following

TABLE I. Ranks of coefficient matrices of three-qubit pure states.

Families \ Ranks of	C_A	C_B	C_C
A-B-C	1	1	1
A-BC	1	2	2
B-AC	2	1	2
C-AB	2	2	1
ABC	2	2	2

recursive formula for the ranks of coefficient matrices of an n-qubit biseparable state:

$$\begin{aligned}&\mathrm{rank}\big(C_{q_1\cdots q_\ell}\big(|\phi\rangle_{j_1\cdots j_k}\otimes|\varphi\rangle_{j_{k+1}\cdots j_n}\big)\big)\\&\quad=\mathrm{rank}\big(C_{q_1^*\cdots q_s^*}\big(|\phi\rangle_{j_1\cdots j_k}\big)\big)\mathrm{rank}\big(C_{q_1'\cdots q_t'}\big(|\varphi\rangle_{j_{k+1}\cdots j_n}\big)\big).\end{aligned}\tag{7}$$

The formula above allows us to calculate recursively the ranks of coefficient matrices of n-qubit biseparable states in terms of the ranks of coefficient matrices of k-qubit states and $(n-k)$-qubit states. To illustrate the use of the recursive formula, we start with the initial values $\mathrm{rank}(C_A(|\phi\rangle_A))=1$ and $\mathrm{rank}(C_\emptyset(|\phi\rangle_A))=1$. It is known that a two-qubit pure state can be either of the form A–B (separable) or the form AB (EPR). Using the recursive formula, we find $\mathrm{rank}(C_A(|\phi\rangle_A|\varphi\rangle_B))=\mathrm{rank}(C_A(|\phi\rangle_A))\times\mathrm{rank}(C_\emptyset(|\varphi\rangle_B))=1$. On the other hand, a direct calculation shows that $\mathrm{rank}(C_A(|\varphi\rangle_{AB}))=2$. Using the results obtained above, we can find the ranks of coefficient matrices of three-qubit pure states. Consider, for example, $\mathrm{rank}(C_C(|\phi\rangle_B|\varphi\rangle_{AC}))$ for biseparable states being of the form B-AC. Using the recursive formula, we have $\mathrm{rank}(C_C(|\phi\rangle_B|\varphi\rangle_{AC}))=\mathrm{rank}(C_\emptyset(|\phi\rangle_B))\times\mathrm{rank}(C_C(\varphi\rangle_{AC}))=2$. In a similar fashion, we can fill in the rest of the entries in Table I, except those in the last row which can be obtained by direct calculations. Proceeding in this way, we can construct Tables II and III for the ranks of coefficient matrices for four and five qubits.

TABLE II. Ranks of coefficient matrices of four-qubit pure states.

Families \ Ranks of	C_A	C_B	C_C	C_D	C_{AB}	C_{AC}	C_{AD}
A-B-C-D	1	1	1	1	1	1	1
A-B-CD	1	1	2	2	1	2	2
A-C-BD	1	2	1	2	2	1	2
A-D-BC	1	2	2	1	2	2	1
B-C-AD	2	1	1	2	2	2	1
B-D-AC	2	1	2	1	2	1	2
C-D-AB	2	2	1	1	1	2	2
A-BCD	1	2	2	2	2	2	2
B-ACD	2	1	2	2	2	2	2
C-ABD	2	2	1	2	2	2	2
D-ABC	2	2	2	1	2	2	2
AB-CD	2	2	2	2	1	4	4
AC-BD	2	2	2	2	4	1	4
AD-BC	2	2	2	2	4	4	1
$ABCD$[a]	2	2	2	2	$\geqslant 2$	$\geqslant 2$	$\geqslant 2$

[a]$ABCD$ can be further partitioned under SLOCC in terms of the ranks of C_{AB}, C_{AC}, and C_{AD}.

TABLE III. Ranks of coefficient matrices of five-qubit pure states.

Families \ Ranks of	C_α	$C_{\beta\gamma(\beta\neq\gamma)}$
i-j-k-ℓ-m	1[b]	1[c]
i-j-k-ℓm	1, if $\alpha=i,j,k$ 2, otherwise	1, if $\beta,\gamma=i,j,k$ or $\beta,\gamma=\ell,m$ 2, otherwise
i-jk-ℓm	1, if $\alpha=i$ 2, otherwise	1, if $\beta,\gamma=j,k$ or $\beta,\gamma=\ell,m$ 2, if $\beta=i$ or $\gamma=i$ 4, otherwise
i-j-$k\ell m$	1, if $\alpha=i$ or j 2, otherwise	1, if $\beta,\gamma=i,j$ 2, otherwise
i-$jk\ell m$	1, if $\alpha=i$ 2, otherwise	2, if $\beta=i$ or $\gamma=i$ 2, 3, or 4, otherwise
ij-$k\ell m$	2[b]	1, if $\beta,\gamma=i,j$ 2, if $\beta,\gamma=k,\ell,m$ 4, otherwise
$ijk\ell m$	2[b]	2, 3, or 4[c]

[a]$\{i,j,k,\ell,m\}$ is any permutation of $\{A,B,C,D,E\}$.
[b]$\alpha=i,j,k,\ell,m$.
[c]$\beta,\gamma=i,j,k,\ell,m$.

Note that in Tables I and II the ranks of only $2^{n-1}-1$ coefficient matrices are shown. This is due to the fact that interchanging two row (respectively, column) bits or exchanging the row and column bits of a coefficient matrix does not alter the rank of the matrix, since the former is equivalent to interchanging two rows (respectively, columns) of the matrix and the latter is equivalent to transposing the matrix. Ignoring $C_\emptyset$ and $C_{1\cdots n}$ which always have rank 1, this amounts to totally $2^{n-1}-1$ potentially different coefficient matrices. For example, the ranks of C_{BA} and C_{BC} are not shown in Table II, since C_{AB} and C_{BA} differ by the interchange of two rows, and C_{BC} is the transpose of C_{AD}. As illustrated in Tables I–III, the ranks of coefficient matrices permit the partitioning of the space of the pure states into inequivalent families under SLOCC (i.e., two states belong to the same family if and only if the ranks of coefficient matrices are all equal). In particular, degenerate families of three, four, and five qubits are inequivalent from one another under SLOCC.

