Entangled symmetric states of N qubits with all positive partial transpositions

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From both theoretical and experimental points of view symmetric states constitute an important class of multipartite states. Still, entanglement properties of these states, in particular those with positive partial transposition (PPT), lack a systematic study. Aiming at filling in this gap, we have recently affirmatively answered the open question of existence of four-qubit entangled symmetric states with PPT and thoroughly characterized entanglement properties of such states [J. Tura *et al.*, Phys. Rev. A **85**, 060302(R) (2012)]. With the present contribution we continue on characterizing PPT entangled symmetric states. On the one hand, we present all the results of our previous work in a detailed way. On the other hand, we generalize them to systems consisting of an arbitrary number of qubits. In particular, we provide criteria for separability of such states formulated in terms of their ranks. Interestingly, for most of the cases, the symmetric states are either separable or typically separable. Then, edge states in these systems are studied, showing in particular that to characterize generic PPT entangled states with four and five qubits, it is enough to study only those that assume few (respectively, two and three) specific configurations of ranks. Finally, we numerically search for extremal PPT entangled states in such systems consisting of up to 23 qubits. One can clearly notice regularity behind the ranks of such extremal states, and, in particular, for systems composed of odd numbers of qubits we find a single configuration of ranks for which there are extremal states.

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I. INTRODUCTION

Characterization of entanglement [1] in composite quantum states with positive partial transposition (PPT states) remains a difficult problem. One of the reasons for that is the lack of a universal separability criterion making it possible to distinguish unambiguously separable from PPT entangled states (see, nevertheless, e.g., Ref. [2] for numerous necessary separability conditions). There are, however, methods providing some insight into the structure of PPT entangled states. One of them exploits the fact that all states that remain positive under partial transposition form a convex set, which as a proper subset contains the PPT entangled states. To fully characterize the latter, it is then enough to know all the extremal points of this convex set. This approach has recently been extensively studied (see Refs. [3–7]). In particular, it made it possible to solve the open problem of existence of four-qubit PPT entangled symmetric states [8], and also, although in an indirect way, disprove the Peres conjecture in the multipartite case [9].

The problem of characterization of PPT entangled states is even more complicated in the multipartite case. Clearly, the set of PPT states arises by intersecting sets of states that remain positive under partial transpositions with respect to single bipartitions; thus, its boundary becomes more complicated with the increasing number of parties. Nevertheless, the complexity can be reduced by imposing some symmetries. For instance, demanding that the states under study commute with multilateral action of unitary or orthogonal groups leads to classes of multipartite states whose full characterization with respect to entanglement becomes possible (see, e.g., Refs. [10]).

Another interesting example of a class of states obtained by imposing some symmetry are those supported on the symmetric subspace of a given multipartite Hilbert space. The so-called *symmetric states* have recently been attracting much attention [11–18]. In particular, the underlying symmetry allowed for the use of the Majorana representation [19] for an identification of SLOCC classes of multipartite symmetric states [14] (see also Refs. [15]). The same symmetry provides advantages in calculating certain entanglement measures [16]. Another motivation comes from the recent experimental realizations of symmetric states of many qubits, as for instance, the six-qubit Dicke states [20] or the eight-qubit GHZ states [21] (see also Ref. [22] in this context).

However, more effort has been devoted to the pure symmetric states, leaving the characterization of entanglement of mixed, in particular PPT, states as an open problem. It is known so far that for $N = 2,3$ all PPT symmetric states are separable [11]. Then, examples of 5- or 6-qubit PPT entangled symmetric states were found in Refs. [17,18]. Recently, the remaining case of $N = 4$ has been studied in Ref. [8], where the open question as to whether partial transposition serves in this case as a necessary and sufficient condition for separability (as this is the case for $N = 2,3$) has been given a negative answer. The main aim of the present paper is to continue the characterization of PPT entanglement in symmetric states. We discuss in detail methods used in Ref. [8] and then generalize them to the case of arbitrary N. We derive separability criteria for PPT symmetric states in terms of their ranks and ranks of their partial transpositions. Then we exclude configurations of ranks for which they are generically not edge. Finally, we adapt to the multipartite case an algorithm making it possible to search for extremal PPT entangled states [3] (see also Ref. [7]). Exploiting it, we study ranks of the extremal PPT entangled symmetric states consisting of up to 23 qubits. Interestingly, we show that there are at most three distinct configurations of ranks for which we find extremal PPT entangled symmetric states, and, in particular, for odd N there is only a single such configuration.

The paper is structured as follows. In the next section (Sec. II) we recall all notions and facts necessary for further considerations. In Sec. III we investigate the entanglement properties of the PPT symmetric states. Then, in Sec. IV we seek extremal entangled PPT symmetric states consisting of even more than 20 qubits and classify them with respect to their ranks. The results obtained for exemplary systems consisting of four, five, and six qubits are collected in Sec. V. We conclude in Sec. VI.

II. PRELIMINARIES

Let us start with a couple of definitions that we use throughout the paper. Let

$$\mathcal{H}_N = \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_N} \tag{1}$$

denote a multipartite product Hilbert space and D the convex set of density operators acting on $\mathcal{H}_N$. By $R(\rho)$, $r(\rho)$, $K(\rho)$, and $k(\rho)$ we denote, respectively, the range, rank, kernel, and the dimension of the kernel of a given $\rho \in D$. Then, $A_1,\ldots,A_N$ stands for the subsystems of a given N-partite ρ, and, in the case of low N, we also denote them by A,B, etc.

PPT and separable states. Let us now split the set $I = \{A_1,\ldots,A_N\}$ into two disjoint subsets S and $\overline{S}$ ($S \cup \overline{S} = I$) and call it bipartition $S|\overline{S}$. We say that a given state ρ acting on $\mathcal{H}_N$ is PPT with respect to this bipartition if and only if $\rho^{T_S} \geqslant 0$. Clearly, states with this property make a convex set denoted D_S. An element of D whose partial transpositions with respect to all bipartitions (notice that for a given bipartition $S|\overline{S}$, partial transpositions with respect to S and $\overline{S}$ are equivalent under the full transposition) are positive are called *fully PPT*, and, since in this paper we deal only with fully PPT states, we call them simply *PPT states*. Clearly, such states make also a convex set which is simply the intersection of D_S for all S.

A particular example of a state that is PPT is the *fully separable state* [23,24]:

$$\rho = \sum_i p_i \rho_{A_1}^i \otimes \cdots \otimes \rho_{A_N}^i, \quad p_i \geqslant 0, \quad \sum_i p_i = 1, \tag{2}$$

where $\rho_{A_j}^i$ denote density matrices representing all subsystems. Clearly, in multipartite systems one may define various types of separability (see, e.g., [24,25]). Nonetheless, as we see later, in the symmetric case a given state ρ is either genuine multipartite entangled, that is, cannot be written as a convex combination of states which are separable with respect to, in general, different bipartitions, or takes the form (2).

Edge states. An important class of entangled PPT states are the so-called *edge states* [26–29]. We call ρ acting on $\mathcal{H}_N$ edge if and only if there does not exist a product vector $|e_1\rangle \otimes \cdots \otimes |e_N\rangle$ with $|e_i\rangle \in \mathbb{C}^{d_i}$ such that $|e_1\rangle \otimes \cdots \otimes |e_N\rangle \in R(\rho)$ and $(|e_1\rangle \otimes \cdots \otimes |e_N\rangle)^{C_S} \in R(\rho^{T_S})$ for all S, where by C_S we denoted partial conjugation with respect to S. The importance of edge states in the separability problem comes from the fact that any PPT state can be decomposed as a mixture of a fully separable and an edge state [26]. Alternatively speaking, these are states from which no product vector can be subtracted without losing the PPT or positivity property, meaning that they lay on the boundary of the set of PPT states. However, they do not have to be extremal, although any extremal state is also edge.

Edge states have been studied in bipartite or tripartite systems and many examples have been found [27,28,30].

Symmetric states. Let us now concentrate on the N-qubit Hilbert space,

$$\mathcal{H}_{2,N} = (\mathbb{C}^2)^{\otimes N}, \tag{3}$$

and consider its subspace $\mathcal{S}_N$ spanned by the un-normalized vectors

$$\left|E_i^N\right\rangle = |\{0,i\},\{1,N-i\}\rangle \quad (i = 0,\ldots,N), \tag{4}$$

which are just symmetric sums of vectors being products of i zeros and $N-i$ ones. These vectors, when normalized, are also known as Dicke states. For further benefits, let us notice that the dimension of $\mathcal{S}_N$ is $N+1$, and therefore it is isomorphic to $\mathbb{C}^{N+1}$, which we denote $\mathcal{S}_N \cong \mathbb{C}^{N+1}$. Also, by $\mathcal{P}_N$ we denote the projector onto $\mathcal{S}_N$.

We call a state ρ acting on $H_{2,N}$ *symmetric* if and only if it is supported on $\mathcal{S}_N$, or, in other words, $R(\rho) \subseteq \mathcal{S}_N$. In yet other words, ρ is symmetric if and only if the equations

$$V_\sigma \rho = \rho V_{\sigma'}^\dagger = \rho \tag{5}$$

are obeyed for any permutations $\sigma,\sigma' \in \Sigma_N$, where Σ_N is the group of all permutations of an N-element set, while V_σ is an operator defined as $V_\sigma|\psi_1\rangle\cdots|\psi_N\rangle = |\psi_{\sigma(1)}\rangle\cdots|\psi_{\sigma(N)}\rangle$ for any vectors $|\psi_i\rangle \in \mathbb{C}^2$.

In the case of symmetric states the number of relevant partial transpositions defining the set of PPT symmetric states $D_{\mathrm{PPT}}^{\mathrm{sym}}$ is significantly reduced. This is because positivity of a partial transposition with respect to some subset S is equivalent to positivity of all partial transpositions with respect to subsystems of the same size $|S|$. Together with the fact that for a given bipartition $S|\overline{S}$, $\rho^{T_S} \geqslant 0 \Leftrightarrow \rho^{T_{\overline{S}}} \geqslant 0$, one has $\lfloor N/2 \rfloor$ partial transpositions defining $D_{\mathrm{PPT}}^{\mathrm{sym}}$. We choose them to be T_{A_1}, $T_{A_1A_2}$, etc.; however, for simplicity we also denote them as $T_1 \equiv T_{A_1}$, $T_2 \equiv T_{A_1A_2}$, and so on. Alternatively, in systems of small size, we use T_A, T_{AB}, T_{ABC}, etc., to denote the relevant partial transpositions.

Let us now notice that since $\mathcal{S}_i \cong \mathbb{C}^{i+1}$, a N-qubit symmetric state ρ can be seen with respect to a bipartition $S|\overline{S}$ as a bipartite state acting on $\mathbb{C}^{|S|+1} \otimes \mathbb{C}^{N-|S|+1}$. This gives us nontrivial bounds on the ranks of partial transpositions with respect to all S, namely,

$$r(\rho^{T_S}) \leqslant (|S|+1)(N-|S|+1) \tag{6}$$

for $|S| = 0,\ldots,\lfloor N/2 \rfloor$, which, in particular, means that $r(\rho) \leqslant N+1$. A very convenient way of classifying PPT states is through their ranks and ranks of their partial transpositions, that is, the $\lfloor N/2 \rfloor$-tuples

$$\begin{aligned}(r(\rho),r(\rho^{T_{A_1}}),r(\rho^{T_{A_1A_2}}),\ldots,r(\rho^{T_{A_1,\ldots,A_{\lfloor N/2\rfloor}}}))\\ \equiv (r(\rho),r(\rho^{T_1}),r(\rho^{T_2}),\ldots,r(\rho^{T_{\lfloor N/2\rfloor}})).\end{aligned} \tag{7}$$

Finally, let us recall that ρ acting on some bipartite Hilbert space $\mathcal{H}_2 = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ is said to be supported on $\mathcal{H}_2$ if and only if $R(\rho_A) = \mathbb{C}^{d_1}$ and $R(\rho_B) = \mathbb{C}^{d_2}$. Alternatively speaking, ρ is not supported on $\mathcal{H}_2$ if either ρ_A or ρ_B has a vector in the kernel.

III. CHARACTERIZING PPT ENTANGLEMENT IN N-QUBIT SYMMETRIC STATES

Here, exploiting the results of Refs. [31,32] we derive separability criteria for the PPT symmetric states in terms of the ranks (7). Recall that in these papers it was shown that any PPT state ρ supported on a Hilbert space $\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}$ is separable if $r(\rho)\leqslant\max\{d_1,d_2\}$.

Then, we study the edge symmetric states and, in particular, we show that symmetric states assuming certain ranks cannot be edge.

A. Separability

To begin with the separability properties let us recall that if a pure N-partite symmetric state $|\psi\rangle$ is separable with respect to some bipartition, then it must be fully separable (2), that is, $|\psi\rangle=|e\rangle^{\otimes N}$ with $|e\rangle\in\mathbb{C}^2$ (see Refs. [11,13]). This straightforwardly implies that entangled symmetric pure and thus mixed states have genuine multipartite entanglement (see also Ref. [13]). Indeed, if a symmetric ρ can be written as a convex combination of density matrices, each separable across some, in general different, bipartition, then ρ has pure separable vectors in its range. Since each such vector is symmetric, it assumes the above form $|e\rangle^{\otimes N}$, meaning that ρ is fully separable. Thus, throughout the paper, by saying that a symmetric state ρ is separable we mean that it is fully separable, that is, it takes the form (2).