IV. DEGENERATE FAMILIES OF GENERAL N QUBITS ARE SLOCC INEQUIVALENT TO ONE ANOTHER

The recursive formula above further gives rise to a criterion for biseparability of an n-qubit pure state. Indeed, we note that Eq. (7) holds particularly true for $\{q_1,\ldots,q_\ell\}=\{j_1,\ldots,j_k\}$. In this case, the coefficient matrices $C_{q_1^*\cdots q_s^*}$ and $C_{q_1'\cdots q_t'}$ reduce to a column vector and a row vector, respectively, and therefore both of them have rank 1. It follows that $\mathrm{rank}(C_{q_1\cdots q_\ell}(|\phi\rangle_{q_1\cdots q_\ell}\otimes|\varphi\rangle_{q_{\ell+1}\cdots q_n}))=1$. Conversely, if $\mathrm{rank}(C_{q_1\cdots q_\ell}(|\psi\rangle_{1\cdots n}))=1$ for an n-qubit pure state $|\psi\rangle_{1\cdots n}$, then $|\psi\rangle_{1\cdots n}$ is biseparable, being of the form $|\psi\rangle_{1\cdots n}=|\phi\rangle_{q_1\cdots q_\ell}\otimes|\varphi\rangle_{q_{\ell+1}\cdots q_n}$. This can be seen as follows. For simplicity, we assume $q_i = i$ with $i = 1,\ldots,n$. If $\mathrm{rank}(C_{12\cdots\ell}(|\psi\rangle_{1\cdots n})) = 1$, then all columns of $C_{12\cdots\ell}$ are proportional to each other and each column can be written into the form $(a_0 b_j, a_1 b_j, \ldots, a_{2^\ell-1} b_j)^T$. Hence, $|\psi\rangle_{1\cdots n}$ can be written as $|\psi\rangle_{1\cdots n} = |\phi\rangle_{1\cdots\ell} \otimes |\varphi\rangle_{(\ell+1)\cdots n}$ with $|\phi\rangle_{1\cdots\ell} = \sum_{i=0}^{2^\ell-1} a_i |i\rangle_{1\cdots\ell}$ and $|\varphi\rangle_{(\ell+1)\cdots n} = \sum_{j=0}^{2^{n-\ell}-1} b_j |j\rangle_{(\ell+1)\cdots n}$. This leads to the following biseparability criterion for n-qubit pure states.

Biseparability criterion for n-qubit pure states. For any coefficient matrix $C_{q_1\cdots q_\ell}$ associated with an n-qubit pure state $|\psi\rangle_{1\cdots n}$, $\mathrm{rank}(C_{q_1\cdots q_\ell}(|\psi\rangle_{1\cdots n})) = 1$ if and only if $|\psi\rangle$ is biseparable, being of the form $|\psi\rangle_{1\cdots n} = |\phi\rangle_{q_1\cdots q_\ell} \otimes |\varphi\rangle_{q_{\ell+1}\cdots q_n}$ (see also [21,23]).

Invoking the fact that an n-qubit pure state is entangled if it is not full separable, we have the following criterion to identify n-qubit entangled (respectively, genuinely entangled) pure states: An n-qubit pure state is entangled (respectively, genuinely entangled) if and only if the rank of at least one of its coefficient matrices is (respectively, the ranks of its all coefficient matrices are) greater than 1.

Note that all the above criteria can be rephrased in terms of the ranks of ℓ-qubit reduced density matrices obtained by tracing out $n-\ell$ qubits [26] or the number of nonzero singular values of coefficient matrices.

Theorem 1 together with the biseparability criterion above yield the following theorem.

Theorem 3. Degenerate families of general n qubits are inequivalent to one another under SLOCC and they can be distinguished in terms of the ranks of coefficient matrices (or in terms of the ranks of ℓ-qubit reduced density matrices obtained by tracing out $n-\ell$ qubits).

The validity of Theorem 3 can be seen as follows. Given an n-qubit pure state, a partition P of the n particles is a collection of disjoint sets in such a way that the particles within any one set are entangled and any two particles from different sets are not entangled. Suppose F_1 and F_2 are two different degenerate families with partitions P_1 and P_2, respectively. Without loss of generality, we assume that there exists a set S such that $S \in P_1$ and $S \notin P_2$. Then the states in F_1 can be written in the biseparable form $|\phi\rangle_S|\varphi\rangle_{\bar{S}}$, where $\bar{S}$ is the set of all particles except those in S. According to the biseparability criterion above, $\mathrm{rank}(C_S) = 1$ for states in F_1. Since the states in F_2 cannot be written in the above biseparable form, $\mathrm{rank}(C_S) > 1$ for states in F_2 . In light of Theorem 1, the two degenerate families are inequivalent to each other under SLOCC.

In addition, we remark that degenerate families of general n qubits can also be distinguished from one another under SLOCC in terms of the ranks of ℓ-qubit reduced density matrices obtained by tracing out $n-\ell$ qubits or the number of nonzero singular values of coefficient matrices.