Let us now establish some conditions for separability in terms of ranks of ρ. We start with the following technical lemma.

Lemma 1. Consider an N-qubit symmetric state and a bipartition $S|\overline{S}$ ($|S|\leqslant N-|S|$). Then, let k_S and $k_{\overline{S}}$ (r_S and $r_{\overline{S}}$) denote the dimensions of the kernels (ranges) of subsystems of ρ with respect to $S|\overline{S}$. The following statements hold:

(i) if $k_{\overline{S}}>0$, then $r(\rho)\leqslant r_{\overline{S}}$;
(ii) if $k_S>0$, then $r(\rho)\leqslant r_S$;
(iii) if $k_S>0$, and $k_{\overline{S}}>0$, then $r(\rho)\leqslant\min\{r_S,r_{\overline{S}}\}$.

Proof. Recall first that with respect to the bipartition $S|\overline{S}$, the N-qubit symmetric state ρ can be seen as a bipartite state acting on $\mathbb{C}^{|S|+1}\otimes\mathbb{C}^{|\overline{S}|+1}=\mathbb{C}^{|S|+1}\otimes\mathbb{C}^{N-|S|+1}$.

We prove the first case and then the remaining two follow. Assume then that $k_{\overline{S}}>0$, meaning that $\rho_{\overline{S}}$ has $k_{\overline{S}}$ linearly independent vectors $|\phi_i\rangle(i=1,\ldots,k_{\overline{S}})$ in the kernel. Consequently, for any $|S|$-qubit symmetric vector $|\psi\rangle$, the projected vectors $\mathcal{P}_N(|\psi\rangle|\phi_i\rangle)(i=1,\ldots,N-r_{\overline{S}}-|S|+1)$ belong to $K(\rho)$. In what follows we show that by choosing properly vectors $|\psi\rangle$, one is able to find $N-r_{\overline{S}}+1$ linearly independent vectors in $K(\rho)$ of this form.

First, we prove that for any $|\psi\rangle$, $N-r_{\overline{S}}-|S|+1$ projected vectors $\mathcal{P}_N(|\psi\rangle|\phi_i\rangle)\in K(\rho)$ are linearly independent. Towards this end, let us assume, on the contrary, that there exists a collection of nonzero numbers $\alpha_i\in\mathbb{C}$ such that $\sum_i\alpha_i\mathcal{P}_N(|\psi\rangle|\phi_i\rangle)=0$. The latter is equivalent to saying that the vector $|\psi\rangle\otimes\sum_i\alpha_i|\phi_i\rangle$ sits in the kernel of $\mathcal{P}_N$. However, since $\sum_i\alpha_i|\phi_i\rangle$ is an $|S|$-qubit symmetric vector, this is possible only if $|\phi_i\rangle$ are linearly dependent, contradicting the fact that they span the kernel of $\mathrm{Tr}_S\rho$.

Now, we consider particular vectors

$$\big|\Phi_i^j\big\rangle=\mathcal{P}_N\big(\big|E_j^{|S|}\big\rangle|\phi_i\rangle\big),\tag{8}$$

with $j=0,\ldots,|S|$ and $i=1,\ldots,|\overline{S}|-r_{\overline{S}}+1$. As already proven, for any j, the vectors $|\Phi_i^j\rangle$ make an $(|\overline{S}|-r_{\overline{S}}+1)$-element linearly independent set. Let us now concentrate on the vectors $|\Phi_i^0\rangle$ and choose the one for which $\langle E_{|\overline{S}|}^N|\Phi_i^0\rangle$ is nonzero, say $|\Phi_l^0\rangle$. Notice that by the construction $\langle E_k^N|\Phi_i^0\rangle=0$ for $k>|\overline{S}|$; to obtain $|\Phi_i^0\rangle$ we symmetrize $|\phi_i\rangle$ with $|E_0^{|S|}\rangle=|0\rangle^{\otimes|S|}$, meaning that $|\Phi_i^0\rangle$ decomposes into the symmetric vectors $|E_k^N\rangle$ with $k\leqslant|\overline{S}|$. If, however, $\langle E_{|\overline{S}|}^N|\Phi_i^0\rangle=0$ for all i, we choose the one for which $\langle E_{|\overline{S}|-1}^N|\Phi_i^0\rangle\neq0$, etc. Clearly, repeating this we must find the desired vector as otherwise $|\Phi_i^0\rangle=0$ for all i, contradicting the fact that the vectors $|\Phi_i^0\rangle$ are linearly independent.

Let us then assume for simplicity that $|\Phi_l^0\rangle$ is such that $\langle E_{|\overline{S}|}^N|\Phi_l^0\rangle\neq0$, meaning that $\langle\phi_l|1\rangle^{\otimes|\overline{S}|}\neq0$ [cf. Eq. (8)]. Consequently, the vectors $|\Phi_l^j\rangle$ $(j=1,\ldots,|S|)$, when decomposed into the symmetric basis of $\mathcal{S}_N$, contain $|E_k^N\rangle$ with $k\geqslant|\overline{S}|+1$ and therefore are linearly independent of the set $\{|\Phi_i^0\rangle\}_i$. Moreover, by the very construction, they make an $|S|$-element set of linearly independent vectors themselves, meaning that the vectors $|\Phi_i^0\rangle(i=1,\ldots,|\overline{S}|-r_{\overline{S}}+1)$ together with $|\Phi_l^j\rangle$ $(j=1,\ldots,|S|)$ make the desired set of $N-r_{\overline{S}}+1$ linearly independent vectors in $K(\rho)$.

Consequently, $k(\rho)\geqslant N-r_{\overline{S}}+1$ which, taking into account the maximal possible rank of a symmetric state ρ, gives the bound $r(\rho)\leqslant N+1-N+r_{\overline{S}}-1=r_{\overline{S}}$. In an analogous way one proves the second case, that is, when $k_S>0$. Precisely, following the above arguments, one sees that k_S linearly independent vectors in the kernel of ρ_S gives at least $N-r_S+1$ linearly independent vectors in $K(\rho)$, imposing the bound $r(\rho)\leqslant r_S$. To prove the third case, one just chooses the tighter of both the above bounds, that is, $r(\rho)\leqslant\min\{r_S,r_{\overline{S}}\}$. ■

Essentially, this lemma says that if the symmetric state ρ is not supported on $\mathbb{C}^{|S|+1}\otimes\mathbb{C}^{N-|S|+1}$ with respect to the bipartition $S|\overline{S}$, its rank is bounded from above by ranks of its subsystems. In the particular case when the subsystem S consists of a single party, it straightforwardly implies that if $r(\rho)\geqslant N$ then ρ has to be supported on $\mathbb{C}^2\otimes\mathbb{C}^N$ with respect to the bipartition one versus the rest $(A_1|A_2,\ldots,A_N)$.

The following fact was already stated in Ref. [11]; however, a detailed proof was not given. We exploit Lemma 1 to demonstrate it rigorously.

Theorem 1. Let ρ be a N-qubit PPT symmetric state. If it is entangled then $r(\rho)=N+1$; that is, ρ is of maximal rank.

Proof. An N-qubit symmetric state can be seen as a bipartite state acting on $\mathbb{C}^2\otimes\mathbb{C}^N$ with respect to the bipartition one qubit versus the rest, as for instance $A_1|A_2,\ldots,A_N$. Let us denote by ρ_{A_1} and $\rho_{A_1,\ldots,A_N}$ the subsystems of ρ with respect to this bipartition. Assuming then that $r(\rho)\leqslant N$, the results of Ref. [31] imply that ρ is separable provided it is supported on $\mathbb{C}^2\otimes\mathbb{C}^N$. If, however, the latter does not hold, there are vectors in the kernel of either ρ_{A_1} or $\rho_{A_2,\ldots,A_N}$. In the first case, Lemma 1 implies that $r(\rho)=1$ and ρ is a pure product vector, while in the second case $r(\rho)$ is upped bounded by the rank of $\rho_{A_2,\ldots,A_N}$. Again, results of Ref. [31] apply here, meaning that ρ is separable. ■

Alternatively speaking, this theorem means that there are no PPT entangled N-qubit symmetric states of rank less than $N+1$. On the other hand, it provides only a sufficient condition for separability, as there are separable symmetric states of rank $N+1$. Another consequence of Theorem 1 is that for an arbitrary bipartition $S|\overline{S}$, a PPT entangled state ρ and its partial transposition ρ^{T_S} are supported on the corresponding Hilbert space $\mathbb{C}^{|S|+1}\otimes\mathbb{C}^{N-|S|+1}$. Specifically, one has the following lemma.

Lemma 2. Consider an N-qubit PPT symmetric state and an arbitrary bipartition $S|\overline{S}$. If ρ is entangled, then ρ^{T_S} is supported on the bipartite Hilbert space corresponding to the bipartition $S|\overline{S}$, that is, $\mathbb{C}^{|S|+1}\otimes\mathbb{C}^{N-|S|+1}$. In other words, if for some bipartition $S|\overline{S}$, ρ^{T_S} is not supported on $\mathbb{C}^{|S|+1}\otimes\mathbb{C}^{N-|S|+1}$, then ρ is separable.

Proof. Assume that the PPT state ρ is entangled but the partial transposition ρ^{T_S} is not supported on the Hilbert space corresponding to the bipartition $S|\overline{S}$, that is, $\mathbb{C}^{|S|+1}\otimes\mathbb{C}^{N-|S|+1}$. This means that one of its subsystems, say the $\overline{S}$ one, contains an $|\overline{S}|$-qubit symmetric vector $|\phi\rangle$ in the kernel. Consequently, for any $|S|$-qubit vector $|\psi\rangle$, the implication

$$\rho^{T_S}|\psi\rangle|\phi\rangle=0\quad\Rightarrow\quad\rho|\psi^*\rangle|\phi\rangle=0 \tag{9}$$

holds. Putting, for instance, $|\psi\rangle=|0\rangle^{\otimes|S|}$, one sees that the symmetrized vector $\mathcal{P}_N(|0\rangle^{\otimes|S|}\otimes|\phi\rangle)$ belongs to $K(\rho)$. As a result $r(\rho)\leqslant N$ and Theorem 1 implies that ρ is separable, leading to the contradiction. ■

Then, with the aid of Lemma 2, we can prove the analog of Theorem 1 for the ranks of partial transpositions of ρ.

Theorem 2. Let us consider an N-qubit PPT symmetric state ρ and a bipartition $S|\overline{S}$ with an arbitrary S ($|S|\leqslant|\overline{S}|$). If it is entangled, then $r(\rho^{T_S})>N-|S|+1$. In particular, if ρ is entangled, then $r(\rho^{T_A})\geqslant N+1$.

Proof. Due to Lemma 2, we can assume that with respect to the bipartition on $S|\overline{S}$, ρ is supported on $\mathbb{C}^{|S|+1}\otimes\mathbb{C}^{N-|S|+1}$ as otherwise it is separable. Then, if $r(\rho^{T_S})\leqslant N-|S|+1$, the results of Ref. [32] imply that ρ is separable. Noting that this reasoning is independent of the bipartition, we complete the proof. ■

In other words, any PPT symmetric states whose rank obeys $r(\rho^{T_S})\leqslant N-|S|+1$ for some bipartition, is separable.

Still, by using more tricky bipartitions we can provide further separability conditions for generic symmetric states in terms of the ranks. In this direction we prove the following theorem.

Theorem 3. Consider an N-qubit PPT symmetric state ρ and a bipartition $S|\overline{S}$. If $r(\rho^{T_S})\leqslant(|S|+1)(N-|S|)$, then the generic ρ is separable.

Proof. Consider an N-qubit state $\sigma=\rho^{T_S}$ which is PPT, but no longer symmetric, and a party which does not belong to S, say A_N. With respect to the bipartition A_N versus the rest, σ can be seen as a bipartite state acting on $\mathbb{C}^{(|S|+1)(N-|S|)}\otimes\mathbb{C}^2$. Since ρ is fully PPT, it clearly follows that $\sigma^{T_{A_N}}\geqslant 0$. This, together with the fact that $r(\rho^{T_S})\leqslant(|S|+1)(N-|S|)$, implies that σ has to be separable across the bipartition $A_1,\dots,A_{N-1}|A_N$, that is,

$$\sigma=\rho^{T_S}=\sum_i|\psi_i\rangle\langle\psi_i|_{A_1,\dots,A_{N-1}}\otimes|e_i\rangle\langle e_i|_{A_N}, \tag{10}$$

provided that it is supported on $\mathbb{C}^{(|S|+1)(N-|S|)}\otimes\mathbb{C}^2$, which generically is the case.