V. SLOCC CLASSIFICATION OF FOUR QUBITS VIA THE RANKS OF COEFFICIENT MATRICES

Suppose that the states $|\psi\rangle$ and $|\psi'\rangle$ of four qubits are SLOCC equivalent to each other, then there are local invertible operators $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, and $\mathcal{A}_4$ such that [2]

$$|\psi'\rangle = \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_3 \otimes \mathcal{A}_4 |\psi\rangle. \tag{8}$$

For a four-qubit state $|\psi\rangle = \sum_{i=0}^{15} a_i|i\rangle$, we consider three coefficient matrices C_{AB}, C_{AC}, and C_{AD} as follows:

$$C_{AB} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 & a_7 \\ a_8 & a_9 & a_{10} & a_{11} \\ a_{12} & a_{13} & a_{14} & a_{15} \end{pmatrix}, \tag{9}$$

$$C_{AC} = \begin{pmatrix} a_0 & a_1 & a_4 & a_5 \\ a_2 & a_3 & a_6 & a_7 \\ a_8 & a_9 & a_{12} & a_{13} \\ a_{10} & a_{11} & a_{14} & a_{15} \end{pmatrix}, \tag{10}$$

$$C_{AD} = \begin{pmatrix} a_0 & a_4 & a_2 & a_6 \\ a_1 & a_5 & a_3 & a_7 \\ a_8 & a_{12} & a_{10} & a_{14} \\ a_9 & a_{13} & a_{11} & a_{15} \end{pmatrix}. \tag{11}$$

The coefficient matrices above satisfy the following equations:

$$C_{AB}(|\psi'\rangle) = \mathcal{A}_1 \otimes \mathcal{A}_2 C_{AB}(|\psi\rangle)(\mathcal{A}_3 \otimes \mathcal{A}_4)^T, \tag{12}$$

$$C_{AC}(|\psi'\rangle) = \mathcal{A}_1 \otimes \mathcal{A}_3 C_{AC}(|\psi\rangle)(\mathcal{A}_2 \otimes \mathcal{A}_4)^T, \tag{13}$$

$$C_{AD}(|\psi'\rangle) = \mathcal{A}_1 \otimes \mathcal{A}_4 C_{AD}(|\psi\rangle)(\mathcal{A}_3 \otimes \mathcal{A}_2)^T. \tag{14}$$

It follows from Eqs. (12)–(14) that if two four-qubit states are SLOCC equivalent then their coefficient matrices C_{AB} (and also C_{AC} and C_{AD}) have the same rank. Conversely, if one of the coefficient matrices C_{AB}, C_{AC}, and C_{AD} differ in the ranks, then the two four-qubit states are SLOCC inequivalent. Let family $F^{C_{AB}}_{r_{AB}}$ be the set of all four-qubit states with the same rank r_{AB} of the coefficient matrix C_{AB}. Here r_{AB} ranges over the values 1, 2, 3, and 4. Clearly, each one of the nine families introduced by Verstraete *et al.* [3] can be further divided into four SLOCC inequivalent subfamilies corresponding to the four possible values of r_{AB}. In a similar manner, we can define the families $F^{C_{AC}}_{r_{AC}}$ and $F^{C_{AD}}_{r_{AD}}$. One can obtain a more refined partition by further dividing the families $F^{C_{AB}}_{r_{AB}}$, $F^{C_{AC}}_{r_{AC}}$, and $F^{C_{AD}}_{r_{AD}}$ into subfamilies $F^{C_{AB}C_{AC}C_{AD}}_{r_{AB}r_{AC}r_{AD}} = F^{C_{AB}}_{r_{AB}} \cap F^{C_{AC}}_{r_{AC}} \cap F^{C_{AD}}_{r_{AD}}$. Clearly, the subfamilies $F^{C_{AB}C_{AC}C_{AD}}_{r_{AB}r_{AC}r_{AD}}$ and $F^{C_{AB}C_{AC}C_{AD}}_{r'_{AB}r'_{AC}r'_{AD}}$ are SLOCC inequivalent when $r_{AB}r_{AC}r_{AD} \neq r'_{AB}r'_{AC}r'_{AD}$.

We now further partition the nine families introduced by Verstraete *et al.* [3] into SLOCC inequivalent subfamilies via the rank of coefficient matrix. For convenience, we rewrite the families G_{abcd} and L_{abc_2} as

$$G_{abcd} = \alpha(|0\rangle + |15\rangle) + \beta(|3\rangle + |12\rangle) + \gamma(|5\rangle + |10\rangle) + \delta(|6\rangle + |9\rangle), \tag{15}$$

$$L_{abc_2} = \alpha'(|0\rangle + |15\rangle) + \beta'(|3\rangle + |12\rangle) + \gamma'(|5\rangle + |10\rangle) + |6\rangle. \tag{16}$$

In Table IV, we show the subfamilies $F^{C_{AB}}_{r_{AB}}$, $F^{C_{AC}}_{r_{AC}}$, and $F^{C_{AD}}_{r_{AD}}$ of G_{abcd}. As illustrated in Table V, G_{abcd} can be further partitioned into nine genuinely entangled subfamilies and three biseparable subfamilies (marked with "*") via r_{AB}, r_{AC}, and r_{AD} (subfamilies not listed in the table are empty). For simplicity, the detailed descriptions of the subfamilies are not shown as they can be easily obtained by taking the intersections of the corresponding descriptions in Table IV. Tables VI and VII illustrate the partitions of the other eight families introduced by Verstraete *et al.* into inequivalent subfamilies. In total, we

TABLE IV. The subfamilies $F^{C_{AB}}_{r_{AB}}$, $F^{C_{AC}}_{r_{AC}}$, and $F^{C_{AD}}_{r_{AD}}$ of G_{abcd}.