Now, let us notice that ρ^{T_S} is still symmetric with respect to subsystem $\overline{S}$, that is, $\mathcal{P}_{\overline{S}}\rho^{T_S}\mathcal{P}_{\overline{S}}=\rho^{T_S}$. Since then all vectors $|\psi_i\rangle|e_i\rangle$ appearing in the decomposition (10) belong to $R(\rho^{T_S})$, they also enjoy the above symmetry, that is, $\mathcal{P}_{\overline{S}}|e_i\rangle_{A_N}|\psi_i\rangle_{A_1,\dots,A_{N-1}}=|e_i\rangle_{A_N}|\psi_i\rangle_{A_1,\dots,A_{N-1}}$ for all i. This, as it is shown below, implies that for any i, $|\psi_i\rangle=|\widetilde{\psi}_i\rangle|e_i\rangle^{\otimes|\overline{S}|}$ with $|\widetilde{\psi}_i\rangle$ being from the Hilbert space corresponding to the subsystem S. Putting these forms to Eq. (10), one arrives at

$$\rho^{T_S}=\sum_i|\widetilde{\psi}_i\rangle\langle\widetilde{\psi}_i|_S\otimes|e_i\rangle\langle e_i|^{\otimes|\overline{S}|}. \tag{11}$$

Now, one can move the partial transposition with respect to S to the right-hand side of the above identity and use the fact that ρ is symmetric, which leads us to the form (2).

To complete the proof let us show that if $\mathcal{P}_{A\cup S}|e\rangle_A|\psi\rangle_{\overline{S}\cup S}=|e\rangle_A|\psi\rangle_{\overline{S}\cup S}$ holds for some one-qubit and N-qubit vectors $|e\rangle$ and $|\psi\rangle$, then the latter assumes the form $|\psi\rangle_{\overline{S}\cup S}=|\widetilde{\psi}\rangle_{\overline{S}}\otimes|e\rangle^{\otimes|S|}$ with $|\widetilde{\psi}\rangle$ belonging to the Hilbert space associated to the subsystem $\overline{S}$. Here by $\mathcal{P}_{A\cup S}$ we denote a projector onto the symmetric subspace of the Hilbert space corresponding to the qubits A and S.

To this end, let us decompose

$$|\psi\rangle=\sum_i|i\rangle_{\overline{S}}|\phi_i\rangle_S, \tag{12}$$

where $\{|i\rangle\}$ denotes any orthogonal basis in the Hilbert space associated to the subsystem $\overline{S}$. It is clear that $\mathcal{P}_{A\cup S}|e\rangle_A|\psi\rangle_{\overline{S}\cup S}=|e\rangle_A|\psi\rangle_{\overline{S}\cup S}$ is equivalent to $\mathcal{P}_{A\cup S}|e\rangle_A|\phi_i\rangle_S=|e\rangle_A|\phi_i\rangle_S$ for all i. By virtue of the results of Refs. [11,13], the latter can hold only if $|\phi_i\rangle=|e\rangle^{\otimes|S|}$ for any i. For completeness, let us recall the proof of this fact. For this purpose, assume that $|e\rangle|\phi\rangle\in\mathcal{S}_{|S|+1}$ for some one-qubit and $|S|$-qubit vectors $|e\rangle$ and $|\phi\rangle$, respectively. Then, one immediately concludes that $|\phi\rangle$ must be symmetric and consequently can be written as

$$|\phi\rangle=\sum_{i=1}^{|S|+1}\alpha_i\left|E_i^{|S|}\right\rangle, \tag{13}$$

with $\alpha_i\in\mathbb{C}$. Putting $|e\rangle=(a,b)$ and utilizing the fact that $\mathcal{P}_{|S|+1}|e\rangle|\phi\rangle=|e\rangle|\phi\rangle$, one has

$$\sum_{i=1}^{|S|+1}\alpha_i\left[\frac{\binom{|S|}{i-1}}{\binom{|S|+1}{i-1}}\left|E_i^{|S|+1}\right\rangle+\frac{\binom{|S|}{i-1}}{\binom{|S|+1}{i}}\left|E_{i+1}^{|S|+1}\right\rangle\right]$$
$$=(a|0\rangle+b|1\rangle)\otimes\sum_{i=1}^{|S|+1}\alpha_i\left|E_i^{|S|}\right\rangle. \tag{14}$$

Projection of the above onto vectors $|0\rangle^{\otimes(|S|+1)}$, $|0,\dots,01\rangle$, $|0,\dots,011\rangle$, etc., leads to equations $a\alpha_j=b\alpha_{j-1}$ ($j=2,\dots,|S|+1$), which, in turn, imply that $\alpha_j=(b/a)^{j-1}\alpha_1$ ($j=2,\dots,|S|+1$).

Putting now $|\phi_i\rangle=|e\rangle^{\otimes|S|}$ for all i to Eq. (12), one gets

$$|\psi\rangle=\sum_i|i\rangle_{\overline{S}}|\psi_i\rangle_S=\sum_i|i\rangle_{\overline{S}}|e\rangle^{\otimes|S|}=|\widetilde{\psi}\rangle_{\overline{S}}|e\rangle^{\otimes|S|}, \tag{15}$$

which finishes the proof. ■

Notice that the above theorems imply that for most of the possible ranks, the symmetric states are either separable or generically separable. The maximal rank of a partial transposition with respect to a given subsystem S is $(|S|+1)(N-|S|+1)$. Taking into account the fact that we can put $r(\rho)=N+1$ (as otherwise the state is separable; cf. Theorem 1), we have in total

$$\prod_{|S|=1}^{\lfloor N/2\rfloor}(|S|+1)(N-|S|+1)=N!(N/2+1)^{2(\lfloor N/2\rfloor-N/2)+1} \tag{16}$$

possible configurations of the relevant ranks (7) that can be assumed by the symmetric states. With Theorems 2 and 3, we see that for a transposition with respect to S, symmetric states of the first $(|S|+1)(N-|S|)$ of the corresponding ranks are either separable or generically separable. This leaves only $|S|+1$ of ranks with respect to this bipartition for which they do not have to be generically separable. Taking into account all the relevant partial transpositions, we have in total $(\lfloor N/2\rfloor+1)!$ of the remaining cases where one can search for PPT entangled symmetric states. This is clearly a small portion (rapidly vanishing for large N) of all the possible ranks [cf. Eq. (16)]. For instance, for $N=4,5$, this gives us 6 treatable configurations of ranks [of all, respectively, 72 and 120 obtained from Eq. (16)], while for $N=6,7$ this number amounts to 24 [Eq. (16) gives in these cases, respectively, 2880 and 5040 possible ranks]. As we see later, if additionally we ask about possibility of being an edge state, these numbers may be further reduced.

B. N-qubit symmetric edge states

Let us now single out the configurations of ranks where the symmetric states can be edge. Clearly, Theorem 3 implies that the ranks of ρ^{T_S} for all S must be larger than $(|S|+1)(N-|S|)$ as otherwise the generic symmetric states are separable. In what follows we will provide a few results making it possible to bound the ranks from above.

In general, as already discussed in Sec. II, to prove that a given ρ is not edge one has to prove that there is $|e\rangle\in\mathbb{C}^2$ such that $|e\rangle^{\otimes N}\in R(\rho)$ and $[|e\rangle^{\otimes N}]^{C_S}\in R(\rho^{T_S})$ for all S. Assuming that the rank of ρ is maximal, $r(\rho)=N+1$, the above is equivalent to solving of a system of $\sum_{|S|=1}^{\lfloor N/2\rfloor}k(\rho^{T_S})$ equations,

$$[\langle e|^{\otimes N}]^{C_S}|\Psi_i^S\rangle=0, \tag{17}$$

where $|\Psi_i^S\rangle\in K(\rho^{T_S})$ and $S=A,AB,\ldots$. By putting $|e\rangle=(1,\alpha)$, one reduces Eqs. (17) to a system of polynomial equations $P(\alpha,\alpha^*)=0$ in α and α^*. This is clearly a hard problem to solve (see, e.g., the discussion in Ref. [31]). Still, under some assumptions and using a method of Ref. [33], it is possible to find a solution to a single equation of that type.

Lemma 3. Consider an equation

$$\sum_{i=0}^{k}(\alpha^*)^iQ_i(\alpha)=0\quad(\alpha\in\mathbb{C}), \tag{18}$$

where Q_i $(i=0,\ldots,k)$ are some polynomials. If $\max_i\{\deg Q_i\}=\deg Q_k=n>k$ and $\deg Q_0=m>k$, then this equation has at least one solution.

Proof. Notice that, via the results of Ref. [31], Eq. (18) has generically at most $2^{k-1}[k+n(n-k+1)]$ complex solutions. To find one, in Eq. (18) we substitute $\alpha=rs$ and $\alpha^*=r/s$ with $r\in\mathbb{R}$ and $s\in\mathbb{C}$, obtaining

$$\sum_{i=0}^{k}s^{k-i}r^iQ_i(rs)=0. \tag{19}$$

Treating r as a parameter and s as a variable, our aim now is to prove that for some r there is s such that $|s|=1$ and Eqs. (18) and (19) is obeyed. For this purpose, let us first put $s=x/r$ with $x\in\mathbb{C}$, which gives us

$$\sum_{i=0}^{k}x^{k-i}r^{2i}Q_i(x)=0. \tag{20}$$

In the limit $r\to\infty$, the left-hand side of the above equation approaches $Q_k(x)$, meaning that Eq. (19) has n solutions $s_i^{\infty}\to0$ $(i=1,\ldots,n)$. Then, in the limit of $r\to0$, the left-hand side of Eq. (20) goes to $Q_0(\alpha)$, implying that Eq. (19) has m solutions $s_i^0\to\infty$ $(i=1,\ldots,m)$.

Then, one sees that Eq. (19) has at most $n+k$ solutions with respect to s. Consequently, for $r\to\infty$ and $r\to0$, Eq. (19) has additional k roots $s_i^{\infty}(i=n+1,\ldots,n+k)$ and $n+k-m$ roots s_i^0 $(i=m+1,\ldots,n+k)$, respectively, which can remain unspecified. As r varies continuously from zero to large values, all roots s_i^0 must continuously tend to s_i^{∞}. However, since $m>k$, at least one of m roots $s_i^0\to\infty$ $(i=1,\ldots,m)$ must tend to one of the n roots $s_i^{\infty}(i=1,\ldots,n)$ which are close to zero. This means that there is at least one pair (r,s) with $|s|=1$ solving Eq. (19) and thus Eq. (18). ■

With the aid of the above lemma we can prove the following theorem.

Theorem 4. Let us consider N-qubit PPT symmetric state and a subsystem S of size $1\leqslant|S|\leqslant\lceil N/2\rceil-1$ and assume that $r(\rho^{T_X})=(|X|+1)(N-|X|+1)$ (maximal) for all X except for $X=S$, for which $r(\rho^{T_S})=(|S|+1)(N-|S|+1)-1$. Then, generically such states are not edge.

Proof. We prove that under the above assumptions it is generically possible to find a product vector $|e\rangle^{\otimes N}\in R(\rho)$ such that $(|e\rangle^{\otimes N})^{C_X}\in R(\rho^{T_X})$ for all subsystems X. Clearly, all ranks of ρ are maximal except for the one corresponding to the partial transposition with respect to the subsystem S, which is $r(\rho^{T_S})=(|S|+1)(N-|S|+1)-1$ (maximal diminished by one). Denoting by $|\Psi\rangle$ the unique vector from the kernel of ρ^{T_S}, one then has to solve a single equation $\langle\Psi|(|e^*\rangle^{\otimes|S|}\otimes|e\rangle^{\otimes(N-|S|)})=0$. After putting $|e\rangle=(1,\alpha)$ with $\alpha\in\mathbb{C}$, the latter can be rewritten as

$$\sum_{i=0}^{|S|}(\alpha^*)^iQ_i(\alpha)=0, \tag{21}$$

where $Q_i(\alpha)\,(i=0,\ldots,|S|)$ are polynomials of degree at most $N-|S|$ and generically they are exactly of degree $N-|S|$. Due to the assumption that $N-|S|>|S|$, Lemma 3 applies here, implying that (21) has at least one solution and generic symmetric ρ of the above ranks is not edge. ■

The above theorem says that generic PPT symmetric states having all ranks maximal except for a single one corresponding to a partial transposition with respect to a subsystem S such that $|S|<N-|S|$, for which the rank is

1 but maximal, are not edge. This method, however, does not work for even N and in the case when the chosen partial transposition is taken with respect to the half of the whole system; that is, $|S| = |\overline{S}| = N - |S|$. This is because in this case the resulting equation (20) is of the same orders in α and α^* and the above method does not apply.

Still, however, using a different approach, we can prove an analogous fact for $N = 4$ and $N = 6$. Specifically, we show that all symmetric states of all ranks are maximal except for the last one, which is one less than maximal, are not edge. Towards this end, let us start with some general considerations.

Let ρ be an N-qubit symmetric state with even N, and let now S denote a particular bipartition consisting of first $N/2$ qubits (half of the state). Assume then that all ranks of ρ are maximal except for the one corresponding to the partial transposition with respect to S, for which it is one but maximal; that is, $(N/2+1)^2 - 1$. Consequently, there is a single vector $|\Psi\rangle$ in the kernel of ρ^{T_S}, meaning that ρ is not edge if and only if a single equation [cf. Eqs. (17)],

$$\langle e|^{\otimes N/2}\langle e^*|^{\otimes N/2}|\Psi\rangle = 0, \tag{22}$$

with $|e\rangle \in \mathbb{C}^2$ has a solution. For this purpose, one notices that

$$V_{S,\overline{S}}\rho^{T_S}V_{S,\overline{S}}^{\dagger} = (\rho^*)^{T_S}, \tag{23}$$

with $V_{S,\overline{S}}$ denoting a unitary operator swapping the subsystems S and $\overline{S}$ (notice that both are of the same size). Then, because $|\Psi\rangle$ is the unique vector in $K(\rho^{T_S})$, it has to enjoy the same symmetry; that is, $V_{S,\overline{S}}|\Psi\rangle = |\Psi^*\rangle$. Denoting now by M_Ψ the matrix representing elements of $|\Psi\rangle$ in the product basis $|E_i^{N/2}\rangle|E_j^{N/2}\rangle$ of $\mathcal{S}_{N/2}\otimes\mathcal{S}_{N/2}$, the latter symmetry implies that $M_\Psi = M_\Psi^\dagger$. Diagonalizing M_Ψ, one sees that $|\Psi\rangle$ can be written in the form

$$|\Psi\rangle = \sum_{l=1}^{N/2+1}\lambda_l|\omega_l^*\rangle|\omega_l\rangle, \tag{24}$$

with $\lambda_l \in \mathbb{R}$ and $|\omega_l\rangle \in \mathcal{S}_{N/2}$ being eigenvalues and eigenvectors of M_Ψ, respectively.

The fact that $|\Psi\rangle \in K(\rho^{T_S})$ implies that for any pair $|x\rangle,|y\rangle \in \mathcal{S}_{N/2}$, one has

$$\begin{aligned}\langle x^*,y|\rho^{T_S}|\Psi\rangle &= \sum_l \lambda_l\langle x^*,y|\rho^{T_S}|\omega_l^*,\omega_l\rangle\\ &= \sum_l \lambda_l\langle\omega_l,y|\rho|x,\omega_l\rangle\\ &= \sum_l \lambda_l\langle\omega_l,y|\rho|\omega_l,x\rangle = 0,\end{aligned} \tag{25}$$

where the third equality follows from the fact that $V_{S,\overline{S}}\rho = \rho$. As a result,

$$\mathrm{Tr}\,[(W\otimes|x\rangle\langle y|)\rho] = 0 \tag{26}$$

holds for any pair of vectors $|x\rangle,|y\rangle \in \mathcal{S}_{N/2}$, where $W = \sum_l \lambda_l|\omega_l\rangle\langle\omega_l|$.

On the other hand, with the aid of Eq. (24), one can rewrite Eq. (22) as

$$\langle e|^{\otimes N/2}W|e\rangle^{\otimes N/2} = 0. \tag{27}$$

Assume, on the contrary, that the above equation [equivalently Eq. (22)] does not have any solution. Then, its left-hand side must have the same sign for any $|e\rangle \in \mathbb{C}^2$, say positive. This means that there is a finite positive number C such that the matrix $W' = W + C(\mathbb{1} - \mathcal{P}_N)$ is an entanglement witness [34], which clearly satisfies Eq. (26). We already know that there are no PPT entangled symmetric states of two and three qubits, and hence for $N = 4$ and $N = 6$, this witness must be decomposable. Precisely,

$$W' = P + Q^{T_A} \tag{28}$$

for $N = 4$, with P,Q being positive matrices acting on $\mathbb{C}^2\otimes\mathbb{C}^2$, while

$$W' = \widetilde{P} + \widetilde{Q}^{T_A} + \widetilde{R}^{T_{AB}} \tag{29}$$

for $N = 6$ with $\widetilde{P},\widetilde{Q},\widetilde{R} \geqslant 0$ acting on $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$. Putting Eqs. (28) and (29) to Eq. (26), one in particular arrives at the conditions

$$\mathrm{Tr}[(P\otimes|x\rangle\langle x|)\,\rho] = \mathrm{Tr}[(Q\otimes|x\rangle\langle x|)\rho^{T_A}] = 0 \tag{30}$$

for $N = 4$ and

$$\begin{aligned}\mathrm{Tr}[(\widetilde{P}\otimes|x\rangle\langle x|)\rho] &= \mathrm{Tr}[(\widetilde{Q}\otimes|x\rangle\langle x|)\rho^{T_A}]\\ &= \mathrm{Tr}[(\widetilde{R}\otimes|x\rangle\langle x|)\rho^{T_{AB}}] = 0\end{aligned} \tag{31}$$

for $N = 6$ and for any $|x\rangle \in \mathcal{S}_{N/2}$. These conditions imply that either ρ or ρ^{T_A}, or $\rho^{T_{AB}}$ is not of full rank, contradicting the assumption. Therefore, Eq. (27) and thus Eq. (22) must have a solution. In this way we have proven the following theorem.

Theorem 5. Four-qubit symmetric states of ranks (5,8,8) and six-qubit symmetric states of ranks (7,12,15,15) are not edge.

We already know that there exist PPT entangled symmetric states consisting of more than three qubits [8,17,18] and thus indecomposable entanglement witnesses detecting them. Consequently, the above method does not apply, in general, for even $N \geqslant 8$. Nevertheless, it provides a necessary condition for being edge; that is, if a symmetric state of all ranks maximal except for $r(\rho^{T_{N/2}})$ which is one less than maximal is edge, the operator $W' = \Sigma_l\lambda_l|\psi_l\rangle\langle\psi_l| + C(\mathbb{1} - \mathcal{P}_N)$ (or $-W'$) constructed from (24) is an indecomposable entanglement witness for some $C > 0$.

Interestingly, for $N = 4$ we can prove an analogous theorem in the case when the rank of $r(\rho^{T_{AB}})$ is two less than maximal.

Theorem 6. Generic four-qubit symmetric states of ranks (5,8,7) are not edge.

Proof. By assumption, ρ and ρ^{T_A} are of full rank, while $K(\rho^{T_{AB}})$ contains two linearly independent vectors $|\Psi_i\rangle$ ($i = 1,2$). Consequently, to find a product vector $|e\rangle^{\otimes 4} \in R(\rho)$ such that $|e^*\rangle^{\otimes 2}|e\rangle^{\otimes 2} \in R(\rho^{T_{AB}})$, one has to solve two equations:

$$\langle e^*|^{\otimes 2}\langle e|^{\otimes 2}|\Psi_i\rangle = 0 \quad (i = 1,2). \tag{32}$$

Let us now briefly characterize the vectors $|\Psi_i\rangle$. With the aid of the identity $\rho^{T_{AB}} = V_{AB,CD}(\rho^*)^{T_{AB}}V_{AB,CD}$, where $V_{AB,CD}$ is a unitary operator swapping AB and CD subsystems, one may show that they can be written as

$$|\Psi_1\rangle = \sum_{k=1}^{2}\lambda_k|e_k\rangle|f_k^*\rangle, \quad |\Psi_2\rangle = \sum_{k=1}^{2}\lambda_k|f_k\rangle|e_k^*\rangle. \tag{33}$$

Indeed, let us first notice that we can assume that one of $|\Psi_i\rangle$ is of Schmidt rank 2. The largest subspace of $\mathbb{C}^3\otimes\mathbb{C}^3$ containing only vectors of Schmidt rank 3 is one-dimensional (see, e.g., Ref. [35]). On the other hand, if one of $|\Psi_i\rangle (i=1,2)$ is of rank 1, that is, is a product with respect to the partition $AB|CD$, $\langle e,f^*|\rho^{T_{AB}}|e,f^*\rangle = \langle e^*,f^*|\rho|e^*,f^*\rangle = 0$, meaning that $\rho|e^*,f^*\rangle = 0$. It is then clear that since $|e\rangle,|f\rangle \in \mathcal{S}_2$, $\mathcal{P}_4|e^*,f^*\rangle \in K(\rho)$, implying that $r(\rho)=4$, which contradicts the assumption.

Assuming then that $|\Psi_1\rangle$ is of rank 2, either $V_{AB,CD}|\Psi_1^*\rangle$ is linearly independent of $|\Psi_1\rangle$, leading to Eq. (33), or $V_{AB,CD}|\Psi_1^*\rangle = \xi|\Psi_1\rangle$ for some $\xi \in \mathbb{C}$. In the latter case, short algebra implies that $|\Psi_i\rangle$ $(i=1,2)$ are not linearly independent, contradicting the fact that they span two-dimensional kernel of $\rho^{T_{AB}}$.

As a result, there is a vector $|e\rangle = (1,\alpha) \in \mathbb{C}^2$ such that $|e^*\rangle^{\otimes 2}|e\rangle^{\otimes 2} \in R(\rho^{T_{AB}})$ if and only if there is $\alpha \in \mathbb{C}$ solving the equation

$$P(\alpha^*)Q(\alpha) + \widetilde{P}(\alpha^*)\widetilde{Q}(\alpha) = 0, \tag{34}$$

where $P,\widetilde{P}$ and $Q,\widetilde{Q}$ are polynomials generically of degree 2. Such α exists if and only if there is $z \in \mathbb{C}$ fulfilling

$$P(\alpha^*) = z\widetilde{P}(\alpha^*) \tag{35}$$

and

$$\widetilde{Q}(\alpha) = -zQ(\alpha). \tag{36}$$

With the aid of the first equation, we can determine α^* as a function of z. There are clearly at most two such solutions. Putting them to Eq. (36) and getting rid of the square root, we arrive at a single equation,

$$(z^*)^2Q_4(z) + z^*Q_4'(z) + Q_4''(z) = 0, \tag{37}$$

where Q_4, Q_4', and Q_4'' stand for polynomials which are generically of fourth degree. It has at least one solution because the polynomial appearing on the left-hand side of Eq. (37) has unequal degrees in z and z^*. This allows for the application of Lemma 3, completing the proof. ■

One notices that this method cannot be directly applied in the case of larger even N. Already for $N=6$, the left-hand side of Eq. (34) contains three terms and therefore the factorization (35) and (36) cannot be done. Let us notice, however, that the numerical search for extremal states of even N done below does not reveal examples of extremal PPT entangled symmetric states of these ranks, suggesting the lack of edge states in generic states of these ranks.

More generally, it should be noticed that the analysis of edge states allows for further reduction of configurations of ranks relevant for characterization of PPT entanglement in symmetric states. This is because a PPT state that is not edge can be written as a mixture of a pure product vector and another PPT state of lower ranks (see also Sec. V).

C. On the Schmidt number of symmetric states

Let us finally comment on the Schmidt number of the symmetric states. Clearly, a pure state $|\psi\rangle \in \mathcal{H}_N$ can be written as a linear combination of fully product vectors from $\mathcal{H}_N$. Following Ref. [36], the smallest number of terms in such decompositions of $|\psi\rangle$ is called the Schmidt rank of $|\psi\rangle$ and denoted $r(|\psi\rangle)$. Then, analogously to Ref. [37], we can define the Schmidt number of ρ to be $\min_{\{|\psi_i\rangle\}}\{\max_i r(|\psi_i\rangle)\}$, where the minimum is taken over all decompositions $\{|\psi_i\rangle\}$ of ρ, that is, $\rho = \sum_i |\psi_i\rangle\langle\psi_i|$.

Below we show that in small symmetric systems consisting of four or five qubits, any entangled state has the Schmidt number 2 or at most 3, respectively. We also comment on the Schmidt number of larger systems.

Before that we need some preparation. Let us introduce the following transformations: $F_n : (\mathbb{C}^2)^{\otimes n} \to \mathbb{C}^2$ and $G_n : (\mathbb{C}^2)^{\otimes(n+1)} \mapsto (\mathbb{C}^2)^{\otimes(2n+1)}$ defined through

$$F_n(1,\alpha)^{\otimes n} = (1,\alpha^n) \tag{38}$$

and

$$G_n[(1,\alpha^{n+1}) \otimes (1,\alpha)^{\otimes n}] = (1,\alpha)^{\otimes(2n+1)}, \tag{39}$$

respectively, for any $\alpha \in \mathbb{C}$ and $n = 1,2,\ldots$. Notice that both maps are of full rank. Then, by $\hat{F}_n$ and $\hat{G}_n$ we denote maps that are defined through the adjoint actions of F_n and G_n, that is, $\hat{X}(\cdot) = X(\cdot)X^\dagger$ $(X = F_n, G_n)$.

Let us comment briefly on the properties of F_n and G_n. First, consider an N-qubit symmetric vector $|\psi\rangle$. An application of $F_{\lceil N/2\rceil}$ to chosen $\lceil N/2\rceil$ qubits of $|\psi\rangle$, say the first ones, brings it to an $(\lfloor N/2\rfloor+1)$-qubit vector $|\psi'\rangle \in \mathbb{C}^2 \otimes \mathcal{S}_{\lfloor N/2\rfloor}$. A subsequent application of $G_{N/2-1}$ to the first $N/2$ qubits of $|\psi'\rangle$ in case of even N and $G_{\lceil N/2\rceil-1}$ to the whole $|\psi'\rangle$ in the case of odd N, returns $|\psi\rangle$.

Analogously, by applying $\hat{F}_{\lceil N/2\rceil}$ to the first $\lceil N/2\rceil$ qubits of an N-qubit mixed symmetric state ρ, one brings it to an $(\lfloor N/2\rfloor+1)$-qubit state σ whose the last $\lfloor N/2\rfloor$ qubits are still supported on the symmetric subspace $\mathcal{S}_{\lfloor N/2\rfloor}$. Let us denote the parties of this state by $B_1,\ldots,B_{\lfloor N/2\rfloor+1}$. With respect to the bipartition $B_1|B_2,\ldots,B_{\lfloor N/2\rfloor+1}$, it can be seen as a bipartite state acting on $\mathbb{C}^2 \otimes \mathbb{C}^{\lfloor N/2\rfloor+1}$. Moreover, this "compressing" operation preserves the rank of any symmetric state, that is, $r(\sigma) = r(\rho)$. This is because $R(\rho)$ is spanned by the symmetric product vectors $(1,\alpha)^{\otimes N}$ for $\alpha \in \mathbb{C}$, which are then mapped by $F_{\lceil N/2\rceil}$ to $(1,\alpha)^{\otimes\lfloor N/2\rfloor} \otimes (1,\alpha^{\lceil N/2\rceil})$. Since all powers of α from the zeroth one to α^N still appear in the projected vectors, the whole information about ρ is encoded in σ. Precisely, by an application of $\hat{G}_{N/2-1}$ to the first $N/2$ qubits of σ for even N and $\hat{G}_{\lceil N/2\rceil-1}$ to all qubits for odd N of σ, one recovers ρ. With all this we are now ready to state and prove the following theorem.