Subfamily	Description
$F_1^{C_{AB}}$	$\alpha=\beta=0\ \&\ \gamma=\pm\delta\neq 0\mid\alpha=\pm\beta\neq 0\ \&\ \gamma=\delta=0$
$F_2^{C_{AB}}$	$\alpha=\beta=0\ \&\ \gamma\neq\pm\delta\mid\gamma=\delta=0\ \&\ \alpha\neq\pm\beta\mid\alpha=\pm\beta\neq 0\ \&\ \gamma=\pm\delta\neq 0$
$F_3^{C_{AB}}$	$\alpha=\pm\beta\neq 0\ \&\ \gamma\neq\pm\delta\mid\gamma=\pm\delta\neq 0\ \&\ \alpha\neq\pm\beta$
$F_4^{C_{AB}}$	$\alpha\neq\pm\beta\ \&\ \gamma\neq\pm\delta$
$F_1^{C_{AC}}$	$\alpha=\gamma=0\ \&\ \beta=\pm\delta\neq 0\mid\alpha=\pm\gamma\neq 0\ \&\ \beta=\delta=0$
$F_2^{C_{AC}}$	$\alpha=\gamma=0\ \&\ \beta\neq\pm\delta\mid\beta=\delta=0\ \&\ \alpha\neq\pm\gamma\mid\alpha=\pm\gamma\neq 0\ \&\ \beta=\pm\delta\neq 0$
$F_3^{C_{AC}}$	$\alpha=\pm\gamma\neq 0\ \&\ \beta\neq\pm\delta\mid\beta=\pm\delta\neq 0\ \&\ \alpha\neq\pm\gamma$
$F_4^{C_{AC}}$	$\alpha\neq\pm\gamma\ \&\ \beta\neq\pm\delta$
$F_1^{C_{AD}}$	$\alpha=\delta=0\ \&\ \beta=\pm\gamma\neq 0\mid\alpha=\pm\delta\neq 0\ \&\ \beta=\gamma=0$
$F_2^{C_{AD}}$	$\alpha=\delta=0\ \&\ \beta\neq\pm\gamma\mid\beta=\gamma=0\ \&\ \alpha\neq\pm\delta\mid\alpha=\pm\delta\neq 0\ \&\ \beta=\pm\gamma\neq 0$
$F_3^{C_{AD}}$	$\alpha=\pm\delta\neq 0\ \&\ \beta\neq\pm\gamma\mid\beta=\pm\gamma\neq 0\ \&\ \alpha\neq\pm\delta$
$F_4^{C_{AD}}$	$\alpha\neq\pm\delta\ \&\ \beta\neq\pm\gamma$

find 28 genuinely entangled subfamilies and all the degenerate classes can be distinguished up to permutations of the four qubits (i.e., A-B-C-D, A-B-CD, AB-CD, $|0\rangle_A|W\rangle_{BCD}$, and $|0\rangle_A|\mathrm{GHZ}\rangle_{BCD}$).

VI. DISCUSSION OF THE COMPLETENESS OF THE NINE FAMILIES OBTAINED BY VERSTRAETE *ET AL.*

The family L_{ab_3} in Ref. [3] was defined as

$$L_{ab_3}=a(|0000\rangle+|1111\rangle)+\frac{a+b}{2}(|0101\rangle+|1010\rangle)+\frac{a-b}{2}(|0110\rangle+|1001\rangle)+\frac{i}{\sqrt{2}}(|0001\rangle+|0010\rangle+|0111\rangle+|1011\rangle). \quad (17)$$

In later work, Chterental *et al.* [3] obtained nine SLOCC inequivalent families of four qubits using invariant theory. Let

TABLE V. SLOCC classification of G_{abcd} via r_{AB}, r_{AC}, and r_{AD}. The subfamilies marked with "*" are biseparable.

$r_{AB}\ r_{AC}\ r_{AD}$	Subfamily description
222	$F_2^{C_{AB}}\cap F_2^{C_{AC}}\cap F_2^{C_{AD}}$
244	$F_2^{C_{AB}}\cap F_4^{C_{AC}}\cap F_4^{C_{AD}}$
333	$F_3^{C_{AB}}\cap F_3^{C_{AC}}\cap F_3^{C_{AD}}$
344	$F_3^{C_{AB}}\cap F_4^{C_{AC}}\cap F_4^{C_{AD}}$
424	$F_4^{C_{AB}}\cap F_2^{C_{AC}}\cap F_4^{C_{AD}}$
434	$F_4^{C_{AB}}\cap F_3^{C_{AC}}\cap F_4^{C_{AD}}$
442	$F_4^{C_{AB}}\cap F_4^{C_{AC}}\cap F_2^{C_{AD}}$
443	$F_4^{C_{AB}}\cap F_4^{C_{AC}}\cap F_3^{C_{AD}}$
444	$F_4^{C_{AB}}\cap F_4^{C_{AC}}\cap F_4^{C_{AD}}$
144*	$F_1^{C_{AB}}$ (i.e., AB-CD)
414*	$F_1^{C_{AC}}$ (i.e., AC-BD)
441*	$F_1^{C_{AD}}$ (i.e., AD-BC)

L'_{ab_3} be defined by

$$L'_{ab_3}=a(|0000\rangle+|1111\rangle)+\frac{a+b}{2}(|0101\rangle+|1010\rangle)+\frac{a-b}{2}(|0110\rangle+|1001\rangle)+\frac{i}{\sqrt{2}}(|0001\rangle+|0010\rangle-|0111\rangle-|1011\rangle), \quad (18)$$

that is, L'_{ab_3} is obtained by replacing the two "+" signs of the last two terms in the formula of L_{ab_3} by "−" signs [6]. It is claimed that there is a perfect correspondence between the nine families obtained by Verstraete *et al.* (with L_{ab_3} replaced by L'_{ab_3}) and the nine families obtained by Chterental *et al.* [6]. Note that the formula of L'_{ab_3} has also been adopted in Ref. [11]. Since both Verstraete *et al.* and Chterental *et al.* claimed that the nine families obtained in their work are inequivalent to each other, a detailed study of the relation between L_{ab_3} and L'_{ab_3} can provide insights into the completeness of their classifications.

A. $L_{ab_3}(a=0)$ is SLOCC equivalent to $L'_{ab_3}(a=0)$

It is readily verified that the following equation holds between $L'_{ab_3}(a=0)$ and $L_{ab_3}(a=0)$:

$$L'_{ab_3}(a=0)=I\otimes I\otimes i\sigma_z\otimes i\sigma_z L_{ab_3}(a=0), \quad (19)$$

where I is the identity and $\sigma_z=\mathrm{diag}\{1,-1\}$.

It follows from Eq. (19) that $L_{ab_3}(a=0)$ and $L'_{ab_3}(a=0)$ are SLOCC equivalent. In particular, setting $b=0$ yields that the states $\frac{i}{\sqrt{2}}(|0001\rangle+|0010\rangle-|0111\rangle-|1011\rangle)$ and $\frac{i}{\sqrt{2}}(|0001\rangle+|0010\rangle+|0111\rangle+|1011\rangle)$ are equivalent under SLOCC.