Theorem 7. Let ρ be an entangled N-qubit symmetric state. If by an application of $\hat{F}_{\lceil N/2\rceil}$ to the first $\lceil N/2\rceil$ qubits of ρ one gets an $(\lfloor N/2\rfloor+1)$-qubit state σ (see above) that is separable with respect to the bipartition $B_1|B_2,\ldots,B_{\lfloor N/2\rfloor+1}$, then ρ can be written as

$$\rho = \sum_{i=1}^{K}\left[\sum_{j=1}^{\lceil N/2\rceil} A_j^{(i)}\left(1,\alpha_j^{(i)}\right)^{\otimes N}\right], \tag{40}$$

where $A_j^i,\alpha_j^i \in \mathbb{C}$ and $[\psi]$ denotes a projector onto $|\psi\rangle$.

Proof. We can clearly assume that $r(\rho) = N+1$. Let the $(\lfloor N/2\rfloor+1)$-qubit state σ, coming from the application of $\hat{F}_{\lceil N/2\rceil}$ to the first $\lceil N/2\rceil$ qubits of ρ, be separable with respect to the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^{\lfloor N/2\rfloor+1}$. It can then be written as

$$\sigma = \sum_{i=1}^{K} p_i|e_i\rangle\langle e_i| \otimes |f_i\rangle\langle f_i|, \quad p_i \geqslant 0, \quad \sum_i p_i = 1, \tag{41}$$

where $|e_i\rangle \in \mathbb{C}^2$ and $|f_i\rangle \in \mathbb{C}^{\lfloor N/2\rfloor+1} \cong \mathcal{S}_{\lfloor N/2\rfloor}$.

As already said, the mapping $\hat{F}_{\lceil N/2\rceil}$ preserves the rank, and therefore $r(\sigma)=r(\rho)=N+1$. On the other hand, σ acts on $\mathbb{C}^2\otimes\mathbb{C}^{\lfloor N/2\rfloor+1}$ and therefore its maximal rank is $2(\lfloor N/2\rfloor+1)$, which means that for odd N, σ is of full rank, while for even N it has a single vector $|\phi\rangle=|10\rangle-|0,N/2\rangle$ in its kernel.

Let us now divide our considerations into the cases of odd and even N. In the first case we put $|e_i\rangle=(1,\alpha_i^{\lceil N/2\rceil})$, and then notice that the isomorphism $\mathbb{C}^{\lfloor N/2\rfloor+1}\cong\mathcal{S}_{\lfloor N/2\rfloor}$ implies that each $|f_i\rangle\in\mathbb{C}^{\lfloor N/2\rfloor+1}$ can be expanded in terms of $\lfloor N/2\rfloor+1$ product symmetric vectors from $\mathcal{S}_{\lfloor N/2\rfloor}$ in the following way:

$$|f_i\rangle=\sum_{j=1}^{\lfloor N/2\rfloor+1}A_j^{(i)}(1,e^{i\varphi_j}\alpha_i)^{\otimes\lfloor N/2\rfloor},\tag{42}$$

with

$$\varphi_j=\frac{2\pi j}{\lceil N/2\rceil},\tag{43}$$

and $A_j^{(i)}\in\mathbb{C}$. Consequently,

$$G_{\lceil N/2\rceil-1}|e_i\rangle|f_i\rangle=\sum_{j=1}^{\lfloor N/2\rfloor+1}A_j^{(i)}(1,e^{i\varphi_j}\alpha_i)^{\otimes N},\tag{44}$$

which, when substituted to Eq. (41) leads directly to (40).

In the case of even N, σ has a single vector $|\phi\rangle=|01\rangle-|N/2,0\rangle$ in $K(\sigma)$. Putting again $|e_i\rangle=(1,\alpha_i^{N/2})$ with $\alpha_i\in\mathbb{C}$, it imposes a constraint on $|f_i\rangle$, that is, $\langle N/2|f_i\rangle=\langle 0|f_i\rangle\alpha_i^{N/2}$, meaning that $|f_i\rangle\in\mathbb{C}^{N/2}$. Therefore, again exploiting the isomorphism $\mathbb{C}^{N/2}\cong\mathcal{S}_{N/2-1}$, one sees that all $|f_i\rangle$ can be written as

$$|f_i\rangle=\sum_{j=1}^{N/2}A_j^{(i)}(1,e^{i\varphi_j}\alpha_i)^{\otimes N/2}.\tag{45}$$

By substituting this to Eq. (41) and applying $\hat{G}_{N/2-1}$ to σ, one gets Eq. (40), completing the proof.

Let us finally comment on the choice of the product symmetric vectors used to expand $|f_i\rangle$'s. In both cases of odd and even N they are chosen to be $(1,e^{i\varphi_j}\alpha)^{\otimes\lfloor N/2\rfloor}$ $(j=0,\ldots,\lceil N/2\rceil-1)$, where φ_j are chosen so that $e^{i\varphi_j}$ are $\lceil N/2\rceil$th roots of the unity. One checks by hand that such vectors span a $\lceil N/2\rceil$-dimensional linear space. ∎

Theorem 7 implies that any PPT entangled symmetric state consisting of four (five) qubits has Schmidt number 2 (at most 3). To be more precise, we prove the following corollaries.

Corollary 1. Any PPT entangled symmetric state ρ of four qubits can be written as

$$\rho=\sum_{i=1}^{K}\left[A_1^{(i)}(1,\alpha_i)^{\otimes 4}+A_2^{(i)}(1,-\alpha_i)^{\otimes 4}\right],\tag{46}$$

while any PPT entangled symmetric state of five qubits can be written as

$$\rho=\sum_{i=1}^{K}\left[A_1^{(i)}(1,\alpha_i)^{\otimes 5}+A_2^{(i)}\left(1,e^{i\frac{2\pi}{3}}\alpha_i\right)^{\otimes 5}+A_3^{(i)}\left(1,e^{i\frac{4\pi}{3}}\alpha_i\right)^{\otimes 5}\right],\tag{47}$$

with $K\leqslant 6$ and $A_j^{(i)},\alpha_i\in\mathbb{C}$.

Proof. By applying $\hat{F}_2$ ($\hat{F}_3$) to the first two (three) qubits of ρ for $N=4$ ($N=5$) we get a state σ acting on $\mathbb{C}^2\otimes\mathbb{C}^3$. It is clearly PPT and due to Ref. [31] also separable. Consequently, Theorem 7 implies that ρ can be written as in Eq. (40), which in particular cases of $N=4$ and $N=5$ leads to (46) and (47), respectively. The number of elements in both the decompositions (46) and (47) follows from the fact that any qubit-qutrit separable state can be written as a convex combination of six product vectors [33]. ∎

It should be noticed that by using the approach developed in Ref. [33], one can obtain decompositions of any PPT entangled four-qubit symmetric state similar to (46), but in which vectors $(1,-\alpha_i)^{\otimes 4}$ are replaced by either $(0,1)^{\otimes 4}$ or $(1,0)^{\otimes 4}$.

Theorem 8. Let ρ be a PPT entangled symmetric four-qubit state. Then it can be written as

$$\rho=\sum_{i=1}^{K}[A_i(1,\alpha_i)^{\otimes 4}+B_i(0,1)^{\otimes 4}],\tag{48}$$

where $K\leqslant 6$, $(1,\alpha_i)\in\mathbb{C}^2$, and A_i,B_i are some complex coefficients, and by $[\psi]$ we denote a projector onto $|\psi\rangle$.

Proof. The proof exploits the method developed in Ref. [33]. First, one notices that any ρ can be written as a sum of rank 1 matrices,

$$\rho=\sum_{i=1}^{K}|\Psi_i\rangle\langle\Psi_i|,\tag{49}$$

where, in particular, $|\Psi_i\rangle$ can be (un-normalized) eigenvectors of ρ, and $K\leqslant 6$ (see Corollary 1). On the other hand, ρ can always be expressed in terms of the symmetric un-normalized basis $\{|E_\mu^4\rangle\}_{\mu=1}^{5}$ spanning $\mathcal{S}_4$ as

$$\rho=\sum_{\mu,\nu=1}^{5}\rho_{\mu\nu}\big|E_\mu^4\big\rangle\big\langle E_\nu^4\big|.\tag{50}$$

Both decompositions (49) and (50) are related via the so-called Gram system of ρ, that is, a collection of K-dimensional vectors $|v_\mu\rangle=(1/\langle E_\mu^4|E_\mu^4\rangle)(\langle\Psi_1|E_\mu^4\rangle,\ldots,\langle\Psi_K|E_\mu^4\rangle)(\mu=1,\ldots,5)$, giving $\rho_{\mu\nu}=\langle v_\mu|v_\nu\rangle$. Putting the latter to Eq. (50) with explicit forms of the vectors $|v_\mu\rangle$, one recovers (49).

Now, by projecting the last party onto $|0\rangle$ we get a three-qubit symmetric PPT state $\widetilde{\rho}$, which, as already stated, is separable. Then, according to Ref. [33], there exists a diagonal matrix $M=\mathrm{diag}[\alpha_1^*,\ldots,\alpha_K^*]$ such that $|v_\mu\rangle=M^{\mu-1}|v_1\rangle$ $(\mu=1,\ldots,4)$. For convenience we can also put $|v_5\rangle=M^4|v_1\rangle+|\widetilde{v}\rangle$ with $|\widetilde{v}\rangle$ being some K-dimensional complex vector. Then, putting $|v_1\rangle=(A_1^*,\ldots,A_K^*)$ and $|\widetilde{v}\rangle=(B_1^*,\ldots,B_K^*)$, one sees that

$$\begin{aligned}|\Psi_i\rangle&=\sum_{\mu=1}^{5}\frac{\langle E_\mu^4|\Psi_i\rangle}{\langle E_\mu^4|E_\mu^4\rangle}\big|E_\mu^4\big\rangle\\&=A_i\sum_{\mu=1}^{4}\alpha_i^{\mu-1}\big|E_\mu^4\big\rangle+B_i\big|E_5^4\big\rangle\\&=A_i(1,\alpha_i)^{\otimes 4}+B_i\big|E_5^4\big\rangle,\end{aligned}\tag{51}$$

where the second equation follows from the explicit form of the vectors $|v_\mu\rangle$. Substituting vectors $|\Psi_i\rangle$ to Eq. (49), one gets (48), which completes the proof. ∎

In order to get the representation (48) with (0,1) replaced with (1,0), one has to project the last party of ρ onto $|1\rangle$ instead of $|0\rangle$.

IV. EXTREMAL PPT ENTANGLED SYMMETRIC STATES

We here seek extremal elements in the convex set of N-qubit symmetric PPT states $D_{\mathrm{PPT}}^{\mathrm{sym}}$. We see that the number of distinct configurations of ranks for which one finds such examples is small and does not increase with N. In particular, if N is even there is only a single such configuration.

Let us start by adapting to the multipartite case an algorithm described in Ref. [3] (see also Ref. [7]) allowing to look for extremal elements of D_{PPT}.

A. An algorithm allowing to seek multipartite extremal PPT states

Let us consider again a product Hilbert space $\mathcal{H}_N = \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_1}$ and the set D_{PPT} of fully PPT states acting on $\mathcal{H}_N$. We call an element ρ of D_{PPT} *extremal* if and only if it does not allow for the decomposition

$$\rho = p\rho_1 + (1-p)\rho_2, \tag{52}$$

with $\rho_i \in D_{\mathrm{PPT}}$ such that $\rho_1 \neq \rho_2$ and $0 < p < 1$. Should ρ be not extremal, Eq. (52) implies that $R(\rho_1) \subseteq R(\rho)$ and $R(\rho_2) \subseteq R(\rho)$ and also $R(\rho_i^{T_k}) \subseteq R(\rho^{T_k})$ for all k. Alternatively speaking, if ρ is not extremal one is able to find another PPT state $\sigma \neq \rho$ such that

$$R(\sigma) \subseteq R(\rho) \tag{53}$$

and

$$R(\sigma^{T_k}) \subseteq R(\rho^{T_k}) \quad (k = 1,\ldots,M), \tag{54}$$

where by M we denote the smallest number of partial transpositions necessary to define the set of PPT states.

On the other hand, if there is a PPT density matrix $\sigma \neq \rho$ fulfilling the conditions (53) and (54), then one finds such a number $x > 0$ that $\rho(x) = (1+x)\rho - x\sigma$ is again a PPT state. Thus, ρ cannot be extremal because it admits the decomposition (52), that is, $\rho = [\rho(x) + x\sigma]/(1+x)$. In this way, one arrives at a natural criterion for extremality saying that ρ is extremal if and only if there is no state $\sigma \neq \rho$ such that (53) and (54) hold.