B. $L'_{ab_3}(a\neq 0)$ [respectively, $L_{ab_3}(a\neq 0)$] is SLOCC inequivalent to L_{ab_3} (respectively, L'_{ab_3})

We first show that the family $L'_{ab_3}(a\neq 0)$ is SLOCC inequivalent to the family L_{ab_3}. In Table VIII we show the partition of L'_{ab_3} into SLOCC inequivalent subfamilies via

TABLE VI. SLOCC classification of L_{abc_2} via r_{AB}, r_{AC}, and r_{AD}. The subfamilies marked with "*" are biseparable.

$r_{AB}\ r_{AC}\ r_{AD}$	Subfamily description
233	$\alpha' = \beta' = 0$ & $\gamma' \neq 0$
244	$\alpha' = \pm\beta' \neq 0$ & $\gamma' = 0$
323	$\alpha' = \gamma' = 0$ & $\beta' \neq 0$
332	$\alpha' \neq 0$ & $\beta' = \gamma' = 0$
333	$\alpha' = \pm\beta' = \pm\gamma' \neq 0$
344	$\gamma' = 0$ & $\alpha'\beta' \neq 0$ & $\alpha' \neq \pm\beta'$ | $\gamma' \neq 0$ & $\alpha' = \pm\beta' \neq 0$ & $\alpha' \neq \pm\gamma'$
424	$\beta' = 0$ & $\alpha' = \pm\gamma' \neq 0$
434	$\beta' = 0$ & $\alpha'\gamma' \neq 0$ & $\alpha' \neq \pm\gamma'$ | $\beta' \neq 0$ & $\alpha' = \pm\gamma' \neq 0$ & $\alpha' \neq \pm\beta'$
442	$\alpha' = 0$ & $\beta' = \pm\gamma' \neq 0$
443	$\alpha' = 0$ & $\beta' \neq \pm\gamma'$ & $\beta'\gamma' \neq 0$ | $\alpha' \neq 0$ & $\beta' = \pm\gamma' \neq 0$ & $\alpha' \neq \pm\beta'$
444	$\gamma' \neq 0$ & $\alpha' \neq \pm\beta'$ & $\beta' \neq 0$ & $\alpha' \neq \pm\gamma'$ & $\alpha' \neq 0$ & $\beta' \neq \pm\gamma'$
111*	$\alpha' = \beta' = \gamma' = 0$ (i.e., *A*-*B*-*C*-*D*)

r_{AB}, r_{AC}, and r_{AD}. Consulting Tables VII and VIII, and using the fact that the subfamilies with different ranks of coefficient matrices are SLOCC inequivalent to each other, it suffices to consider the following six cases.

Case 1. $L'_{ab_3}(a = b \neq 0)$ is SLOCC inequivalent to $L_{ab_3}(b = -3a \neq 0)$.

In this case, we can resort to D_{xy}, a degree 6 polynomial invariant of four qubits [25] (see the Appendix for the expression of D_{xy}). Indeed, it can be verified that if $|\psi\rangle$ and $|\psi'\rangle$ are any two SLOCC equivalent states, that is, they satisfy Eq. (2), then the following equation holds:

$$D_{xy}(|\psi'\rangle) = D_{xy}(|\psi\rangle)\left[\prod_{i=1}^{4}\det\mathcal{A}_i\right]^3. \tag{20}$$

It follows from Eq. (20) that for any two SLOCC equivalent states $|\psi\rangle$ and $|\psi'\rangle$, either $D_{xy}(|\psi'\rangle)$ and $D_{xy}(|\psi\rangle)$ both vanish or neither vanishes.

A direct calculation shows that

$$D_{xy} = -\tfrac{1}{32}(a-b)^3(a+b)^3 \tag{21}$$

for both L_{ab_3} and L'_{ab_3}. The desired result then follows by noting that $D_{xy} = 16a^6 \neq 0$ for $L_{ab_3}(b = -3a \neq 0)$ whereas $D_{xy} = 0$ for $L'_{ab_3}(a = b \neq 0)$.

Case 2. $L'_{ab_3}(a = -b \neq 0)$ is SLOCC inequivalent to $L_{ab_3}(b = 3a \neq 0)$.

This case can be dealt with similarly as case 1 by noting that $D_{xy} = 16a^6 \neq 0$ for $L_{ab_3}(b = 3a \neq 0)$ whereas $D_{xy} = 0$ for $L'_{ab_3}(a = -b \neq 0)$.

Case 3. $L'_{ab_3}(b = -3a \neq 0)$ is SLOCC inequivalent to $L_{ab_3}(b = -3a \neq 0)$.

In this case, the semi-invariants defined in Ref. [7] turn out to be useful. More specifically, for any four-qubit state $|\psi\rangle = \sum_{i=0}^{15} c_i|i\rangle$, the semi-invariants F_1 and F_2 are defined in Ref. [7] as

$$F_1(\psi) = (c_0c_7 - c_2c_5 + c_1c_6 - c_3c_4)^2, \\ -4(c_2c_4 - c_0c_6)(c_3c_5 - c_1c_7), \tag{22}$$

$$F_2(\psi) = (c_8c_{15} - c_{11}c_{12} + c_9c_{14} - c_{10}c_{13})^2 \\ -4(c_{11}c_{13} - c_9c_{15})(c_{10}c_{12} - c_8c_{14}). \tag{23}$$

TABLE VII. SLOCC classifications of L_{ab_3}, $L_{a_2b_2}$, L_{a_4}, $L_{a_20_{3\oplus\bar{1}}}$, $L_{0_{5\oplus\bar{3}}}$, $L_{0_{7\oplus\bar{1}}}$, and $L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}}$ via r_{AB}, r_{AC}, and r_{AD}. The subfamilies marked with "*" are biseparable.

Family	$r_{AB}\ r_{AC}\ r_{AD}$	Subfamily description	Family	$r_{AB}\ r_{AC}\ r_{AD}$	Subfamily description
$L_{a_2b_2}$	333	$ab = 0$ & $a \neq b$	L_{ab_3}	222	$a = b = 0$ (i.e., $\|W\rangle_{ABCD}$)
	424	$a = \pm b \neq 0$		344	$ab = 0$ & $a \neq b$
	434	$ab \neq 0$ & $a \neq \pm b$		424	$a = b \neq 0$
	212*	$a = b = 0$ (i.e., *A*-*C*-*BD*)		434	$b = -3a \neq 0$
L_{a_4}	323	$L_{a_4}(a = 0)$		442	$a = -b \neq 0$
	434	$L_{a_4}(a \neq 0)$		443	$b = 3a \neq 0$
$L_{a_20_{3\oplus\bar{1}}}$	333	$L_{a_20_{3\oplus\bar{1}}}(a \neq 0)$		444	$ab \neq 0$ & $b \neq \pm a$ & $b \neq \pm 3a$
	222*	$a = 0$ (i.e., $\|0\rangle_A\|W\rangle_{BCD}$)	$L_{0_{5\oplus\bar{3}}}$	333	$L_{0_{5\oplus\bar{3}}}$
$L_{0_{7\oplus\bar{1}}}$	333	$L_{0_{7\oplus\bar{1}}}$	$L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}}$	222*	$\|0\rangle_A\|\mathrm{GHZ}\rangle_{BCD}$

TABLE VIII. SLOCC classification of L'_{ab_3} via r_{AB}, r_{AC}, and r_{AD}.

$r_{AB}\, r_{AC}\, r_{AD}$	Subfamily description
222	$a=b=0$ (i.e., $\vert W\rangle_{ABCD}$)
344	$ab=0\ \&\ a\neq b$
424	$\emptyset$
434	$a=b\neq 0 \mid b=-3a\neq 0$
442	$\emptyset$
443	$a=-b\neq 0 \mid b=3a\neq 0$
444	$ab\neq 0\ \&\ b\neq\pm a\ \&\ b\neq\pm 3a$

Let $|\phi\rangle$ be any four-qubit state SLOCC equivalent to L_{ab_3} [i.e., they satisfy Eq. (2)]. Let

$$\mathcal{A}_1=\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{pmatrix}. \tag{24}$$

A tedious but straightforward calculation yields

$$F_1(\phi)=\frac{1}{2}(a^2-b^2)\alpha_1^4\left[\prod_{i=2}^{4}\det\mathcal{A}_i\right]^2, \tag{25}$$

$$F_2(\phi)=\frac{1}{2}(a^2-b^2)\alpha_3^4\left[\prod_{i=2}^{4}\det\mathcal{A}_i\right]^2. \tag{26}$$

In view of Eqs. (25) and (26) and the fact that $\mathcal{A}_1$ is invertible, it follows at once that if $|\phi\rangle$ is SLOCC equivalent to $L_{ab_3}(a\neq\pm b)$, then the following equation holds:

$$|F_1(\phi)|+|F_2(\phi)|\neq 0. \tag{27}$$

Let $|\varphi\rangle$ be any state SLOCC equivalent to L'_{ab_3} [i.e., they satisfy Eq. (2)]. Again, a tedious but straightforward calculation yields

$$F_1(\varphi)=\frac{-1}{2\sqrt{2}}i\alpha_1^3(-i\sqrt{2}(3a^2+b^2)\alpha_1+8a(a^2-b^2)\alpha_2)\times\left[\prod_{i=2}^{4}\det\mathcal{A}_i\right]^2, \tag{28}$$

$$F_2(\varphi)=\frac{-1}{2\sqrt{2}}i\alpha_3^3(-i\sqrt{2}(3a^2+b^2)\alpha_3+8a(a^2-b^2)\alpha_4)\times\left[\prod_{i=2}^{4}\det\mathcal{A}_i\right]^2. \tag{29}$$

When $a(a^2-b^2)\neq 0$, consider the operator,

$$\mathcal{A}_1^*=\begin{pmatrix}\alpha_1 & \frac{i\sqrt{2}(3a^2+b^2)}{8a(a^2-b^2)}\alpha_1\\ 0 & \alpha_4\end{pmatrix}, \tag{30}$$

where $\alpha_1\alpha_4\neq 0$. Clearly, $\mathcal{A}_1^*$ is invertible. In view of Eqs. (28)–(30), it follows that there exists a state $|\varphi^*\rangle$ equivalent to $L'_{ab_3}(a(a^2-b^2)\neq 0)$ under local invertible operators $\mathcal{A}_1^*$, $\mathcal{A}_2$, $\mathcal{A}_3$, and $\mathcal{A}_4$, such that

$$|F_1(\varphi^*)|+|F_2(\varphi^*)|=0. \tag{31}$$

From Eqs. (27) and (31), $|\varphi^*\rangle$ is SLOCC inequivalent to the state $L_{ab_3}(a\neq\pm b)$. Therefore, $L'_{ab_3}(a(a^2-b^2)\neq 0)$ is SLOCC inequivalent to $L_{ab_3}(a\neq\pm b)$. In particular, $L'_{ab_3}(b=-3a\neq 0)$ is SLOCC inequivalent to $L_{ab_3}(b=-3a\neq 0)$.

Case 4. $L'_{ab_3}(b=3a\neq 0)$ is SLOCC inequivalent to $L_{ab_3}(b=3a\neq 0)$.

This case can be treated analogously to case 3.

Case 5. $L'_{ab_3}(a\neq 0\ \&\ b=0)$ is SLOCC inequivalent to $L_{ab_3}(ab=0\ \&\ a\neq b)$.

In Ref. [10], we proved that $L_{ab_3}(a=0\ \&\ b\neq 0)$ and $L_{ab_3}(a\neq 0\ \&\ b=0)$ are SLOCC inequivalent. A proof analogous to that of Ref. [10] shows that $L'_{ab_3}(a=0\ \&\ b\neq 0)$ and $L'_{ab_3}(a\neq 0\ \&\ b=0)$ are SLOCC inequivalent. Using the fact that $L_{ab_3}(a=0\ \&\ b\neq 0)$ is SLOCC equivalent to $L'_{ab_3}(a=0\ \&\ b\neq 0)$ [see Eq. (19)] yields that $L'_{ab_3}(a\neq 0\ \&\ b=0)$ is SLOCC inequivalent to $L_{ab_3}(a=0\ \&\ b\neq 0)$. Furthermore, an argument analogous to case 3 shows that $L'_{ab_3}(a\neq 0\ \&\ b=0)$ is inequivalent to $L_{ab_3}(a\neq 0\ \&\ b=0)$. Indeed, we can further conclude that $L_{ab_3}(a=0)$ and $L_{ab_3}(a\neq 0)$ are SLOCC inequivalent and $L'_{ab_3}(a=0)$ and $L'_{ab_3}(a\neq 0)$ are SLOCC inequivalent.