Interestingly, one can even relax the assumption of σ being a PPT state to being a Hermitian matrix satisfying a set of equations following from Eqs. (53) and (54):

$$\mathcal{P}_k h^{T_k} \mathcal{P}_k = h^{T_k} \quad (k = 0,\ldots,M), \tag{55}$$

where T_0 is an identity map. This is because if such h exists, one considers a one-parameter class of matrices $\rho(x) = (1 + x\mathrm{Tr}h)\rho - xh$ (Trh is added in order to ensure that $\mathrm{Tr}[\rho(x)] = 1$ for any x). It is straightforward to show that there exist $x_1 < 0$ and $x_2 > 0$ such that for any $x \in [x_1,x_2]$, $\rho(x_i)$ $(i = 1,2)$ are PPT states. Consequently, $\rho = [1/(|x_1| + x_2)][x_2\rho(x_1) + x_1\rho(x_2)]$, meaning that it cannot be extremal.

Let us also notice that the system (55) is equivalent to a single equation,

$$[\hat{\mathcal{P}}_M \circ \cdots \circ \hat{\mathcal{P}}_1 \circ \hat{\mathcal{P}}](h) = h, \tag{56}$$

where the maps $\hat{\mathcal{P}}_k$ are defined through $\hat{\mathcal{P}}_k(\cdot) = [\mathcal{P}_k(\cdot)^{T_k}\mathcal{P}_k]^{T_k}$ $(k = 0,\ldots,M)$. Indeed, if h satisfies the system (55) then it also obeys (56). On the other hand, if the condition (56) is satisfied with some h, then it has to obey (55); if one of the conditions (55) does not hold, a simple comparison of norms of both sides of (56) shows that (55) neither can be satisfied.

As a result, we have just reached an operational criterion for extremality of elements of D_{PPT} [3] (see also Ref. [7]).

Theorem 9. A given PPT state ρ acting on $\mathcal{H}_N$ is extremal in D_{PPT} if and only if there does not exists a Hermitian solution to the system (55) which is linearly independent of ρ.

This also leads to a necessary condition for extremality that can be formulated in terms of the ranks $r(\rho^{T_k})(k = 1,\ldots,M)$. Namely, each equation in (55) imposes $[\dim \mathcal{H}_N]^2 - [r(\rho^{T_k})]^2$ linear constraints on the matrix h. The maximal number of the constraints imposed by the system is thus $\sum_{k=0}^{M}([\dim \mathcal{H}_N]^2 - [r(\rho^{T_k})]^2)$. On the other hand, a Hermitian matrix acting on $\mathcal{H}_N$ is specified by $[\dim \mathcal{H}_N]^2$ real parameters and therefore if

$$\sum_k [r(\rho^{T_k})]^2 \geqslant M[\dim \mathcal{H}_N]^2 + 1, \tag{57}$$

the system (55) has a solution and ρ is not extremal.

Importantly, the above considerations imply an algorithm making it possible to seek extremal elements of D_{PPT} [3]. Given a PPT state ρ and a solution h to the system (55), one considers $\rho(x) = (1 + x\mathrm{Tr}h)\rho - xh$. It is fairly easy to see that there is $x = x_*$ for which $\rho_1 \equiv \rho(x_*) \in D_{\mathrm{PPT}}$ but $r(\rho_1^{T_k}) = r(\rho^{T_k}) - 1$ for some k. In other words, we can choose x in such a way that one of the ranks of the resulting state ρ_1 is diminished by one.

For the resulting state ρ_1 we again look for solutions to (55). If the only solution is ρ_1 itself (up to normalization), then it is already extremal. If not, one again considers $\rho_1(x) = (1 + x\mathrm{Tr}h)\rho_1 - xh$ and finds such $x = x_*$ that $\rho_2 \equiv \rho_1(x_*)$ is a PPT state with one of the ranks diminished by one. We proceed in this way until we obtain an extremal state. If the latter has rank 1 it is separable; otherwise it has to be entangled. Clearly, since we deal with finite-dimensional Hilbert space, the final state of this algorithm is reached in a finite number of steps.

In our implementation of this algorithm we obtain the Hermitian matrix h by solving a slightly different system of equations than (55) [or the single one (56)], that is,

$$h^{T_k}|\Psi_i^{(k)}\rangle = 0 \quad (i = 1,\ldots,k(\rho^{T_k})) \tag{58}$$

for all $k = 0,\ldots,M$, with $|\Psi_i^{(k)}\rangle$ denoting vectors spanning the kernel of ρ^{T_k}. Clearly, both systems (55) and (58) are equivalent.

A general Hermitian matrix acting on $\mathcal{H}_N$ is fully characterized by $(\dim \mathcal{H}_N)^2$ real parameters h_i, which we consider elements of a vector $|h\rangle \in \mathbb{C}^{(\dim \mathcal{H}_N)^2}$. Then, the set (58) can be easily reformulated as a single matrix equation $\mathcal{R}|h\rangle = 0$, where $\mathcal{R}$ is a $[2\dim \mathcal{H}_N \sum_{l=0}^{M} k(\rho^{T_l})] \times (\dim \mathcal{H}_N)^2$ matrix with real entries. The number of rows of $\mathcal{R}$ stems from the fact that each group of equations in (58) corresponding to a given k gives $2k(\rho^{T_k}) \dim \mathcal{H}_N$ linear and real conditions for h_i, which when summed over all partial transpositions (including also T_0) results in the aforementioned number of rows. Clearly, this gives more conditions for h_i than those following from Eqs. (55), that is, $\sum_{k=0}^{M}[(\dim \mathcal{H})^2 - [r(\rho_k^{T})]^2]$ (some of them are clearly linearly dependent). Nevertheless, even if the number of equations is larger in comparison to Eq. (56), our approach does not require multiplying many matrices, which in the case of many parties makes the implementation a bit faster.

B. Extremal PPT entangled symmetric states

Let us now apply the above considerations to the symmetric states. First of all, one notices that the condition (57) has to be slightly modified because different partial transpositions of a symmetric state and the state itself act on Hilbert spaces of different dimensions (see Ref. [7] for the case of four qubits). Specifically, an equation in (55) corresponding to the partial transposition with respect to S gives $(|S|+1)^2(N-|S|+1)^2-[r(\rho^{T_S})]^2(|S|=0,\ldots,\lfloor N/2\rfloor)$ linear equations. Then, the condition (57) becomes

$$\sum_{|S|=0}^{\lfloor N/2\rfloor}[r(\rho^{T_S})]^2 \geqslant \sum_{|S|=0}^{\lfloor N/2\rfloor}(|S|+1)^2(N-|S|+1)^2-(N+1)^2+1. \quad (59)$$

Taking into account the fact that among symmetric states only those of rank $N+1$ can be entangled allows us to rewrite the above inequality as

$$\sum_{|S|=1}^{\lfloor N/2\rfloor}[r(\rho^{T_S})]^2 \geqslant \sum_{|S|=1}^{\lfloor N/2\rfloor}(|S|+1)^2(N-|S|+1)^2-(N+1)^2+1. \quad (60)$$

In Table I we collect all the ranks, singled out with the aid of inequality (60), for which there are no extremal PPT states for exemplary cases of low number of qubits $N=4,5,6$.

We have applied the above algorithm to the PPT symmetric states with the number of qubits varying from $N=4$ to $N=23$. As the initial state we took the projector onto the symmetric space $\mathcal{P}_N$. Clearly, it has all the ranks maximal, which is important from the point of view of the algorithm; if the ranks of the initial state ρ are too low (cf. Theorems 2 and 3), the algorithm cannot produce an entangled PPT state out of ρ by lowering its ranks. Then, we have searched for random extremal states by choosing randomly at each stage of the protocol the matrices h resulting from solving (55). On the other hand, to find other examples of ranks than those obtained through a random search and not excluded by the analysis above, we designed the matrices h in such a way that they lower specific ranks. The obtained ranks are collected in Table II. Interestingly, there are always at most three different configurations of ranks assumed by the found extremal PPT entangled symmetric states and it seems that the number of configurations does not increase with N. Moreover, in the case of odd N, there is only a single such configuration (all ranks are maximal except for the last one which is two less than maximal). It is an interesting problem to confirm these findings analytically. If this is the case, the problem of characterization of PPT entanglement in symmetric states reduces significantly to the characterization of extremal states assuming few different configurations of ranks, in particular a single one for odd N. Notice that in the case of symmetric qubits, there cannot be PPT entangled states of lower ranks than those assumed by extremal one as this is the case in higher-dimensional Hilbert spaces (cf. Ref. [4]). This is because in order to construct such states one needs PPT extremal entangled states supported on lower-dimensional Hilbert spaces, and in our case such states are always separable.

TABLE I. Inequality (60) (second column) for exemplary cases of $N=4,5,6$ together with the ranks (third column) excluded with its aid for which there are no extremal entangled symmetric states. Notice that the fact that there are no extremal states of maximal ranks can be inferred without restoring to inequality (60).

N	Inequality (60)	Ranks excluded with (60)
4	$[r(\rho^{T_1})]^2+[r(\rho^{T_2})]^2\geqslant 121$	(5,7,9),(5,8,8),(5,8,9)
5	$[r(\rho^{T_1})]^2+[r(\rho^{T_2})]^2\geqslant 209$	(6,9,12),(6,10,11),(6,10,12)
6	$[r(\rho^{T_1})]^2+[r(\rho^{T_2})]^2$ $+[r(\rho^{T_3})]^2\geqslant 577$	(7,10,15,16),(7,11,15,16), (7,12,14,16),(7,12,15,15), (7,12,15,16)

Let us study in detail extremal entangled states in the exemplary case of $N=4$. From Theorems 2 and 3 it follows that PPT states of ranks $(5,r(\rho^{T_A}),r(\rho^{T_{AB}}))$ with $r(\rho^{T_A})\leqslant 6$ or $r(\rho^{T_{AB}})\leqslant 6$ are either all separable or generically separable. Then, Theorem 6 states that generic PPT states of ranks (5,8,7) are not edge and thus not extremal. Finally, inequality (56) implies that PPT states of ranks (5,7,9), (5,8,8), and (5,8,9) cannot be extremal. As a result, the natural candidates for extremal states that can be obtained with the aid of the above algorithm have ranks (5,7,7) and (5,7,8).

We have run the algorithm 30 000 times and 19.2% of the generated examples were extremal entangled states of ranks (5,7,8). In the remaining 80.8% of cases we arrived at states of ranks (5,7,7), all being separable. Also, when lowering the ranks from the initial state of ranks (5,8,9), 99.4% of the times we have obtained an intermediate (5,8,8) state, whereas intermediate states of ranks (5,7,9) have appeared in 0.6% of remaining cases. In conclusion, it should be noticed that with the aid of the above algorithm we have generated PPT entangled extremal states assuming only a single configuration of ranks. All states of ranks (5,7,7) appeared to be separable and there is an indication suggesting that generic four-qubit PPT symmetric states of these ranks are separable (see the Appendix).

Let us finally notice that to make the application of the algorithm to systems consisting of even 23 qubits possible, one has to take an advantage of the underlying symmetry and try to avoid representing a symmetric ρ and its partial transpositions in the full Hilbert space $\mathcal{H}_{2,N}=(\mathbb{C}^2)^{\otimes N}$ (see also Ref. [12]). Indeed, since $\mathcal{S}_N\cong\mathbb{C}^{N+1}$, one can represent ρ or a general Hermitian matrix supported on $\mathcal{S}_N$, as a $(N+1)\times(N+1)$ matrix, which we further denote ρ_{red}. In order to move from one representation to the other one we use a $(N+1)\times 2^N$ matrix $B_N:\mathcal{S}_N\mapsto\mathbb{C}^{N+1}$ given by

$$B_N=\sum_{m=0}^{N}|m\rangle\langle\widetilde{E}_m^N|, \quad (61)$$

TABLE II. The ranks of extremal states found by using the algorithm described in Sec. IV A. The first column contains the number of qubits, while the next six columns the ranks of ρ and its partial transpositions $r(\rho^{T_i})$ $(i = 1,\dots,11)$. Notice that there are no PPT entangled states with fewer than four qubits [11] (cf. Sec. III A). The negative numbers in parentheses denote the difference between the given rank and its maximal value (the lack of parentheses means that the rank is maximal). For all N there are at most three possible configurations of ranks with extremal states, and, interestingly, in the case of odd N there is always only one such configuration.