Case 6. $L'_{ab_3}(ab\neq 0\ \&\ a\neq\pm b\ \&\ b\neq\pm 3a)$ is SLOCC inequivalent to $L_{ab_3}(ab\neq 0\ \&\ a\neq\pm b\ \&\ b\neq\pm 3a)$.

This case can be treated analogously to case 3.

As a consequence, $L'_{ab_3}(a\neq 0)$ is SLOCC inequivalent to L_{ab_3}. An analogous argument shows that $L_{ab_3}(a\neq 0)$ is SLOCC inequivalent to L'_{ab_3}.

C. The relation between L'_{ab_3} and L_{ab_3} under permutations

Let $|\gamma\rangle$ be the state of the subfamily $L'_{ab_3}(a\neq 0\ \&\ b=0)$, $|\eta\rangle$ be the state of the subfamily $L'_{ab_3}(b=3a\neq 0)$, $|\vartheta\rangle$ be the state of the subfamily $L'_{ab_3}(b=-3a\neq 0)$, and $|\nu\rangle$ be the state of the subfamily $L'_{ab_3}(ab\neq 0\ \&\ a\neq\pm b\ \&\ b\neq\pm 3a)$. We argue that the above four subfamilies are SLOCC inequivalent to L_{ab_3} under any permutation of qubits. This can be seen as follows. Let (i,j) be the transposition of qubits i and j. A tedious calculation shows that the permutations giving rise to different $|\gamma\rangle$ are $\kappa_1=I$, $\kappa_2=(1,3)$, $\kappa_3=(1,4)$, $\kappa_4=(1,2)(1,3)$, $\kappa_5=(1,2)(1,4)$, and $\kappa_6=(1,4)(1,2)(1,3)$. Similarly, the permutations giving rise to different $|\eta\rangle$, $|\vartheta\rangle$, and $|\nu\rangle$ are $\pi_1=I$, $\pi_2=(1,2)$, $\pi_3=(1,3)$, $\pi_4=(1,4)$, $\pi_5=(1,3)(1,2)$, $\pi_6=(1,4)(1,2)$, $\pi_7=(1,2)(1,3)$, $\pi_8=(1,2)(1,4)$, $\pi_9=(1,2)(1,3)(1,2)$, $\pi_{10}=(1,2)(1,4)(1,2)$, $\pi_{11}=(1,4)(1,2)(1,3)$, and $\pi_{12}=(1,4)(1,2)(1,3)(1,2)$. The result that $\kappa_i|\gamma\rangle(i=1,\ldots,6)$, $\pi_j|\eta\rangle$, $\pi_j|\vartheta\rangle$, and $\pi_j|\nu\rangle(j=1,\ldots,12)$ are all SLOCC inequivalent to L_{ab_3} then follows by calculating the ranks r_{AB}, r_{AC}, and r_{AD} of $\kappa_i|\gamma\rangle$, $\pi_j|\eta\rangle$, $\pi_j|\vartheta\rangle$ and $\pi_j|\nu\rangle$, and using an argument analogous to that of case 3 in the previous section.

Remark 2. By using Tables VII and VIII, one can verify that $(1,4)L'_{ab_3}(a=b\neq 0)$ is SLOCC equivalent to $L_{ab_3}(a=0\ \&\ b\neq 0)$ under the invertible local operator $\sigma_x\otimes\sigma_z\otimes iI\otimes\sigma_y$, and $(1,3)L'_{ab_3}(a=-b\neq 0)$ is SLOCC equivalent to $L_{ab_3}(a=0\ \&\ b\neq 0)$ under the invertible local operator $\sigma_x\otimes\sigma_z\otimes\sigma_y\otimes iI$.

D. $L'_{ab_3}(a\neq 0)$ is SLOCC inequivalent to the other eight families by Verstraete *et al.*

Here we show that $L'_{ab_3}(a\neq 0)$ is not only SLOCC inequivalent to L_{ab_3} but also SLOCC inequivalent to the other eight families by Verstraete *et al.* For simplicity, we only show that $L'_{ab_3}(a=-b\neq 0)$ is SLOCC inequivalent to the other eight families obtained by Verstraete *et al.* From Table VIII, $r_{AB}r_{AC}r_{AD} = 443$ for $L'_{ab_3}(a = -b \neq 0)$. Consulting Tables V–VII, and using the fact that the subfamilies with different ranks of coefficient matrices are SLOCC inequivalent to each other, it suffices to show that $L'_{ab_3}(a = -b \neq 0)$ is SLOCC inequivalent to the subfamilies with $r_{AB}r_{AC}r_{AD} = 443$ of G_{abcd} and L_{abc_2}.

To show that $L'_{ab_3}(a = -b \neq 0)$ is SLOCC inequivalent to the subfamily with $r_{AB}r_{AC}r_{AD} = 443$ of G_{abcd}, we use the degree 6 polynomial invariant D_{xy} given in Eq. (20). It is readily seen from Eq. (21) that $D_{xy} = 0$ for $L'_{ab_3}(a = -b \neq 0)$. A simple calculation shows that

$$D_{xy} = (\alpha\beta - \gamma\delta)(\alpha\beta + \gamma\delta)(\alpha^2 + \beta^2 - \gamma^2 - \delta^2) \quad (32)$$

for G_{abcd} [as defined in Eq. (15)]. It is readily seen from Eq. (32) that $D_{xy} \neq 0$ for the subfamily with $r_{AB}r_{AC}r_{AD} = 443$ of G_{abcd} and then the desired result follows.

Next we show that $L'_{ab_3}(a = -b \neq 0)$ is SLOCC inequivalent to the subfamily with $r_{AB}r_{AC}r_{AD} = 443$ of L_{abc_2} [as defined in Eq. (16)]. A calculation shows that

$$D_{xy} = (\alpha'\beta')^2(\alpha'^2 - \gamma'^2 + \beta'^2) \quad (33)$$

for L_{abc_2}. From Table VI, we distinguish the following two cases.

Case 1. $\alpha' \neq 0$ & $\beta' = \pm\gamma' \neq 0$ & $\alpha' \neq \pm\beta'$.

In this case $D_{xy} \neq 0$ and then the desired result follows.

Case 2. $\alpha' = 0$ & $\beta' \neq \pm\gamma'$ & $\beta'\gamma' \neq 0$.

In this case $D_{xy} = 0$. We can resort to the semi-invariants given in Eqs. (22) and (23). Let $|\varphi\rangle$ be any state SLOCC equivalent to $L'_{ab_3}(a = -b \neq 0)$ with $\mathcal{A}_1$ given by Eq. (24). A tedious but straightforward calculation yields

$$F_1(|\varphi\rangle) = -2a^2\alpha_1^4\left[\prod_{i=2}^{4}\det\mathcal{A}_i\right]^2, \quad (34)$$

$$F_2(|\varphi\rangle) = -2a^2\alpha_3^4\left[\prod_{i=2}^{4}\det\mathcal{A}_i\right]^2. \quad (35)$$

In view of Eqs. (34) and (35) and the fact that $\mathcal{A}_1$ is invertible, it follows at once that if $|\varphi\rangle$ is SLOCC equivalent to $L'_{ab_3}(a = -b \neq 0)$, then the following equation holds:

$$|F_1(\varphi)| + |F_2(\varphi)| \neq 0. \quad (36)$$

The desired result then follows by noting that $F_1 = F_2 = 0$ for L_{abc_2} with $\alpha' = 0$ & $\beta' \neq \pm\gamma'$ & $\beta'\gamma' \neq 0$.

As a consequence, $L'_{ab_3}(a = -b \neq 0)$ is SLOCC inequivalent to the nine families obtained by Verstraete *et al.* [3].

The discussion suggests that the partition in Ref. [3] is incomplete. For completeness, one may add the family L'_{ab_3} to the family L_{ab_3} in Ref. [3]. An analogous argument shows that the partition in Ref. [6] is incomplete as well, and for completeness, one may add the family L_{ab_3} to the family 6 in Ref. [6].

VII. CONCLUSION

We have recast the necessary and sufficient condition for two n-qubit states to be equivalent under SLOCC into an equivalent form in terms of the coefficient matrices associated with the states. As a direct consequence of the new necessary and sufficient condition, we have shown that the rank of the coefficient matrix as well as the rank of the ℓ-qubit reduced density matrix is invariant under SLOCC. We have also presented a recursive formula for the calculation of the rank of coefficient matrix of an n-qubit biseparable state. The recursive formula further gives rise to a biseparability criterion in terms of the rank of coefficient matrix to determine if an arbitrary n-qubit pure state is biseparable. The invariance of the rank of coefficient matrix together with the biseparability criterion reveals that all the degenerate families of general n qubits are inequivalent under SLOCC.

We have then classified four-qubit states under SLOCC via the ranks of coefficient matrices and the nine families introduced by Verstraete *et al.* were further partitioned into inequivalent subfamilies. In particular, we have found 28 genuinely entangled families and all the degenerate classes can be distinguished up to permutations of the four qubits. We have performed a detailed study of the relation between the family L_{ab_3} and the family L'_{ab_3} with corrections to the signs of the last two terms in the formula of L_{ab_3} via the ranks of coefficient matrices. By using a degree 6 polynomial invariant and two semi-invariants of four qubits, we have found that $L'_{ab_3}(a = 0)$ is SLOCC equivalent to $L'_{ab_3}(a = 0)$ whereas $L'_{ab_3}(a \neq 0)$ is SLOCC inequivalent to $L_{ab_3}(a \neq 0)$. We have also demonstrated that $L'_{ab_3}(a \neq 0$ & $b = 0)$, $L'_{ab_3}(b = \pm 3a \neq 0)$, and $L'_{ab_3}(ab \neq 0$ & $a \neq \pm b$ & $b \neq \pm 3a)$ are SLOCC inequivalent to L_{ab_3} under any permutation of qubits, whereas $L'_{ab_3}(a = \pm b \neq 0)$ are SLOCC equivalent to $L_{ab_3}(a = 0$ & $b \neq 0)$ under some permutations. This suggests that the partition of four-qubit states into the nine families by Verstraete *et al.* is incomplete, and for completeness, one may simply add the family L'_{ab_3} to the family L_{ab_3}.

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APPENDIX

Following [25], D_{xy} can be constructed as

$$D_{xy} = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix}, \quad (A1)$$

where the entries of D_{xy} are given by

$$\begin{aligned} d_{11} &= a_0a_3 - a_1a_2, \\ d_{12} &= a_0a_7 - a_1a_6 - a_2a_5 + a_3a_4, \\ d_{13} &= a_4a_7 - a_5a_6, \\ d_{21} &= a_0a_{11} - a_1a_{10} - a_2a_9 + a_3a_8, \\ d_{22} &= a_0a_{15} - a_1a_{14} - a_2a_{13} + a_3a_{12} \\ &\quad + a_4a_{11} - a_5a_{10} - a_6a_9 + a_7a_8, \\ d_{23} &= a_4a_{15} - a_5a_{14} - a_6a_{13} + a_7a_{12}, \\ d_{31} &= a_8a_{11} - a_9a_{10}, \\ d_{32} &= a_8a_{15} - a_9a_{14} - a_{10}a_{13} + a_{11}a_{12}, \\ d_{33} &= a_{12}a_{15} - a_{13}a_{14}. \end{aligned} \quad (A2)$$

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