N	$r(\rho)$	$r(\rho^{\Gamma_1})$	$r(\rho^{\Gamma_2})$	$r(\rho^{\Gamma_3})$	$r(\rho^{\Gamma_4})$	$r(\rho^{\Gamma_5})$	$r(\rho^{\Gamma_6})$	$r(\rho^{\Gamma_7})$	$r(\rho^{\Gamma_8})$	$r(\rho^{\Gamma_9})$	$r(\rho^{\Gamma_{10}})$	$r(\rho^{\Gamma_{11}})$
4	5	7 (−1)	8 (−1)									
5	6	10	10 (−2)									
6	7	12	14 (−1)	14 (−2)								
			14 (−1)	13 (−3)								
7	8	14	18	18 (−2)								
8	9	16	21	23 (−1)	23 (−2)							
				23 (−1)	22 (−3)							
9	10	18	24	28	28 (−2)							
10	11	20	27	32	34 (−1)	33 (−3)						
					34 (−1)	34 (−2)						
					35 (+0)	32 (−4)						
11	12	22	30	36	40	40 (−2)						
12	13	24	33	40	45	47 (−1)	47 (−2)					
						47 (−1)	46 (−3)					
						48 (+0)	45 (−4)					
13	14	26	36	44	50	54	54 (−2)					
14	15	28	39	48	55	60	62 (−1)	62 (−2)				
							62 (−1)	61 (−3)				
							63 (+0)	60 (−4)				
15	16	30	42	52	60	66	70	70 (−2)				
16	17	32	45	56	65	72	77	79 (−1)	79 (−2)			
								79 (−1)	78 (−3)			
								80 (+0)	77 (−4)			
17	18	34	48	60	70	78	84	88	88 (−2)			
18	19	36	51	64	75	84	91	96	98 (−1)	98 (−2)		
									98 (−1)	97 (−3)		
									99 (+0)	96 (−4)		
19	20	38	54	68	80	90	98	104	108	108 (−2)		
20	21	40	57	72	85	96	105	112	117	119 (−1)	119 (−2)	
										119 (−1)	118 (−3)	
										120 (+0)	117 (−4)	
21	22	42	60	76	90	102	112	120	126	130	130 (−2)	
22	23	44	63	80	95	108	119	128	135	140	142 (−1)	142 (−2)
											142 (−1)	141 (−3)
											143 (+0)	140 (−4)
23	24	46	66	84	100	114	126	136	144	150	154	154 (−2)

which gives $\rho_{\text{red}} = B_N \rho B_N$. It is straightforward to check that for any N, $B_N^T B_N = \mathcal{P}_N$ and $B_N B_N^T = \mathbb{1}_{N+1}$.

Then, accordingly, the partial transposition of ρ with respect to T_k can be represented as a $(k+1)(N-k+1) \times (k+1)(N-k+1)$ matrix $\rho_{\text{red}}^{T_k}$ acting on $\mathbb{C}^{k+1} \otimes \mathbb{C}^{N-k+1}$. To get the latter from ρ_{red} without restoring to the representation of ρ in the full Hilbert space $\mathcal{H}_{2,\underset{\sim}{N}}$, one can utilize a $(k+1)(N-k+1) \times (N+1)$ matrix $\widetilde{B}_k = (B_k \otimes B_{N-k})B_N^T$, that is,

$$
\begin{aligned}
\rho_{\text{red}}^{T_k} &= \left[(B_k \otimes B_{N-k})\rho\left(B_k^T \otimes B_{N-k}^T\right)\right]^{T_k} \\
&= \left[(B_k \otimes B_{N-k})B_N^T \rho_{\text{red}} B_N \left(B_k^T \otimes B_{N-k}^T\right)\right]^{T_k} \\
&= \left(\widetilde{B}_k \rho_{\text{red}} \widetilde{B}_k^T\right)^{T_k}.
\end{aligned} \tag{62}
$$

Short algebra shows that the elements of $\widetilde{B}_k (k = 1,\dots,\lfloor N/2 \rfloor)$ are given by

$$
\langle i,j|\widetilde{B}_k|n\rangle = \sqrt{\binom{N}{i}\binom{N}{j}\bigg/\binom{N}{n}}\,\delta_{i+j=n}. \tag{63}
$$

with $i = 0,\dots,k$, $j = 0,\dots,N-k$, and $n = 0,\dots,N$.

Consequently, to get effectively a partial transposition of ρ, one maps a $(N+1) \times (N+1)$ matrix ρ_{red} with $\widetilde{B}_k$, and subsequently performs a simple partial transposition on the resulting bipartite matrix [cf. Eq. (62)]. Accordingly, one also transforms the Hermitian matrices h appearing in the system (58). Notice that this approach allows us to reduce the algorithm complexity from exponential to polynomial in N both in time and memory. Precisely, the estimated time complexity of our approach amounts to $O(N^6)$.

V. SPECIAL CASES

Here we summarize the obtained results for particular systems consisting of four, five, and six qubits.

$N = 4$. It follows from Theorem 2 that the four-qubit symmetric states are separable if either $r(\rho) \leqslant 4$ or $r(\rho^{T_A}) \leqslant 4$, or $r(\rho^{T_{AB}}) \leqslant 3$. Then, Theorem 3 implies that if either $r(\rho^{T_A}) \leqslant 6$ or $r(\rho^{T_{AB}}) \leqslant 6$, generic symmetric ρ is separable. This leaves only six configuration of ranks [of all possible 72 assuming that $r(\rho) = 5$] among which one may seek PPT entangled symmetric states: (5,7,7), (5,7,8), (5,7,9), (5,8,7), (5,8,8), and (5,8,9).

Passing to edgeness, it is known that states of ranks (5,8,9) cannot be edge. Then, it follows from Theorem 5 that all states of ranks (5,8,8) are not edge, while from Theorems 4 and 6 that generic states of ranks (5,7,9) and (5,8,7) are not edge. These theorems also show that a typical PPT entangled state assuming one of the above four configurations of ranks can always be brought, by subtracting properly chosen symmetric fully product vector, to a PPT entangled state of ranks either (5,7,7) or (5,7,8). Interestingly, with the half-analytical–half-numerical method presented in Ref. [8], as well as the numerical algorithm described in Sec. IV B, we found solely examples of PPT entangled states of ranks (5,7,8) [due to inequality (60) PPT states of ranks (5,7,9), (5,8,8), and clearly (5,8,9) cannot be extremal]. All the states of ranks (5,7,7) found with the above algorithm were separable. This together with the analytical considerations enclosed in the Appendix ranks are generically separable. Provided this is the case, the analysis of PPT entangled symmetric states of four qubits could be reduced significantly to the characterization of states with a single configuration of ranks (5,7,8).

$N = 5$. In this case Theorem 2 implies that five-qubit PPT symmetric states are separable if either $r(\rho) \leqslant 5$ or $r(\rho^{T_A}) \leqslant 5$, or $r(\rho^{T_{AB}}) \leqslant 4$ are separable. Then, Theorem 3 says that if either $r(\rho^{T_A}) \leqslant 8$ or $r(\rho^{T_{AB}}) \leqslant 9$, they are generically separable. Similarly to the case of $N = 4$, this leaves 6 [of 120 possible under the assumption that $r(\rho) = 6$] ranks for which typical PPT symmetric states need not be separable: (6,9,10), (6,9,11), (6,9,12), (6,10,10), (6,10,11), and (6,10,12).

With the aid of Theorem 4, one sees that five-qubit PPT symmetric states of ranks (6,9,12), (6,10,11), and (6,10,12) are generically not edge [notice that due to inequality (60) for the same ranks PPT states cannot be extremal]. Hence, analysis of PPT entanglement in in this case reduces to three configurations of ranks (6,9,10), (6,10,10), and (6,9,11). Interestingly, only in the second case we found examples of extremal states with the above numerical algorithm (see Table I).

$N = 6$. Let us finally consider the case of six qubits. Theorem 2 states that such PPT states are separable provided that either $r(\rho) \leqslant 6$ or $r(\rho^{T_A}) \leqslant 6$, or $r(\rho^{T_{AB}}) \leqslant 5$, or $r(\rho^{T_{ABC}}) \leqslant 4$. Moreover, Theorem 3 implies that they are generically separable if either $r(\rho^{T_A}) \leqslant 10$ or $r(\rho^{T_{AB}}) \leqslant 12$, or $r(\rho^{T_{ABC}}) \leqslant 12$. The number of the remaining configurations of ranks among which one may seek PPT entangled states is then 24 [of all possible 2880 when assumed that $r(\rho) = 7$], which is considerably larger than the corresponding numbers for $N = 4,5$.

Then, from Theorem 5 it follows that six-qubit PPT states of ranks (7,12,15,15) are not edge and Theorem 4 says that generic PPT states of ranks (7,12,14,16) and (7,11,15,16) are also not edge. This, together with the fact that states of maximal ranks; that is, (7,12,15,16) are not edge, makes it possible to reduce the problem of characterization of six-qubit PPT states to still quite large number of 20 configurations. There are, nevertheless, only two sets of ranks for which, using the algorithm from Sec. IV A, we found extremal PPT entangled states.

VI. CONCLUSION

Let us briefly summarize the obtained results. Our aim was to characterize PPT entanglement in symmetric states. We have made a significant step towards reaching this goal, yet the complete characterization for the general case remains open.

First, we have derived simple separability criteria for PPT symmetric states in terms of their ranks, complementing the criterion stated in Ref. [11]. Interestingly, these criteria imply that for most of the possible configurations of ranks, PPT symmetric states are generically separable, and PPT entanglement may appear only in a small fraction of cases, vanishing for large number of parties. Putting $r(\rho) = N + 1$, there are precisely $(\lfloor N/2 \rfloor + 1)!$ such configurations of $N!(N/2 + 1)^{2(\lfloor N/2 \rfloor - N/2)+1}$ all possible ones. For the exemplary cases of four and five qubits this gives 6 different sets (of, respectively, 72 and 120 all possible ones) of ranks for which typical PPT states need not be separable.

Second, we have singled out some of the configurations of ranks for which PPT symmetric states are generically not edge, allowing for further reduction of relevant configurations of ranks. This is because if a PPT entangled state is not edge it can be decomposed as a convex combination of a pure product vector and a PPT symmetric state of lower ranks. From this point of view the relevant configurations of ranks are those than cannot be further reduced by subtracting a product vector from the state. Again, in the particular case of small systems consisting of four and five qubits, PPT states of higher ranks are generically not edge, lowering the number of the configurations to treatable two and three, respectively, for $N = 4$ and $N = 5$.

Finally, with the aid of the algorithm proposed in Ref. [3], we have searched for extremal PPT symmetric states. We have investigated systems consisting of 4 to 23 qubits and encountered a clear pattern behind the configurations of ranks for which we have found examples of extremal states. In particular, for even N, except for the cases of $N = 6$ and $N = 8$, there are always three configurations (following the same pattern) of ranks. Interestingly, for odd N there is only a single such configuration; that is, all ranks of the state and its partial transpositions are maximal except for the last one (the partial transposition with respect to half of the qubits), which amounts to two less than maximal. This is somehow contrary to the intuition that the number of different sets of ranks for which one finds extremal states should grow. On the other hand, it indicates that the problem of characterization of PPT entanglement in symmetric states could further be reduced to just few different types of states.

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APPENDIX: THE CASE OF (5,7,7)

We consider here four-qubit PPT symmetric states of ranks (5,7,7) and provide a possible way of proving that generic states of these ranks are separable. For this purpose we use the approach developed in Ref. [33]. First, recall that any state ρ acting on $\mathcal{H}$ can be written as a convex combination of rank 1 projectors

$$\rho = \sum_{k=1}^{l} |\psi_k\rangle\langle\psi_k|, \tag{A1}$$

where the un-normalized vectors $|\psi_k\rangle$ are, in general, nonorthognal (a particular example of such decomposition is the eigendecomposition of ρ).

Denoting by $|e_k\rangle$ orthonormal vectors spanning $\mathcal{H}$, we see that any element of ρ in this basis can be written as

$$\langle e_i|\rho|e_j\rangle = \sum_{k=1}^{l}\langle e_i|\psi_k\rangle\langle\Psi_k|e_j\rangle = \langle v_i|v_j\rangle \tag{A2}$$

for any $i,j = 1,\ldots,\dim\mathcal{H}$, where the l-dimensional vectors $|v_i\rangle$ are defined as

$$|v_i\rangle = \begin{pmatrix} \langle\psi_1|e_i\rangle \\ \vdots \\ \langle\psi_l|e_i\rangle \end{pmatrix} \quad (i = 1,\ldots,\dim\mathcal{H}). \tag{A3}$$

Consequently, ρ is the so-called Gram matrix, that is, the matrix of scalar products of a set of vectors $\{|v_i\rangle\}$ called further the Gram system of ρ. A different decomposition in Eq. (A1) leads to a different Gram system and all Gram systems of ρ are related via unitary matrices (if extended to a properly large Hilbert space).

Let us now consider a four-qubit PPT symmetric state ρ of ranks (5,7,7) and find its Gram system together with the Gram systems of ρ^{T_A}, ρ^{T_B}, and $\rho^{T_{AB}}$, starting from ρ^{T_A}. Since, by assumption, the latter is positive and $r(\rho^{T_A}) = 7$, it admits a decomposition as in Eq. (A1) (for instance, the eigendecomposition) with seven rank 1 components, that is,

$$\rho^{T_A} = \sum_{k=1}^{7} |\Psi_k\rangle\langle\Psi_k|, \tag{A4}$$

where the vectors $|\Psi_k\rangle$ are subnormalized.

Denoting by $\mathcal{B}$ the standard product basis $\{|i,j,k,l\rangle\}$ of $(\mathbb{C}^2)^{\otimes 4}$ and utilizing the fact that every $|\Psi_k\rangle$ in (A1) belongs to $\mathbb{C}^2\otimes\mathcal{S}_3$, one finds that the Gram system of ρ^{T_A} with respect to $\mathcal{B}$ consists of eight seven-dimensional vectors $|a\rangle,\ldots,|d\rangle$ and $|\widetilde{a}\rangle,\ldots,|\widetilde{d}\rangle$ whose elements are given by

$$\begin{aligned}
a_k &= \langle\Psi_k|0000\rangle,\\
b_k &= \langle\Psi_k|0001\rangle = \langle\Psi_k|0010\rangle = \langle\Psi_k|0100\rangle,\\
c_k &= \langle\Psi_k|0011\rangle = \langle\Psi_k|0101\rangle = \langle\Psi_k|0110\rangle,\\
d_k &= \langle\Psi_k|0111\rangle,\\
\widetilde{a}_k &= \langle\Psi_k|1000\rangle,\\
\widetilde{b}_k &= \langle\Psi_k|1001\rangle = \langle\Psi_k|1010\rangle = \langle\Psi_k|1100\rangle,\\
\widetilde{c}_k &= \langle\Psi_k|1011\rangle = \langle\Psi_k|1101\rangle = \langle\Psi_k|1110\rangle,\\
\widetilde{d}_k &= \langle\Psi_k|1111\rangle.
\end{aligned} \tag{A5}$$

Let us then introduce the following 7×4 matrices $A = (|a\rangle,|b\rangle,|b\rangle,|c\rangle)$, $B = (|b\rangle,|c\rangle,|c\rangle,|d\rangle)$, $\widetilde{B} = (|\widetilde{a}\rangle,|\widetilde{b}\rangle,|\widetilde{b}\rangle,|\widetilde{c}\rangle)$, and $\widetilde{C} = (|\widetilde{b}\rangle,|\widetilde{c}\rangle,|\widetilde{c}\rangle,|\widetilde{d}\rangle)$, with columns given by the vectors $|a\rangle,\ldots,|\widetilde{d}\rangle$. Then ρ^{T_A} can be written as

$$\begin{aligned}
\rho^{T_A} &= \begin{pmatrix} A^\dagger A & A^\dagger B & A^\dagger\widetilde{B} & A^\dagger\widetilde{C} \\ B^\dagger A & B^\dagger B & B^\dagger\widetilde{B} & B^\dagger\widetilde{C} \\ \widetilde{B}^\dagger A & \widetilde{B}^\dagger B & \widetilde{B}^\dagger\widetilde{B} & \widetilde{B}^\dagger\widetilde{C} \\ \widetilde{C}^\dagger A & \widetilde{C}^\dagger B & \widetilde{C}^\dagger\widetilde{B} & \widetilde{C}^\dagger\widetilde{C} \end{pmatrix}\\
&= \begin{pmatrix} A^\dagger \\ B^\dagger \\ \widetilde{B}^\dagger \\ \widetilde{C}^\dagger \end{pmatrix}(A\ \ B\ \ \widetilde{B}\ \ \widetilde{C}).
\end{aligned} \tag{A6}$$

where $X^\dagger Y$ $(X,Y = A,B,\widetilde{A},\widetilde{B})$ denotes a 4×4 matrix consisting of scalar products of vectors defining X and Y. We then symbolically denote $\rho^{T_A} = (A\ \ B\ \ \widetilde{B}\ \ \widetilde{C})$.

In the same way we can represent ρ. Since it is symmetric it admits the form $\rho = (A'\ \ B'\ \ B'\ \ C)$ with A', B', and C constructed in the same way as A, B, etc., from the Gram system of ρ. Now, since both three-qubit matrices $\langle 0|\rho|0\rangle$ and $\langle 0|\rho^{T_A}|0\rangle$ arising by projecting the first qubit of ρ and ρ^{T_A} onto $|0\rangle$, are equal, they have the same Gram systems and therefore there is a unitary U such that $A' = UA$ and $B' = UB$. Then, since by multiplying by a unitary operator all the vectors of the Gram system of ρ one gets another Gram system, we can always set $A' = A$ and $B' = B$.

Analogously, one sees that the matrices $\langle 1|\rho|1\rangle = \langle 1|\rho^{T_A}|1\rangle$ (arising by projecting the first qubit onto $|1\rangle$), and therefore there is a unitary U such that $\widetilde{B} = UB$ and $\widetilde{C} = UC$. In conclusion, we see that ρ and ρ^{T_A} can be represented as

$$\rho = (A\ \ B\ \ B\ \ C), \qquad \rho^{T_A} = (A\ \ \ B\ \ \ UB\ \ \ UC). \tag{A7}$$

By taking partial transposition of ρ with respect to A and comparing it with the above representation of ρ^{T_A}, one gets some conditions for U:

$$\begin{aligned}
A^\dagger UB &= B^\dagger A, \qquad A^\dagger UC = B^\dagger B,\\
B^\dagger UB &= C^\dagger A, \qquad B^\dagger UC = C^\dagger B.
\end{aligned} \tag{A8}$$

The same reasoning allows us to represent ρ^{T_B} and $\rho^{T_{AB}}$ in the following way:

$$\begin{aligned}
\rho^{T_B} &= (A\ \ \ UB\ \ \ B\ \ \ UC),\\
\rho^{T_{AB}} &= (A\ \ \ UB\ \ \ VB\ \ \ VUC),
\end{aligned} \tag{A9}$$

with V being a unitary matrix and $UB = VB$. Again, comparison of $\rho^{T_{AB}}$ given by Eq. (A9) to the partial transposition of ρ^{T_B} with respect to A, gives further conditions

$$A^\dagger VB = B^\dagger A, \qquad A^\dagger VUC = B^\dagger UB, \\ B^\dagger U^\dagger VB = C^\dagger U^\dagger A, \qquad B^\dagger U^\dagger VUC = C^\dagger B. \tag{A10}$$

Now, having Gram systems of ρ and its partial transpositions, we introduce the matrix

$$Q = \sum_{k=1}^{7} [|0\rangle|\Psi_k\rangle + |1\rangle|\Phi_k\rangle], \tag{A11}$$

with $|\Psi_k\rangle$ and $|\Phi_k\rangle$ denoting decompositions of ρ^{T_B} and $\rho^{T_{AB}}$ [as in (A4)] corresponding to their Gram systems introduced above, and $[|\psi\rangle]$ standing for a projection onto $|\psi\rangle$. In terms of the Gram systems (A9), Q assumes the form

$$Q = (A \;\; UB \;\; B \;\; UC \;\; A \;\; UB \;\; VB \;\; VUC). \tag{A12}$$

By careful counting of the dimensions, one notices that Q is an un-normalized density matrix acting on $\mathbb{C}^7 \otimes \mathbb{C}^3$ with respect to the bipartition $aAB|CD$, where a denotes the auxiliary subsystem [cf. (A11)], while $ABCD$ stand forthe subsystems of ρ. Also, by definition Q is of rank 7, and hence according to Ref. [32] Q is separable with respect to this bipartition provided that it is supported on $\mathbb{C}^7 \otimes \mathbb{C}^3$ and $Q^{T_{aAB}} \geqslant 0$. Although we cannot prove the former condition, it is clear that generic Q is supported on $\mathbb{C}^7 \otimes \mathbb{C}^3$. Then, after direct algebra and with the aid of conditions (A8) and (A10), one sees that the latter condition; that is, $Q^{T_{aAB}} \geqslant 0$ holds if the two equations

$$B^\dagger VUC = C^\dagger UB, \quad C^\dagger U^\dagger VUC = C^\dagger UC \tag{A13}$$

are obeyed. Still, exploiting the explicit forms of B and C, the above conditions can be further simplified, leading to a set of equations for scalar product of vectors composing the Gram system of ρ. Some of them, by virtue of the Eqs. (A8) and (A10) are immediately satisfied. Provided that the remaining equations also hold, one has that indeed $Q^{T_{aAB}} \geqslant 0$ and generic Q is separable.

Then, it is clear that by projecting the auxiliary qubit a of Q onto $|0\rangle$, one recovers ρ. So, if Q is separable across $aAB|CD$, then ρ is separable across $AB|CD$, implying that, due to the underlying symmetry, it is fully separable.

[1] R. Horodecki, M. Horodecki, P. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).

[2] O. Gühne and G. Tóth, Phys. Rep. **474**, 1 (2009).

[3] J. M. Leinaas, J. Myrheim, and E. Ovrum, Phys. Rev. A **76**, 034304 (2007).

[4] J. M. Leinaas, J. Myrheim, and P. Ø. Sollid, Phys. Rev. A **81**, 062329 (2010).

[5] P. Ø. Sollid, J. M. Leinaas, and J. Myrheim, Phys. Rev. A **84**, 042325 (2011).

[6] Ł. Skowronek, J. Math. Phys. **52**, 122202 (2011); L. Chen and D. Ž. Đoković, *ibid.* **52**, 122203 (2011).

[7] R. Augusiak, J. Grabowski, M. Kuś, and M. Lewenstein, Opt. Commun. **283**, 805 (2010).

[8] J. Tura, R. Augusiak, P. Hyllus, M. Kuś, J. Samsonowicz, and M. Lewenstein, Phys. Rev. A **85**, 060302(R) (2012).

[9] T. Vértesi and N. Brunner, Phys. Rev. Lett. **108**, 030403 (2012).

[10] T. Eggeling and R. F. Werner, Phys. Rev. A **63**, 042111 (2001); D. Chrusciński and A. Kossakowski, *ibid.* **73**, 062315 (2006); **73**, 062314 (2006); Open Sys. Information Dyn. **14**, 25 (2007).

[11] K. Eckert, J. Schliemann, D. Bruß, and M. Lewenstein, Ann. Phys. **299**, 88 (2002).

[12] J. K. Stockton, J. M. Geremia, A. C. Doherty, and H. Mabuchi, Phys. Rev. A **67**, 022112 (2003).

[13] T. Ichikawa, T. Sasaki, I. Tsutsui, and N. Yonezawa, Phys. Rev. A **78**, 052105 (2008).

[14] T. Bastin, S. Krins, P. Mathonet, M. Godefroid, L. Lamata, and E. Solano, Phys. Rev. Lett. **103**, 070503 (2009); T. Bastin, P. Mathonet, and E. Solano, arXiv:1011.1243.

[15] P. Mathonet, S. Krins, M. Godefroid, L. Lamata, E. Solano, and T. Bastin, Phys. Rev. A **81**, 052315 (2010); C. D. Cenci, D. W. Lyons, L. M. Snyder, and S. N. Walck, Quantum Inf. Comput. **10**, 1029 (2010).

[16] M. Hayashi, D. Markham, M. Murao, M. Owari, and S. Virmani, Phys. Rev. A **77**, 012104 (2008); R. Hübener, M. Kleinmann, T.-C. Wei, C. González-Guillen, and O. Gühne, *ibid.* **80**, 032324 (2009); M. Aulbach and D. Markham, and M. Murao, New J. Phys. **12**, 073025 (2010); J. Martin, O. Giraud, P. A. Braun, D. Braun, and T. Bastin, Phys. Rev. A **81**, 062347 (2010); D. J. H. Markham, *ibid.* **83**, 042332 (2011).

[17] G. Tóth and O. Gühne, Phys. Rev. Lett. **102**, 170503 (2009).

[18] G. Tóth and O. Gühne, Appl. Phys. B **98**, 617 (2010).

[19] E. Majorana, Nuovo Cimento **9**, 43 (1932).

[20] R. Prevedel, G. Cronenberg, M. S. Tame, M. Paternostro, P. Walther, M. S. Kim, and A. Zeilinger, Phys. Rev. Lett. **103**, 020503 (2009); W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, G. Tóth, and H. Weinfurter, *ibid.* **103**, 020504 (2009).

[21] Yun-Feng Huang, B.-H. Liu, L. Peng, Y.-H. Li, L. Li, C.-F. Li, and G.-C. Guo, Nat. Commun. **2**, 546 (2011).

[22] T. Bastin, C. Thiel, J. von Zanthier, L. Lamata, E. Solano, and G. S. Agarwal, Phys. Rev. Lett. **102**, 053601 (2009).

[23] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).

[24] W. Dür and J. I. Cirac, Phys. Rev. A **61**, 042314 (2000).

[25] A. Acín, D. Bruß, M. Lewenstein, and A. Sanpera, Phys. Rev. Lett. **87**, 040401 (2001).

[26] M. Lewenstein and A. Sanpera, Phys. Rev. Lett. **80**, 2261 (1998).

[27] M. Lewenstein, B. Kraus, J. I. Cirac, and P. Horodecki, Phys. Rev. A **62**, 052310 (2000).

[28] S. Karnas and M. Lewenstein, Phys. Rev. A **64**, 042313 (2001).

[29] A. Sanpera, D. Bruß, and M. Lewenstein, Phys. Rev. A **63**, 050301(R) (2001).

[30] S.-H. Kye and H. Osaka, J. Math. Phys. **53**, 052201 (2012), and references therein.
[31] B. Kraus, J. I. Cirac, S. Karnas, and M. Lewenstein, Phys. Rev. A **61**, 062302 (2000).
[32] P. Horodecki, M. Lewenstein, G. Vidal, and I. Cirac, Phys. Rev. A **62**, 032310 (2000).
[33] J. Samsonowicz, M. Kuś, and M. Lewenstein, Phys. Rev. A **76**, 022314 (2007).
[34] P. Badziąg, P. Horodecki, R. Horodecki, and R. Augusiak, arXiv:quant-ph/0703097.
[35] T. Cubitt, A. Montanaro, and A. Winter, J. Math. Phys. **49**, 022107 (2008).
[36] J. Eisert and H. J. Briegel, Phys. Rev. A **64**, 022306 (2001).
[37] B. M. Terhal and P. Horodecki, Phys. Rev. A **61**, 040301(R) (2000).